Computers and Geotechnics 11 (1991) 159–170

TECHNICAL NOTE

A SPURIOUS ZERO-ENERGY MODE IN THE NUMERICAL ANALYSIS OF PILED RAFT FOUNDATIONS

P. Clancy and D.V. Griffiths
Department of Engineering
University of Manchester
Manchester M13 9PL
U.K.

ABSTRACT

Numerical analysis of piled raft foundations has been performed using 4-node quadrilateral plate bending elements for the slab and axial elements for the piles. Selective reduced integration in the plate stiffness matrix formation was shown generally to improve the performance of the element, but led to the formation of a spurious zero-energy mode in some circumstances. Several numerical examples are presented to illustrate the reasons for such contrasting behaviour, and suggestions are made as to how this zero-energy mode can be avoided.

INTRODUCTION

When loads are applied to a pile group, additional settlement of the piles occurs (over and above that which would occur with a similarly loaded single pile) due to interaction effects in the soil mass (1). Mindlin's solution for displacements due to a point load in a semi-infinite elastic half-space has been applied to this problem, and lends itself to numerical analysis in a boundary element form (2). In the present analysis, the piles are modelled by one-dimensional rod finite elements, while the soil next to the piles is represented by 't-z' springs attached at the nodes, and the Mindlin equation is used to model interaction between piles.

If a raft is used to cap the pile group, some redistribution of loads occurs, dependent on the stiffness of the raft. Analytical solutions are available for the case of a rigid raft (3), and the present work attempts to quantify the effect of raft stiffness on the distribution of load between the piles. To achieve this, the raft is modelled by plate-bending finite elements attached to the top of the pile group. However, the particular choice of plate element gave an interesting example of zero-energy mode formation in a problem of practical interest. This has been demonstrated previously, but only by means of a somewhat contrived example (4).

Computers and Geotechnics 0266-352X/91/$03·50

An eigenvalue analysis of the plate element under both full integration and selective reduced integration revealed the root of the problem, with the latter case introducing two extra zero-energy modes.

Plate Bending Finite Element

To model the raft, a four-node quadrilateral isoparametric plate bending element (4,5) was considered. Although in this application a 'thin plate' was being considered (typically, side length : raft thickness was greater than 100), the element was based on 'thick' plate theory such that transverse shear strains were accounted for.

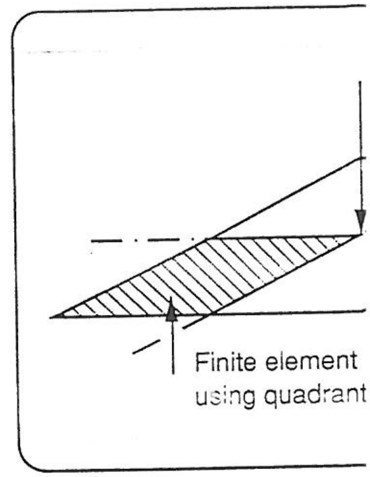
The plate bending problem was represented by an irreducible formulation, with the freedoms of the transverse displacement and two rotations at each node. The shear and bending components of the stiffness matrix were formed separately so that 'full' integration (with 2x2 Gauss points), 'reduced' integration (1x1 Gauss point), or a selective combination (i.e. fully integrated bending with reduced integrated shear) could be employed.

It has been demonstrated that under full integration the element would give excessively stiff solutions in thin plate applications. This 'shear locking' phenomenon is due to too many constraints on the shear deformation and can be overcome by using selective reduced integration on the shear term.

To demonstrate this, a square simply-supported plate carrying a central point load was considered, with the following dimensions and material properties :

Square side length	= 10.
Plate thickness	= 0.1
Young's modulus	= 10.92 E5
Poisson's ratio	= 0.3
Applied load	= 1.

