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GENERALIZED NUMERICAL INTEGRATION OF MOMENTS

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SUMMARY

Sampling points and weighting coefficients of the Gaussian type are presented for integrands typically encountered in axisymmetric finite elements. The proposed method is a generalization of *Gaussian Integration of Moments* for non-zero limits of integration. The method achieves one extra order of accuracy in the integration of polynomials as compared with the Gauss–Legendre method with the same number of sampling points. Although the locations of sampling points require the solution of non-linear equations, analytical solutions are presented for the cases of one and two sampling points. Special cases of these general expressions are shown to include both *Gauss–Legendre* integration corresponding to an integration range at a considerable distance from the axis of symmetry, and *Fishman* integration corresponding to an integration range whose lower limit lies on the axis of symmetry.

INTRODUCTION

Integration of finite element matrices in axisymmetry frequently involves evaluation of expressions of the form:

$$\int_{r_0}^{r_f} r f(r)\,\mathrm{d}r \tag{1}$$

where r is the radial distance from the axis of symmetry.

The best known example of this type in axisymmetric elasticity is the formation of the element stiffness matrix (see e.g. Zienkiewicz and Taylor[3]):

$$\mathbf{K}_{ij}^{e} = \iint \mathbf{B}_i^{\mathrm{T}} \mathbf{D} \mathbf{B}_j r\,\mathrm{d}r\,\mathrm{d}z \tag{2}$$

where r and z are the radial and axial co-ordinates, $\mathbf{K}^e$ is the element stiffness matrix, $\mathbf{B}$ is the axisymmetric strain/displacement matrix and $\mathbf{D}$ is the constitutive matrix. In axisymmetric elasticity, the $\mathbf{B}$ matrix depends on its radial co-ordinate with $1/r$ terms appearing in the stiffness formulation. Owing to the appearance of these reciprocal terms, the axisymmetric stiffness matrix cannot be integrated exactly using conventional quadrature.

This paper considers numerical integration formulae suitable for evaluating expressions of the type given in equation (1). The aim is to find the radial sampling points and weighting coefficients such that accuracy is optimized. Advantage will be taken of the fact that r always appears explicitly in the integrand.

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In contrast to Gauss–Legendre quadrature, it will be shown that this can be achieved by locating the sampling points non-symmetrically about the centre of the range of integration.

INTEGRATION OF MOMENTS

Sampling points and weighting coefficients for integrands of the type

$$\int_0^1 r f(r)\,\mathrm{d}r \approx \sum_{i=1}^{n} w_i f(r_i) \tag{3}$$

have been published[1] under the heading *Gaussian Integration of Moments*. Fishman's formulae assume limits of integration of $[0, 1]$, whereas in the present work, these formulae are extended to cover general limits of integration $[r_0, r_f]$, as indicated by equation (1), but in the modified form

$$\int_{r_0}^{r_f} r f(r)\,\mathrm{d}r \approx \sum_{i=1}^{n} W_i r_i f(r_i) \tag{4}$$

where the new weighting coefficients are equivalent to those given by Fishman divided by the corresponding sampling point, i.e.

$$W_i = \frac{w_i}{r_i} \quad \text{for all} \quad i = 1, 2, \ldots, n \tag{5}$$

This rearrangement leads to a more general formulation as will be shown, and is also more convenient for implementation in finite element codes such as those published by Smith and Griffiths.[2]

CALCULATION OF SAMPLING POINTS AND WEIGHTS

It is readily shown that the locations of the sampling points are given as the roots of a family of orthogonal polynomials. The order of the polynomial depends on the number of sampling points required.

In the case of n sampling points, we define the following polynomial whose roots will be the required sampling points:

$$G_n(r) = a_0 + a_1 r + a_2 r^2 + \cdots + a_{n-1} r^{n-1} + r^n \tag{6}$$

We then perform analytical integration on the following expressions, equating the result in each case to zero:

$$\begin{gathered} \int_{r_0}^{r_f} r G_n(r)\,\mathrm{d}r = 0 \\ \int_{r_0}^{r_f} r^2 G_n(r)\,\mathrm{d}r = 0 \\ \vdots \\ \int_{r_0}^{r_f} r^n G_n(r)\,\mathrm{d}r = 0 \end{gathered} \tag{7}$$

leading to n linear equations in the unknown coefficients $a_0, a_1, a_2, \ldots, a_{n-1}$. Once these coefficients are found, the polynomial $G_n(r)$ is defined and its roots can be estimated using any suitable numerical method. Analytical solutions for the roots of $G_n(r)$ are readily found only for n equal to one or two, and these are described in subsequent sections.

Having found the roots of $G_n(r)$ (i.e. $r_1, r_2, \ldots, r_n$), the corresponding weighting coefficients (i.e. $W_1, W_2, \ldots, W_n$) are found by solving the set of linear equations

$$\begin{aligned} W_1 r_1 + W_2 r_2 + \cdots + W_n r_n &= \frac{1}{2}(r_f^2 - r_0^2) \\ W_1 r_1^2 + W_2 r_2^2 + \cdots + W_n r_n^2 &= \frac{1}{3}(r_f^3 - r_0^3) \\ &\vdots \\ W_1 r_1^n + W_2 r_2^n + \cdots + W_n r_n^n &= \frac{1}{n+1}(r_f^{n+1} - r_0^{n+1}) \end{aligned} \tag{8}$$

The results of applying this method to the cases where $n = 1, 2$ and 3 are now described below.

One-point formula (global)

The polynomial is of the form

$$G_1(r) = a_0 + r \tag{9}$$

where

$$a_0 = -\frac{2(r_f^2 + r_0 r_f + r_0^2)}{3(r_f + r_0)} \tag{10}$$

leading to a sampling point and weighting coefficient given by

$$r_1 = \frac{2(r_f^2 + r_0 r_f + r_0^2)}{3(r_f + r_0)} \tag{11}$$

and

$$W_1 = \frac{3(r_f - r_0)(r_f + r_0)^2}{4(r_f^2 + r_f r_0 + r_0^2)} \tag{12}$$

Two-point formula (global)