Quadrant symmetry implies that only a quarter of the plate needs to be analysed (fig. 1). Both full integration and selective reduced integration were used, and the results for the displacement under the load are given in table 1 for various mesh refinements. These compare with the analytical solution of 0.01160 given by



Fi

Simply supp
under centr

Mesh Refinement	Ful Inte
1x1 element	0.4
2x2 elements	0.1
3x3 elements	0.2
4x4 elements	0.4
5x5 elements	0.7

Compare
anal
(

Central tran
of simply-su
under central

both full integration and selective
:m, with the latter case

isoparametric plate bending
pplication a 'thin plate' was being
was greater than 100), the element
verse shear strains were accounted

d by an irreducible formulation,
t and two rotations at each node.
ess matrix were formed separately
, 'reduced' integration (1x1 Gauss
grat[illegible] bending with reduced

ntegration the element would give
ons. This 'shear locking'
the shear deformation and can be
on the shear term.
ported plate carrying a central point
ions and material properties :

= 10.
= 0.1
= 10.92 E5
= 0.3
= 1.

quarter of the plate needs to be
lective reduced integration were used,
load are given in table 1 for various
nalytical solution of 0.01160 given by

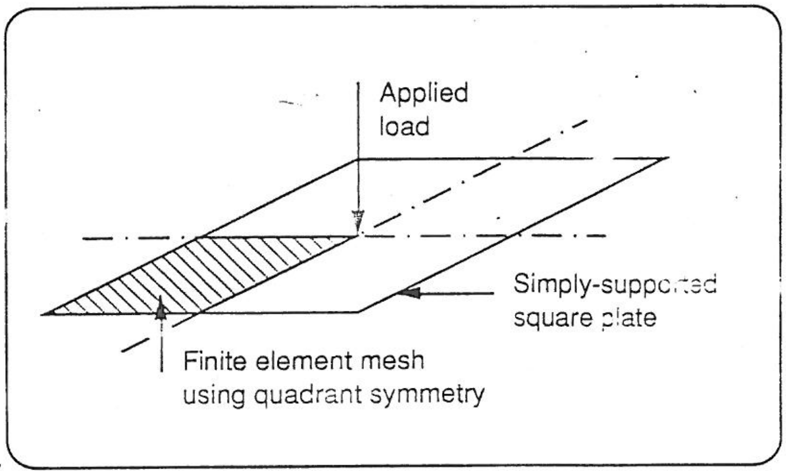


Figure 1

Simply supported square plate
under centrally applied load

	Displacement	
Mesh Refinement	Full Integration	Reduced Shear Integration
1x1 element	0.4278 E-4	0.1277 E-1
2x2 elements	0.1343 E-3	0.1153 E-1
3x3 elements	0.2832 E-3	0.1154 E-1
4x4 elements	0.4846 E-3	0.1156 E-1
5x5 elements	0.7328 E-3	0.1159 E-1

Compare displacements with
analytical solution
of 0.01160

Table 1

Central transverse displacement
of simply-supported square plate
under centrally applied point load

Timoshenko and Woinowsky-Krieger (6).

Computed Raft-Pile Behaviour

To model a piled raft foundation, the plate bending finite elements were attached to the piles so that vertical degrees of freedoms were common at the connected nodes. At this stage, no account was made of raft-soil contact. To test the adequacy of this arrangement, the raft was given a very high stiffness and a load was applied which would give unit displacement of an equivalent free-standing pile group (3).

Two pile group configurations were investigated: 2x2 and 3x3 piles, with an overall square geometry. The number of plate element between each pile was successively increased, and in each case the results from selective reduced integration were compared.

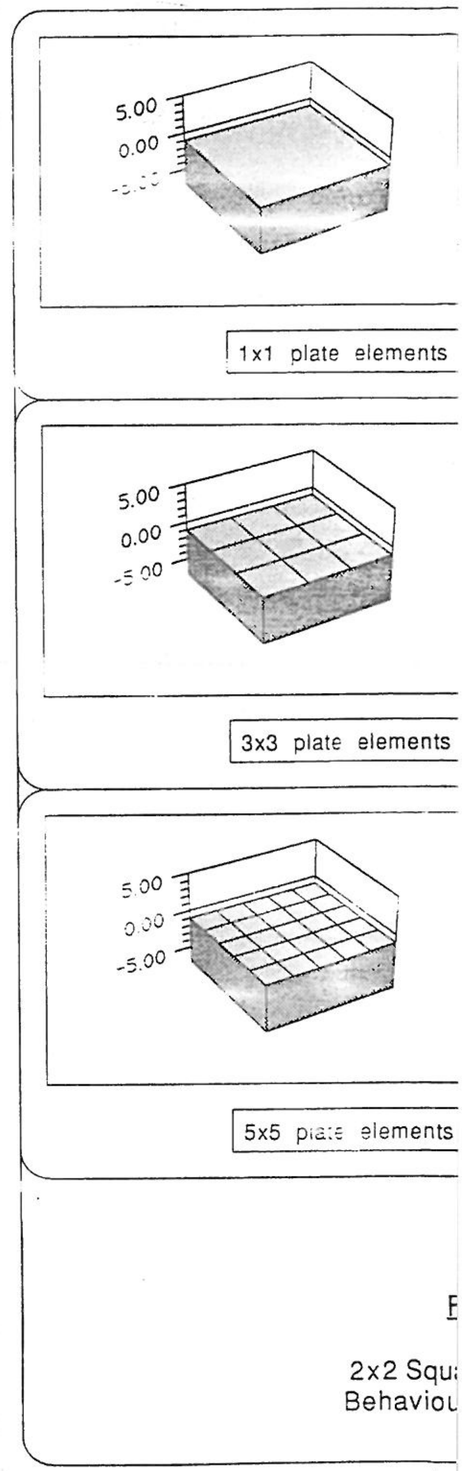
Fig. 2 shows the behaviour of the raft mesh attached to the 2x2 pile group, and fig. 3 is a similar set of plots for the 3x3 group. The 2x2 group demonstrates a spurious zero-energy mode which appears to be dependent on the number of plate elements between each pile. When this modeshape doesn't form, a unit displacement of the raft occurs, as anticipated.

The 3x3 pile group also demonstrates the spurious zero-energy mode, but in this case it occurs for all mesh refinements.

Eigenvalue Analysis

An eigenvalue analysis of an isolated plate bending element was undertaken to investigate its zero-energy modes, under both full integration and selective reduced integration. In each case plots were made of all twelve of the modeshapes (the element has twelve degrees of freedom, three at each of its four nodes). Fig. 4 shows the fully integrated element, and fig. 5 the results of using selective reduced integration.

The analysis was performed on a roughly square element, with one side being 1% longer than the other. This was to try and prevent modeshapes of equal eigenvalue from combining with each other. The eigenvalues given adjacent to each



bending finite elements were
reedoms were common at th
made of raft-soil contact. To test
iven a very high stiffness and a
nent of an equivalent free-standing

ated: 2x2 and 3x3 piles, with an
ement between each pile was
s from selective reduced

attached to the 2x2 pile group,
ıp. The 2x2 group demonstrates a
ependent on the number of plate
e n't form, a unit

urious zero-energy mode, but in

ysis

nding element was undertaken to
ntegration and selective reduced
velve of the modeshapes (the
h of its four nodes). Fig. 4
sults of using selective reduced

re element, with one side being
ent modeshapes of equal
genvalues given adjacent to each

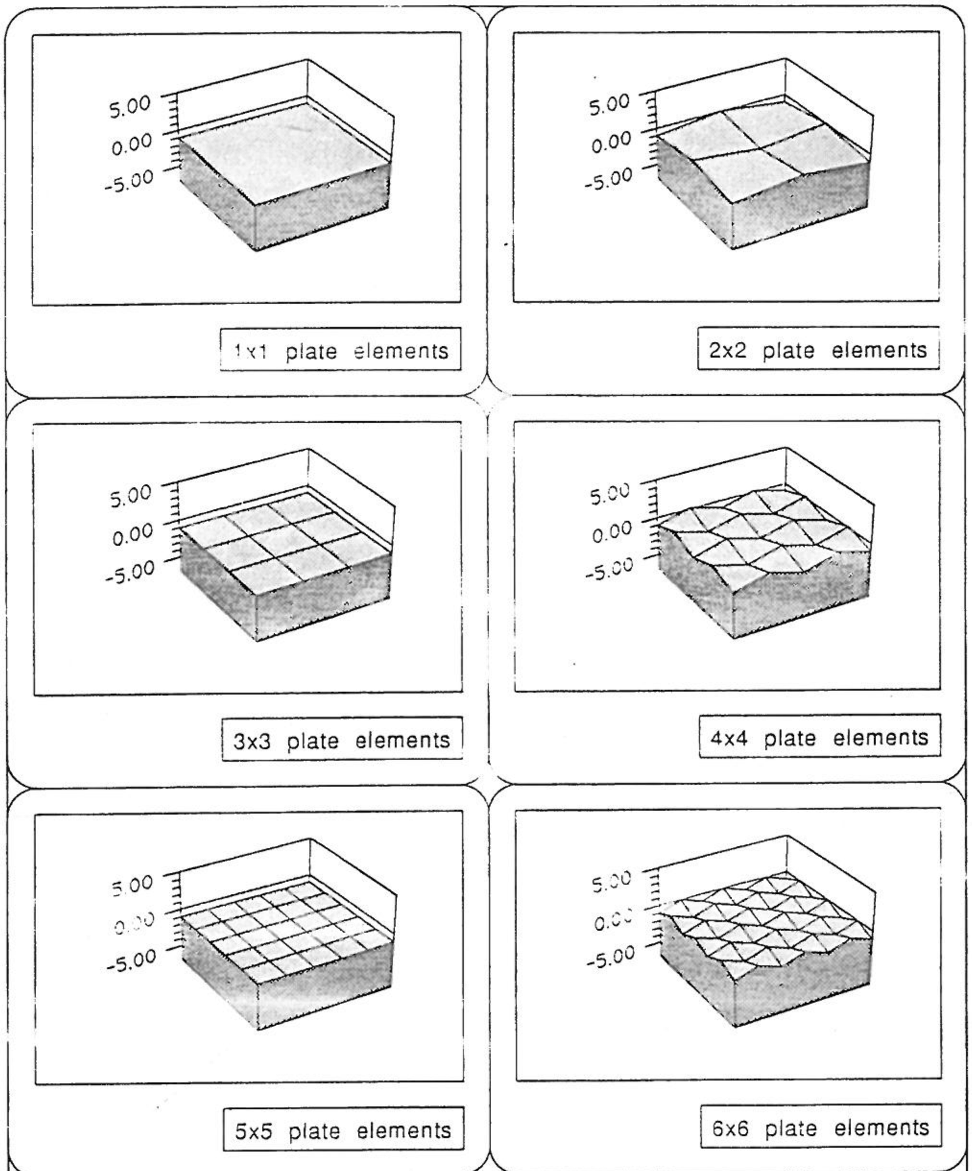


Figure 2

2x2 Square Pile Group
Behaviour of Raft Mesh

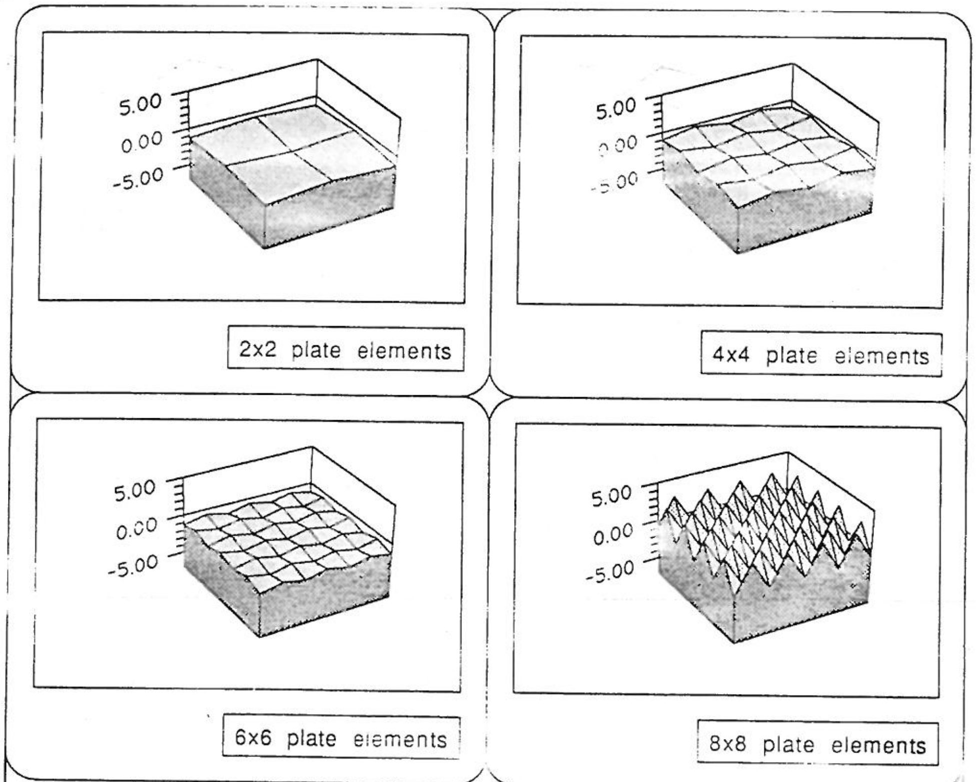


Figure 3

3x3 Square Pile Group
Behaviour of Raft Mesh

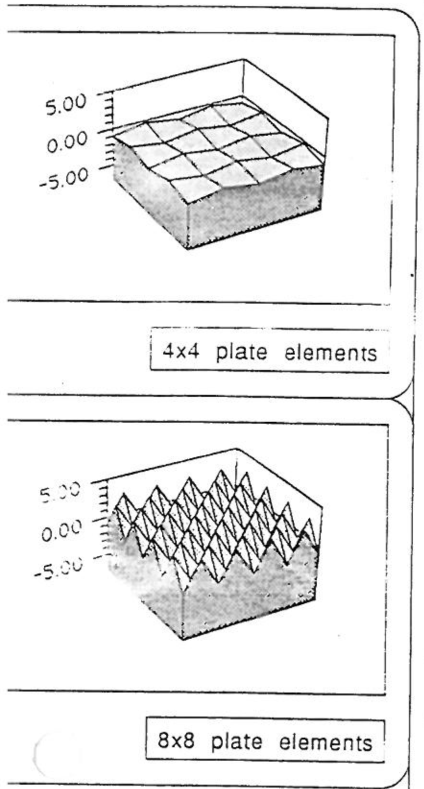
of the modeshapes are for a square elem