The polynomial is of the form

$$G_2(r) = a_0 + a_1 r + r^2 \tag{13}$$

where

$$a_0 = \frac{3(r_f^4 + 4r_f^3 r_0 + 10r_f^2 r_0^2 + 4r_f r_0^3 + r_0^4)}{10(r_f^2 + 4r_f r_0 + r_0^2)} \tag{14}$$

and

$$a_1 = -\frac{6(r_f + r_0)(r_f^2 + 3r_f r_0 + r_0^2)}{5(r_f^2 + 4r_f r_0 + r_0^2)} \tag{15}$$

leading to sampling points and weighting coefficients given by

$$r_1, r_2 = \frac{3(\sqrt{6}D \pm F\sqrt{B})}{5\sqrt{6}E} \tag{16}$$

and

$$W_1, W_2 = \frac{5(3\sqrt{6}\sqrt{B}C \pm 2A)}{36(\sqrt{6}\sqrt{B}D \pm FB)} \tag{17}$$

where

$$\begin{aligned} A &= r_f^6 + 9r_f^5 r_0 + 9r_f^4 r_0^2 - 38r_f^3 r_0^3 + 9r_f^2 r_0^4 + 9r_f r_0^5 + r_0^6 \\ B &= r_f^4 + 10r_f^3 r_0 + 28r_f^2 r_0^2 + 10r_f r_0^3 + r_0^4 \\ C &= r_f^4 + 4r_f^3 r_0 - 4r_f r_0^3 - r_0^4 \\ D &= r_f^3 + 4r_f^2 r_0 + 4r_f r_0^2 + r_0^3 \\ E &= r_f^2 + 4r_f r_0 + r_0^2 \\ F &= r_f - r_0 \end{aligned} \tag{18}$$

Three-point formula (global)

The analytical expressions are becoming very messy, so for the three-point formula only the polynomial coefficients are provided. The polynomial is of the form

$$G_3(r) = a_0 + a_1 r + a_2 r^2 + r^3 \tag{19}$$

where

$$a_0 = -\frac{4(r_f^6 + 9r_f^5 r_0 + 45r_f^4 r_0^2 + 65r_f^3 r_0^3 + 45r_f^2 r_0^4 + 9r_f r_0^5 + r_0^6)}{35(r_f^2 + 8r_f r_0 + r_0^2)(r_f + r_0)} \tag{20}$$

$$a_1 = \frac{6(r_f^2 + 5r_f r_0 + r_0^2)(r_f^2 + 3r_f r_0 + r_0^2)}{7(r_f^2 + 8r_f r_0 + r_0^2)} \tag{21}$$

and

$$a_2 = -\frac{12(r_f^4 + 9r_f^3 r_0 + 15r_f^2 r_0^2 + 9r_f r_0^3 + r_0^4)}{7(r_f + r_0)(r_f^2 + 8r_f r_0 + r_0^2)} \tag{22}$$

and a suitable numerical technique can be used to find the roots of $G_3(r)$.

NORMALIZATION

A further refinement to the weights and sampling points described here is to normalize them with respect to a local radial co-ordinate system varying between -1 and $+1$. This is the local co-ordinate system used in many finite element codes involving quadrilateral finite elements, and is particularly convenient for Gauss–Legendre quadrature in which the weight and sampling points are symmetrical about the mid point of the element.

This transformation leads to *local* sampling points given by

$$\xi_i = \frac{(r_i - (r_f + r_0)/2)}{(r_f - r_0)/2} \quad \text{for all} \quad i = 1, 2, \ldots, n \tag{23}$$

and *local* weighting coefficients given by

$$H_i = \frac{W_i}{(r_f - r_0)/2} \quad \text{for all } i = 1, 2, \ldots, n \tag{24}$$

One of the advantages of this normalization process is that the local weights and sampling points can be expressed in dimensionless form in terms of the *ratio* of the global limits of integration r_0 and r_f where

$$R = \frac{r_0}{r_f} \tag{25}$$

With reference to the analytical expressions obtained in the previous section, the weights and sampling points in local co-ordinates are now obtained for the cases where $n = 1$ and 2. Numerical solutions are also provided for the case where $n = 3$.

One-point formula (local)

The local sampling point is given by

$$\xi_1 = \frac{1 - R}{3(1 + R)} \tag{26}$$

and the local weighting coefficient by

$$H_1 = \frac{3(1 + R)^2}{2(1 + R + R^2)} \tag{27}$$

The generality of these expressions is demonstrated by considering limiting cases of R. As the lower limit of integration r_0 approaches the axis of symmetry, the ratio R tends to zero, and the local sampling point and weight from equations (26) and (27) take the values

$$\xi_1 \approx \tfrac{1}{3} \tag{28}$$

and

$$H_1 \approx \tfrac{3}{2} \tag{29}$$

Reverting to limits of integration in the range [0, 1], the global sampling point and weighting coefficient become

$$r_1 \approx \tfrac{2}{3} \tag{30}$$

and

$$W_1 \approx \tfrac{3}{4} \quad \text{hence} \quad w_1 \approx \tfrac{1}{2} \tag{31}$$

which are identical to the Fishman values for one sampling point with limits of integration [0, 1].

Alternatively, at a considerable distance from the axis of symmetry, the effects of axisymmetry become negligible, the ratio R tends to unity, and the local sampling point and weight from equations (26) and (27) take the values

$$\xi_1 \approx 0 \tag{32}$$

and

$$H_1 \approx 2 \tag{33}$$

which are identical to the Gauss–Legendre values for one sampling point with limits of integration [− 1, 1].

The variations of ξ_1 and H_1 in the range $0 \leqslant R \leqslant 1$ for one-point integration are given in Table I and shown graphically in Figure 1.

Two-point formula (local)

The local sampling points are given by

$$\xi_1, \xi_2 = \frac{e \pm \sqrt{6}\sqrt{c}}{5a} \tag{34}$$

and the local weighting coefficients by

$$H_1, H_2 = \frac{125\sqrt{6}a^4}{18[4\sqrt{6}bc \pm 3de\sqrt{c}]} \tag{35}$$

where

$$\begin{aligned} a &= 1 + 4R + R^2 \\ b &= 2 + 10R + 21R^2 + 10R^3 + 2R^4 \\ c &= 1 + 10R + 28R^2 + 10R^3 + R^4 \\ d &= 1 + 10R + 38R^2 + 10R^3 + R^4 \\ e &= 1 - R^2 \end{aligned} \tag{36}$$

Considering the same limiting values of R discussed in the previous section, when the ratio R

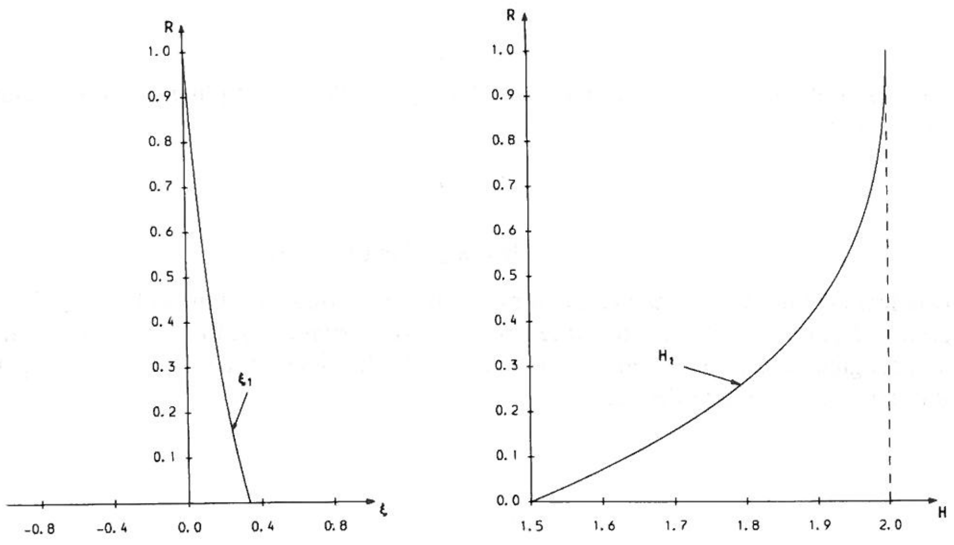

Figure 1. Sampling point ξ and weighting coefficient H vs. ratio $R = r_0/r_f$ (one-point integration)

Table I. One-point integration ($n = 1$)

R	ξ_1	H_1
0·00	0·333333	1·500000
0·02	0·320261	1·529400
0·04	0·307692	1·557604
0·06	0·295597	1·584618
0·08	0·283951	1·610457
0·10	0·272727	1·635135
0·12	0·261905	1·658674
0·14	0·251462	1·681097
0·16	0·241379	1·702429
0·18	0·231638	1·722699
0·20	0·222222	1·741935
0·22	0·213115	1·760170
0·24	0·204301	1·777435
0·26	0·195767	1·793763
0·28	0·187500	1·809187
0·30	0·179487	1·823741
0·32	0·171717	1·837458
0·34	0·164179	1·850371
0·36	0·156863	1·862513
0·38	0·149758	1·873918
0·40	0·142857	1·884615
0·42	0·136150	1·894638
0·44	0·129630	1·904016
0·46	0·123288	1·912778
0·48	0·117117	1·920954
0·50	0·111111	1·928571
0·52	0·105263	1·935657
0·54	0·099567	1·942236
0·56	0·094017	1·948335
0·58	0·088608	1·953976
0·60	0·083333	1·959184
0·62	0·078189	1·963979
0·64	0·073171	1·968384
0·66	0·068273	1·972418
0·68	0·063492	1·976102
0·70	0·058824	1·979452
0·72	0·054264	1·982487
0·74	0·049808	1·985225
0·76	0·045455	1·987680
0·78	0·041199	1·989868
0·80	0·037037	1·991803
0·82	0·032967	1·993500
0·84	0·028986	1·994972
0·86	0·025090	1·996230
0·88	0·021277	1·997288
0·90	0·017544	1·998155
0·92	0·013889	1·998843
0·94	0·010309	1·999363
0·96	0·006803	1·999722
0·98	0·003367	1·999932
1·00	0·000000	2·000000

$$R = r_0/r_f$$

$$\left.\begin{aligned} r_i &= (r_f + r_0)/2 + \xi_i(r_f - r_0)/2 \\ W_i &= H_i(r_f - r_0)/2 \end{aligned}\right\} \text{ for all } i = 1, 2, \ldots, n$$

tends to zero, the local sampling points and weights from equations (34) and (35) take the values

$$\xi_1, \xi_2 \approx \frac{1 \pm \sqrt{6}}{5} \tag{37}$$

and

$$H_1, H_2 \approx \frac{125\sqrt{6}}{18(8\sqrt{6} \pm 3)} \tag{38}$$

Reverting to limits of integration in the range [0, 1], the global sampling points and weighting coefficients become

$$r_1, r_2 \approx \frac{6 \pm \sqrt{6}}{10} \tag{39}$$

and

$$W_1, W_2 \approx \frac{125\sqrt{6}}{36(8\sqrt{6} \pm 3)} \quad \text{hence} \quad w_1, w_2 \approx \frac{25(\sqrt{6} \pm 1)}{12(8\sqrt{6} \pm 3)} \tag{40}$$

which are identical to the Fishman values for two sampling point with limits of integration [0, 1].

When the ratio R tends to unity, the local sampling points and weights from equations (34) and (35) take the values

$$\xi_1, \xi_2 \approx \pm \frac{1}{\sqrt{3}} \tag{41}$$

and

$$H_1, H_2 \approx 1 \tag{42}$$

which are identical to the Gauss–Legendre values for two sampling points with limits of integration [− 1, 1].

The variations of ξ_1, ξ_2 and H_1, H_2 in the range $0 \leqslant R \leqslant 1$ for two-point integration are given in Table II and shown graphically in Figure 2.

Three-point formula (*local*)

The coefficients given in equations (20)–(22) have been obtained in the range $0 \leqslant R \leqslant 1$ and the roots of $G_3(r)$ found numerically. The weighting coefficients were then computed from equations (8). Following normalization, the variations of ξ_1, ξ_2, ξ_3 and H_1, H_2, H_3 in the range $0 \leqslant R \leqslant 1$ for three-point integration are given in Table III and shown graphically in Figure 3.

DISCUSSION OF METHOD

The weights and sampling points described in this paper are specially designed for integrands of the type described in equation (1). The formulae are a generalization of *Gaussian Integration of Moments* for arbitrary limits of integration $[r_0, r_f]$.