From fig. 4, it can be seen that un
zero energy modes, corresponding to th
displacement; rotation about the x-axis;
Unfortunately, it was not possible to iso
because they have a zero eigenvalue for
on a mesh must always be sufficient to

Fig. 5 shows that the element has
reduced integration, i.e. there are now t
form. Again, there is interference betw
fig. 4 it is possible to predict which of t
become zero energy under reduced shea
this: they must have zero rotation at th
curvature at the four bending Gauss po
conditions are the fourth and seventh, i
respectively, under full integration.

Under full integration, the plate
demonstrate shear 'locking' when used
integration on the shear terms, retainin
this excessive stiffness. However this
extra zero-energy modes, which the pl
are insufficient.

For an element to assume one o
two rotations at the single shear Gaus
derivative ($\delta^2 w/\delta x \delta y$) is free to take a
nature of this term, i.e. $\delta/\delta x(\delta w/\delta y)$ = δ
element twist by an equal amount. C
a common edge ensures that all adjac
the mesh) must therefore deform in a
element is constrained from twisting,



roup
Mesh

of the modeshapes are for a square element.

From fig. 4, it can be seen that under full integration the element has three zero energy modes, corresponding to the rigid-body motions, i.e. transverse displacement; rotation about the x-axis; and rotation about the y-axis. Unfortunately, it was not possible to isolate each of these zero-energy modes because they have a zero eigenvalue for any ratio of side lengths. The restraints on a mesh must always be sufficient to prevent these from forming.