Sampling points and weights given by Fishman and Gauss–Legendre are both special cases of the proposed formulation. This is clearly indicated in Figures 1, 2 and 3 which show the transition from Fishman ($R = 0$) to Gauss–Legendre ($R = 1$) sampling points and weighting coefficients. In

Table II. Two-point integration ($n = 2$)

R	ξ_1	H_1	ξ_2	H_2
0·00	− 0·289898	1·024972	0·689898	0·752806
0·02	− 0·314007	1·044020	0·684092	0·766898
0·04	− 0·335148	1·060176	0·678949	0·779363
0·06	− 0·353833	1·073568	0·674322	0·790554
0·08	− 0·370466	1·084420	0·670104	0·800726
0·10	− 0·385366	1·092998	0·666217	0·810067
0·12	− 0·398791	1·099578	0·662602	0·818720
0·14	− 0·410949	1·104426	0·659215	0·826795
0·16	− 0·422014	1·107789	0·656019	0·834377
0·18	− 0·432126	1·109890	0·652988	0·841535
0·20	− 0·441404	1·110926	0·650100	0·848323
0·22	− 0·449948	1·111069	0·647335	0·854785
0·24	− 0·457844	1·110469	0·644680	0·860957
0·26	− 0·465162	1·109252	0·642121	0·866871
0·28	− 0·471965	1·107528	0·639650	0·872552
0·30	− 0·478305	1·105390	0·637257	0·878020
0·32	− 0·484230	1·102917	0·634935	0·883293
0·34	− 0·489779	1·100173	0·632678	0·888389
0·36	− 0·494988	1·097217	0·630480	0·893319
0·38	− 0·499887	1·094093	0·628336	0·898097
0·40	− 0·504504	1·090843	0·626243	0·902731
0·42	− 0·508863	1·087498	0·624197	0·907231
0·44	− 0·512985	1·084087	0·622194	0·911606
0·46	− 0·516890	1·080633	0·620233	0·915861
0·48	− 0·520595	1·077155	0·618310	0·920004
0·50	− 0·524115	1·073669	0·616423	0·924040
0·52	− 0·527465	1·070188	0·614571	0·927974
0·54	− 0·530655	1·066724	0·612751	0·931811
0·56	− 0·533699	1·063284	0·610961	0·935555
0·58	− 0·536605	1·059877	0·609201	0·939211
0·60	− 0·539384	1·056508	0·607470	0·942781
0·62	− 0·542044	1·053182	0·605764	0·946268
0·64	− 0·544593	1·049902	0·604085	0·949677
0·66	− 0·547037	1·046672	0·602430	0·953009
0·68	− 0·549383	1·043494	0·600799	0·956268
0·70	− 0·551638	1·040370	0·599190	0·959456
0·72	− 0·553806	1·037300	0·597604	0·962574
0·74	− 0·555893	1·034285	0·596039	0·965626
0·76	− 0·557903	1·031326	0·594494	0·968613
0·78	− 0·559841	1·028422	0·592969	0·971536
0·80	− 0·561711	1·025574	0·591463	0·974399
0·82	− 0·563516	1·022781	0·589976	0·977202
0·84	− 0·565260	1·020042	0·588507	0·979948
0·86	− 0·566946	1·017357	0·587056	0·982637
0·88	− 0·568577	1·014726	0·585621	0·985271
0·90	− 0·570156	1·012146	0·584204	0·987852
0·92	− 0·571685	1·009618	0·582802	0·990381
0·94	− 0·573166	1·007141	0·581416	0·992859
0·96	− 0·574603	1·004713	0·580046	0·995287
0·98	− 0·575997	1·002333	0·578691	0·997667
1·00	− 0·577350	1·000000	0·577350	1·000000

$$R = r_0/r_f$$

$$\left.\begin{aligned} r_i &= (r_f + r_0)/2 + \xi_i(r_f - r_0)/2 \\ W_i &= H_i(r_f - r_0)/2 \end{aligned}\right\} \text{ for all } \quad i = 1, 2, \ldots, n$$

Table III. Three-point integration ($n = 3$)

R	ξ_1	H_1	ξ_2	H_2	ξ_3	H_3
0·00	− 0·575319	0·657689	0·181066	0·776387	0·822824	0·440924
0·02	− 0·602947	0·668858	0·166627	0·790048	0·819697	0·448705
0·04	− 0·624503	0·676001	0·155039	0·800900	0·817174	0·454976
0·06	− 0·641782	0·679637	0·145314	0·809855	0·815040	0·460274
0·08	− 0·655941	0·680546	0·136889	0·817456	0·813173	0·464901
0·10	− 0·667759	0·679461	0·129418	0·824042	0·811499	0·469042
0·12	− 0·677777	0·676981	0·122677	0·829836	0·809971	0·472814
0·14	− 0·686380	0·673560	0·116512	0·834994	0·808556	0·476297
0·16	− 0·693853	0·669534	0·110817	0·839625	0·807233	0·479548
0·18	− 0·700407	0·665146	0·105513	0·843813	0·805985	0·482608
0·20	− 0·706206	0·660565	0·100541	0·847619	0·804799	0·485508
0·22	− 0·711375	0·655913	0·095855	0·851094	0·803667	0·488270
0·24	− 0·716015	0·651273	0·091419	0·854276	0·802581	0·490913
0·26	− 0·720203	0·646700	0·087204	0·857197	0·801535	0·493452
0·28	− 0·724006	0·642231	0·083188	0·859885	0·800525	0·495897
0·30	− 0·727475	0·637891	0·079350	0·862363	0·799547	0·498260
0·32	− 0·730654	0·633693	0·075674	0·864648	0·798597	0·500548
0·34	− 0·733579	0·629645	0·072147	0·866759	0·797673	0·502767
0·36	− 0·736280	0·625750	0·068757	0·868709	0·796772	0·504924
0·38	− 0·738783	0·622008	0·065493	0·870512	0·795893	0·507023
0·40	− 0·741110	0·618414	0·062346	0·872179	0·795034	0·509068
0·42	− 0·743279	0·614966	0·059308	0·873720	0·794193	0·511064
0·44	− 0·745307	0·611658	0·056373	0·875145	0·793370	0·513014
0·46	− 0·747208	0·608485	0·053533	0·876462	0·792562	0·514920
0·48	− 0·748993	0·605440	0·050783	0·877678	0·791770	0·516785
0·50	− 0·750674	0·602517	0·048118	0·878801	0·790991	0·518612
0·52	− 0·752260	0·599712	0·045534	0·879837	0·790226	0·520401
0·54	− 0·753759	0·597016	0·043025	0·880792	0·789473	0·522156
0·56	− 0·755178	0·594426	0·040588	0·881670	0·788733	0·523879
0·58	− 0·756524	0·591936	0·038220	0·882478	0·788003	0·525569
0·60	− 0·757803	0·589540	0·035917	0·883219	0·787284	0·527229
0·62	− 0·759020	0·587233	0·033677	0·883898	0·786576	0·528861
0·64	− 0·760179	0·585012	0·031495	0·884518	0·785877	0·530465
0·66	− 0·761285	0·582871	0·029370	0·885084	0·785187	0·532042
0·68	− 0·762342	0·580806	0·027299	0·885598	0·784506	0·533594
0·70	− 0·763352	0·578813	0·025280	0·886064	0·783834	0·535121
0·72	− 0·764320	0·576889	0·023310	0·886485	0·783171	0·536625
0·74	− 0·765248	0·575030	0·021389	0·886864	0·782515	0·538105
0·76	− 0·766138	0·573234	0·019512	0·887202	0·781866	0·539564
0·78	− 0·766993	0·571496	0·017680	0·887503	0·781225	0·541001
0·80	− 0·767815	0·569814	0·015890	0·887769	0·780591	0·542417
0·82	− 0·768607	0·568186	0·014141	0·888002	0·779964	0·543813
0·84	− 0·769369	0·566608	0·012431	0·888203	0·779344	0·545189
0·86	− 0·770104	0·565079	0·010758	0·888375	0·778730	0·546546
0·88	− 0·770813	0·563595	0·009122	0·888519	0·778122	0·547885
0·90	− 0·771497	0·562156	0·007521	0·888638	0·777521	0·549206
0·92	− 0·772159	0·560759	0·005953	0·888731	0·776925	0·550509
0·94	− 0·772798	0·559402	0·004419	0·888802	0·776335	0·551795
0·96	− 0·773417	0·558084	0·002916	0·888851	0·775750	0·553065
0·98	− 0·774016	0·556802	0·001443	0·888880	0·775171	0·554318
1·00	− 0·774597	0·555556	0·000000	0·888889	0·774597	0·555556

$$R = r_0/r_f$$

$$\left.\begin{aligned} r_i &= (r_f + r_0)/2 + \xi_i(r_f - r_0)/2 \\ W_i &= H_i(r_f - r_0)/2 \end{aligned}\right\} \quad \text{for all} \quad i = 1, 2, \ldots, n$$

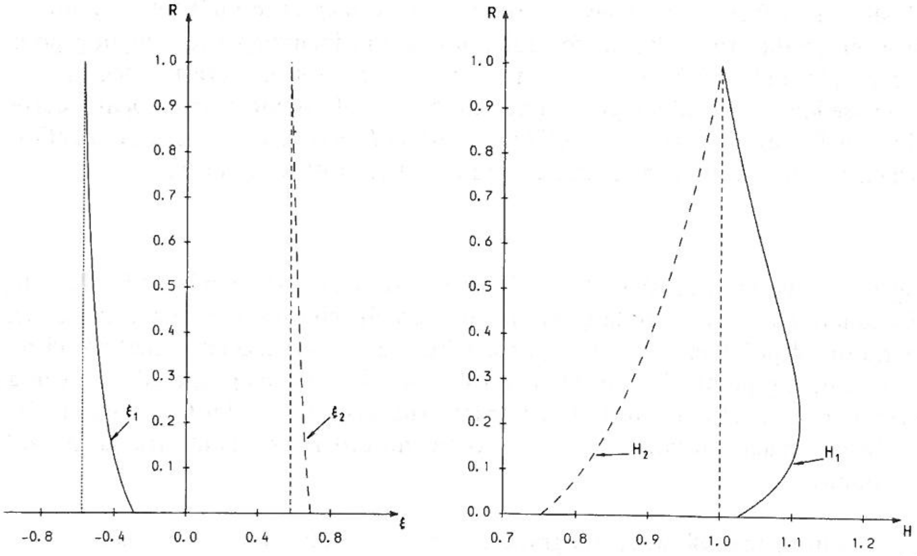

Figure 2. Sampling points ξ and weighting coefficients H vs. ratio $R = r_0/r_f$ (two-point integration)

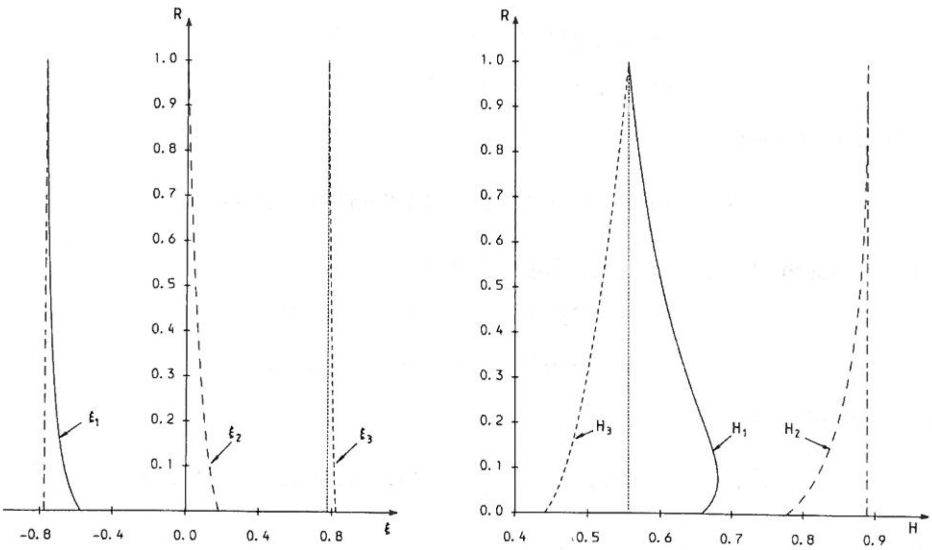

Figure 3. Sampling points ξ and weighting coefficients H vs. ratio $R = r_0/r_f$ (three-point integration)

both the two- and three-point integration formulae (Figures 2 and 3), the weighting coefficient corresponding to the smallest sampling point exhibits a maximum value. In Figure 2, the maximum occurs in H_1 when $R \approx 0{\cdot}22$, and in Figure 3 when $R \approx 0{\cdot}08$. The significance of these stationary points is not fully understood at present.

The Fishman coefficients have been normalized with respect to limits of integration $[-1, 1]$ and are given in the Appendix for formulae up to and including five sampling points. These sampling points and weighting coefficients are suitable for numerical integration over any domain whose lower limit of integration lies on the axis of symmetry ($r_0 = 0$) and correspond to those given in Tables I–III with $R = 0$. The case when $R = 0$ deserves a special mention, because this is when the influence of the radial co-ordinate r is most pronounced.

Accuracy

Owing to the assumption that the variable r always appears explicitly in the integrand as shown in equation (1), the formula given in equation (4) achieves one extra order of accuracy in the integration of polynomials as compared with the Gauss–Legendre method with the same number of sampling points. For example, a two-point formula as proposed here will give exact solutions for the function $f(r)$ up to third order, whereas a two-point Gauss–Legendre formula would only give exact solutions for $f(r)$ up to second order. This is illustrated in the following simple example.

Example. Evaluate the following integrand using a two-point integration rule:

$$\int_1^2 rr^3 \, \mathrm{d}r$$

Comparing with equation (1), $f(r) = r^3$ with limits of integration $r_0 = 1$ and $r_f = 2$. By Gauss–Legendre,

$$r_1 = 1{\cdot}21132 \quad \text{and} \quad r_2 = 1{\cdot}78868$$
$$W_1 = 0{\cdot}5 \quad \text{and} \quad W_2 = 0{\cdot}5$$

hence from equation (4),

$$\int_1^2 rr^3 \, \mathrm{d}r \approx 0{\cdot}5(1{\cdot}21132^4 + 1{\cdot}78868^4) = 6{\cdot}19444$$

By the present method and equations (16) and (17),

$$r_1 = 1{\cdot}23794 \quad \text{and} \quad r_2 = 1{\cdot}80821$$
$$W_1 = 0{\cdot}53683 \quad \text{and} \quad W_2 = 0{\cdot}46202$$

hence from equation (4),

$$\int_1^2 rr^3 \, \mathrm{d}r = 0{\cdot}53683(1{\cdot}23794^4) + 0{\cdot}46202(1{\cdot}80821^4) = 6{\cdot}20000$$

which, as expected, agrees with the exact solution.

FINITE ELEMENT IMPLICATIONS

In this section, the Integration of Moments described previously is compared with Gauss–Legendre integration with respect to the integration of some simple finite element matrices. Stiffness and mass matrices are considered as given by the expressions

$$\mathbf{K}_{ij}^e = \iint \mathbf{B}_i^{\mathrm{T}} \mathbf{D} \mathbf{B}_j r \, \mathrm{d}r \, \mathrm{d}z \tag{43}$$

and

$$\mathbf{M}_{ij}^{e} = \rho \int\int \mathbf{N}_i^{\mathrm{T}} \mathbf{N}_j r \, \mathrm{d}r \, \mathrm{d}z \tag{44}$$

respectively, where ρ is the mass density and $\mathbf{N}$ represents the shape functions.

Discussion is limited to the case where rectangular finite elements are oriented parallel to the global r- and z-axes. The sampling points in the z-direction are of the Gauss–Legendre type and are symmetrical about the centreline. The sampling points in the r-direction are of the Moments type and are shifted away from the axis of symmetry. For the case in which the lower radial limit of integration lies on the axis of symmetry ($R = 0$), the sampling point and weighting coefficient are given in the Appendix.

Four-node element

A four-node axisymmetric element with one integrating point in each direction is shown in Figure 4. This one-point method is now used to form the element stiffness and mass matrices.

The stiffness matrix given by equation (43) requires integration of expressions which include terms with powers ranging from r^3 to r^{-1}. Assuming symmetry, the maximum and minimum powers of r in each term of the matrix $\mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}$ are given below:

$$\begin{bmatrix} -2,2 & -1,1 & -2,2 & -1,1 & -1,2 & 0,1 & -1,2 & 0,1 \\ & 0,2 & -1,1 & 0,2 & 0,1 & 0,2 & 0,1 & 0,2 \\ & & -2,2 & -1,1 & -1,2 & 0,1 & -1,2 & 0,1 \\ & & & 0,2 & 0,1 & 0,2 & 0,1 & 0,2 \\ & & & & 0,2 & 0,1 & 0,2 & 0,1 \\ & & & & & 0,2 & 0,1 & 0,2 \\ & & & & & & 0,2 & 0,1 \\ & & & & & & & 0,2 \end{bmatrix} \tag{45}$$

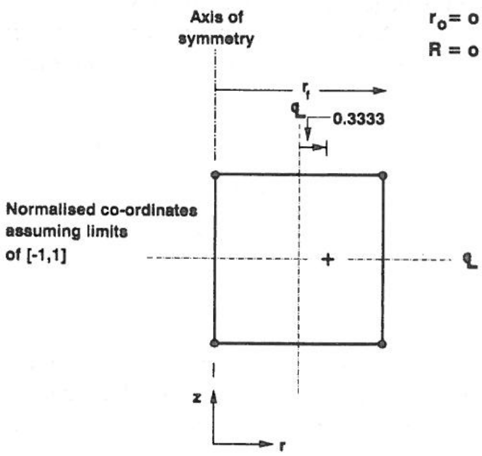

Figure 4. One-point integration of a four-node element

Noting that all terms are multiplied by r before integration, all integrands include at least r^2 terms. Hence, a one-point Gauss–Legendre approach would be unable to exactly integrate any of the components of this matrix.

When using the Moments approach, the radial multiplier r is already taken into account, so only the powers of r present in the $\mathbf{B}^{\mathrm{T}}\mathbf{DB}$ component of the integrand need to be considered. One-point Moments integration will exactly integrate linear terms in $\mathbf{B}^{\mathrm{T}}\mathbf{DB}$ which correspond to those marked with the letter E below:

$$\begin{bmatrix} \ast & \ast & \ast & \ast & \ast & \mathrm{E} & \ast & \mathrm{E} \\ & \ast & \ast & \ast & \mathrm{E} & \ast & \mathrm{E} & \ast \\ & & \ast & \ast & \ast & \mathrm{E} & \ast & \mathrm{E} \\ & & & \ast & \mathrm{E} & \ast & \mathrm{E} & \ast \\ & & & & \ast & \mathrm{E} & \ast & \mathrm{E} \\ & & & & & \ast & \mathrm{E} & \ast \\ & & & & & & \ast & \mathrm{E} \\ & & & & & & & \ast \end{bmatrix} \tag{46}$$

The improved performance of the present method over Gauss–Legendre regarding linear terms in $\mathbf{B}^{\mathrm{T}}\mathbf{DB}$ should be set against an inferior performance when integrating the terms with negative powers of r. For example, the terms in $\mathbf{B}^{\mathrm{T}}\mathbf{DB}$ containing r^{-1} can be integrated exactly by Gauss–Legendre after multiplication by r provided a sufficient number of sampling points are included. Moments integration will never be able to exactly integrate the terms containing r^{-1}.

Gauss–Legendre will also perform better on the terms containing r^{-2} because these are increased by one order after multiplication by r. This means that, when comparing the performance of the two methods with two sampling points, Gauss–Legendre performs better, integrating 60 terms exactly (out of 64) compared with 44 by Moments integration. Neither method is capable of exact integration on the r^{-2} terms, however, irrespective of the number of sampling points included.

Every term in the mass matrix given by equation (44) requires integration of terms which contains positive powers up to r^3. Assuming symmetry, the maximum and minimum powers of r in each term of the matrix $\mathbf{N}^{\mathrm{T}}\mathbf{N}$ are given below (radial terms only included):

$$\begin{bmatrix} 0,2 & 0,2 & 1,2 & 1,2 \\ & 0,2 & 1,2 & 1,2 \\ & & 2,2 & 2,2 \\ & & & 2,2 \end{bmatrix} \tag{47}$$

Although in this case Moments integration would be more accurate than Gauss–Legendre, neither method is capable of exact integration of any of the terms of the mass matrix with only one sampling point. Both methods with two points, however, will integrate the mass matrix exactly.

Eight-node element

An eight-node axisymmetric element with two integrating points in each direction is shown in Figure 5. This two-point method is now used to form the element stiffness and mass matrices.

The stiffness matrix given by equation (43) requires integration of expressions which include terms with powers ranging from r^5 to r^{-1}. Assuming symmetry, the maximum and minimum powers of r in each term of the matrix $\mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}$ are given below:

$$
\begin{bmatrix}
-2,4 & -1,3 & -2,3 & -1,2 & -2,4 & -1,3 & -1,4 & 0,3 & -1,4 & 0,3 & -1,3 & 0,2 & -1,4 & 0,3 & -1,4 & 0,3 \\
 & 0,4 & -1,2 & 0,3 & -1,3 & 0,4 & 0,3 & 0,4 & 0,3 & 0,4 & 0,2 & 0,3 & 0,3 & 0,4 & 0,3 & 0,4 \\
 & & -2,2 & -1,1 & -2,3 & -1,2 & -1,3 & 0,2 & -1,3 & 0,2 & -1,2 & 0,1 & -1,3 & 0,2 & -1,3 & 0,2 \\
 & & & 0,2 & -1,2 & 0,3 & 0,2 & 0,3 & 0,2 & 0,3 & 0,1 & 0,2 & 0,2 & 0,3 & 0,2 & 0,3 \\
 & & & & -2,4 & -1,3 & -1,4 & 0,3 & -1,4 & 0,3 & -1,3 & 0,2 & -1,4 & 0,3 & -1,4 & 0,3 \\
 & & & & & 0,4 & 0,3 & 0,4 & 0,3 & 0,4 & 0,2 & 0,3 & 0,3 & 0,4 & 0,3 & 0,4 \\
 & & & & & & 0,4 & 1,3 & 0,4 & 1,3 & 0,3 & 1,2 & 0,4 & 1,3 & 0,4 & 1,3 \\
 & & & & & & & 0,4 & 1,3 & 0,4 & 1,2 & 0,3 & 1,3 & 0,4 & 1,3 & 0,4 \\
 & & & & & & & & 0,4 & 1,3 & 0,3 & 1,2 & 0,4 & 1,3 & 0,4 & 1,3 \\
 & & & & & & & & & 0,4 & 1,2 & 0,3 & 1,3 & 0,4 & 1,3 & 0,4 \\
 & & & & & & & & & & 0,2 & 0,1 & 0,3 & 1,2 & 0,3 & 1,2 \\
 & & & & & & & & & & & 0,2 & 1,2 & 0,3 & 1,2 & 0,3 \\
 & & & & & & & & & & & & 0,4 & 1,3 & 0,4 & 1,3 \\
 & & & & & & & & & & & & & 0,4 & 1,3 & 0,4 \\
 & & & & & & & & & & & & & & 0,4 & 1,3 \\
 & & & & & & & & & & & & & & & 0,4
\end{bmatrix} \qquad (48)
$$

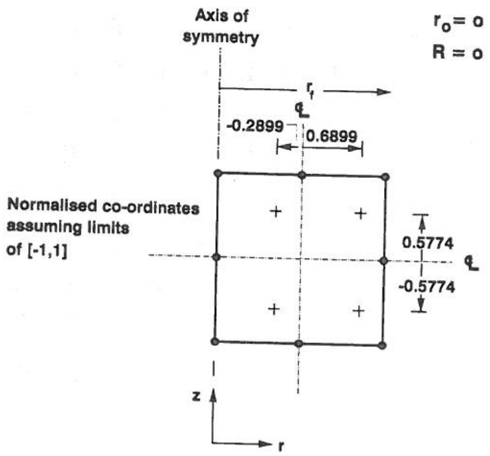

Figure 5. Two-point integration of an eight-node element

Noting that all terms are multiplied by r before integration, two-point Gauss–Legendre will exactly integrate terms in $\mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}$ containing powers of r ranging from r^{-1} to r^2, as indicated with the letter E below:

$$
\begin{bmatrix}
\ast & \ast & \ast & E & \ast & \ast & \ast & \ast & \ast & \ast & \ast & E & \ast & \ast & \ast & \ast \\
 & \ast & E & \ast & \ast & \ast & \ast & \ast & \ast & E & \ast & \ast & \ast & \ast & \ast & \ast \\
 & & \ast & \ast & \ast & E & \ast & E & \ast & E & E & E & \ast & E & \ast & E \\
 & & & E & E & \ast & E & \ast & E & \ast & E & \ast & E & \ast & E & \ast \\
 & & & & \ast & \ast & \ast & \ast & \ast & \ast & \ast & E & \ast & \ast & \ast & \ast \\
 & & & & & \ast & \ast & \ast & \ast & \ast & E & \ast & \ast & \ast & \ast & \ast \\
 & & & & & & \ast & \ast & \ast & \ast & \ast & E & \ast & \ast & \ast & \ast \\
 & & & & & & & \ast & \ast & \ast & E & \ast & \ast & \ast & \ast & \ast \\
 & & & & & & & & \ast & \ast & \ast & E & \ast & \ast & \ast & \ast \\
 & & & & & & & & & \ast & E & \ast & \ast & \ast & \ast & \ast \\
 & & & & & & & & & & E & E & \ast & E & \ast & E \\
 & & & & & & & & & & & E & E & \ast & E & \ast \\
 & & & & & & & & & & & & \ast & \ast & \ast & \ast \\
 & & & & & & & & & & & & & \ast & \ast & \ast \\
 & & & & & & & & & & & & & & \ast & \ast \\
 & & & & & & & & & & & & & & & \ast
\end{bmatrix} \tag{49}
$$

Two-point Moments integration will exactly integrate terms in $\mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}$ containing positive powers up to r^3, as indicated in equation (50).

$$
\begin{bmatrix}
\ast & \ast & \ast & \ast & \ast & \ast & \ast & E & \ast & E & \ast & E & \ast & E & \ast & E \\
 & \ast & \ast & E & \ast & \ast & E & \ast & E & \ast & E & \ast & E & \ast & E & \ast \\
 & & \ast & \ast & \ast & \ast & \ast & E & \ast & E & \ast & E & \ast & E & \ast & E \\
 & & & E & \ast & E & E & E & E & E & E & E & E & E & E & E \\
 & & & & \ast & \ast & \ast & E & \ast & E & \ast & E & \ast & E & \ast & E \\
 & & & & & \ast & E & \ast & E & \ast & E & E & E & \ast & E & \ast \\
 & & & & & & \ast & E & \ast & E & E & E & \ast & E & \ast & E \\
 & & & & & & & \ast & E & \ast & E & E & E & \ast & E & \ast \\
 & & & & & & & & \ast & E & E & E & \ast & E & \ast & E \\
 & & & & & & & & & \ast & E & E & E & \ast & E & \ast \\
 & & & & & & & & & & E & E & E & E & E & E \\
 & & & & & & & & & & & E & E & E & E & E \\
 & & & & & & & & & & & & \ast & E & \ast & E \\
 & & & & & & & & & & & & & \ast & E & \ast \\
 & & & & & & & & & & & & & & \ast & E \\
 & & & & & & & & & & & & & & & \ast
\end{bmatrix} \tag{50}
$$

Considering the whole matrix of 256 terms, two-point Gaus–Legendre integrates 59 terms exactly compared with 147 by two-point Moments integration.

As noted previously, the terms in $\mathbf{B}^T\mathbf{D}\mathbf{B}$ containing negative powers of r can never be integrated exactly by the Moments approach. Gauss–Legendre can integrate exactly those terms containing r^{-1} due to the axisymmetric multiplier r, but not those containing r^{-2}. This means that, when comparing the performance of the two methods with three sampling points, Gauss–Legendre performs better, integrating 249 terms exactly compared with 199 by Moments integration.

The consistent mass matrix given by equation (44) for eight-node elements requires integration of terms with positive powers up to r^5. Assuming symmetry, the maximum and minimum powers of r in each term of the matrix $\mathbf{N}^T\mathbf{N}$ are given below (radial terms only included):

$$\begin{bmatrix} 0,4 & 0,3 & 0,4 & 1,4 & 1,4 & 1,3 & 1,4 & 1,4 \\ & 0,2 & 0,3 & 1,3 & 1,3 & 1,2 & 1,3 & 1,3 \\ & & 0,4 & 1,4 & 1,4 & 1,3 & 1,4 & 1,4 \\ & & & 2,4 & 2,4 & 2,3 & 2,4 & 2,4 \\ & & & & 2,4 & 2,3 & 2,4 & 2,4 \\ & & & & & 2,2 & 2,3 & 2,3 \\ & & & & & & 2,4 & 2,4 \\ & & & & & & & 2,4 \end{bmatrix} \tag{51}$$

Two-point Gauss–Legendre will exactly integrate terms in $\mathbf{N}^T\mathbf{N}$ with positive powers up to r^2, accounting for the terms marked with the letter E in equation (52).

$$\begin{bmatrix} * & * & * & * & * & * & * & * \\ & \mathrm{E} & * & * & * & \mathrm{E} & * & * \\ & & * & * & * & * & * & * \\ & & & * & * & * & * & * \\ & & & & * & * & * & * \\ & & & & & \mathrm{E} & * & * \\ & & & & & & * & * \\ & & & & & & & * \end{bmatrix} \tag{52}$$

Two-point integration by Moments will exactly integrate terms with positive powers up to r^3, as marked in equation (53).

$$\begin{bmatrix} \ast & E & \ast & \ast & \ast & E & \ast & \ast \\ & E & E & E & E & E & E & E \\ & & \ast & \ast & \ast & E & \ast & \ast \\ & & & \ast & \ast & E & \ast & \ast \\ & & & & \ast & E & \ast & \ast \\ & & & & & E & E & E \\ & & & & & & \ast & \ast \\ & & & & & & & \ast \end{bmatrix} \tag{53}$$

Consideration of the whole matrix of 64 terms shows that two-point Gauss–Legendre integrates 4 terms exactly compared with 28 by two-point Moments integration. Both methods with three sampling points will integrate the mass matrix exactly.

CONCLUDING REMARKS

Numerical integration of moments is a specialized technique suitable for integrands of the form given in equation (1). The method can exactly integrate polynomials of one order higher than the corresponding Gauss–Legendre approach with the same number of sampling points. In the present work, the method has been generalized to allow arbitrary limits of integration. By presenting the results in a normalized form, it was shown that the sampling points and weighting coefficients could be expressed in terms of the ratio of the limits of integration $R = r_0/r_f$. These expressions tended to be quite complicated for higher order integration, but for the cases of one and two sampling points analytical solutions have been derived. It was shown that Gauss–Legendre weights and sampling points were given as special cases of the formulation corresponding to $R = 1$.

To illustrate the technique, it was applied to the integration of simple mass and stiffness matrices of rectangular axisymmetric finite elements. It was shown that, for low order integration formulae, the Moments approach led to a more accurate representation of both the stiffness and mass matrices. For higher order integration, however, the Gauss–Legendre method led to a more accurate stiffness matrix owing to the presence of terms containing r^{-1} and r^{-2}.

APPENDIX

Normalized abscissae and weights for Gaussian Integration of Moments

$$\int_0^{r_f} r f(r)\,\mathrm{d}r \approx \sum_{i=1}^{n} W_i r_i f(r_i)$$

where $r_i = r_f(1 + \xi_i)/2$ and $W_i = r_f H_i/2$ for all $i = 1, 2, \ldots, n$

n	ξ_i	H_i
1	0·33333 33333	1·50000 00000
2	− 0·28989 79486	1·02497 16524
	0·68989 79486	0·75280 61254
3	− 0·57531 89236	0·65768 86399
	0·18106 62711	0·77638 69375
	0·82282 40810	0·44092 44225
4	− 0·72048 02714	0·44620 78022
	− 0·16718 08647	0·62365 30459
	0·44631 39728	0·56271 20303
	0·88579 16078	0·28742 71216
5	− 0·80292 98284	0·31964 07533
	− 0·39092 85468	0·48538 71883
	0·12405 03795	0·52092 67831
	0·60397 31642	0·41690 13342
	0·92038 02860	0·20158 83853

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