Fig. 5 shows that the element has five zero energy modes under selective reduced integration, i.e. there are now two spurious zero energy modes that can form. Again, there is interference between the zero energy modes. Returning to fig. 4 it is possible to predict which of the modeshapes under full integration will become zero energy under reduced shear integration. There are two conditions for this: they must have zero rotation at the single shear Gauss point; and zero curvature at the four bending Gauss points. The two modes which satisfy these conditions are the fourth and seventh, i.e. those with Eigenvalues of 570 and 26709 respectively, under full integration.

Discussion

Under full integration, the plate bending element has been shown to demonstrate shear 'locking' when used in thin plate applications. Selective reduced integration on the shear terms, retaining full integration on the bending, eliminates this excessive stiffness. However this under-integration technique introduces two extra zero-energy modes, which the plate can form if the mesh boundary conditions are insufficient.

For an element to assume one of the spurious zero-energy modeshapes, the two rotations at the single shear Gauss point must be zero, but the second order derivative ($\delta^2 w/\delta x \delta y$) is free to take a constant value. Due to the complimentary nature of this term, i.e. $\delta/\delta x(\delta w/\delta y) = \delta/\delta y(\delta w/\delta x) = \delta^2 w/\delta x \delta y$, all four edges of the element twist by an equal amount. Continuity of displacement and rotations along a common edge ensures that all adjacent elements (and by extension, all element in the mesh) must therefore deform in a similar way. Conversely, if one edge of an element is constrained from twisting, the spurious zero-energy mode cannot occur

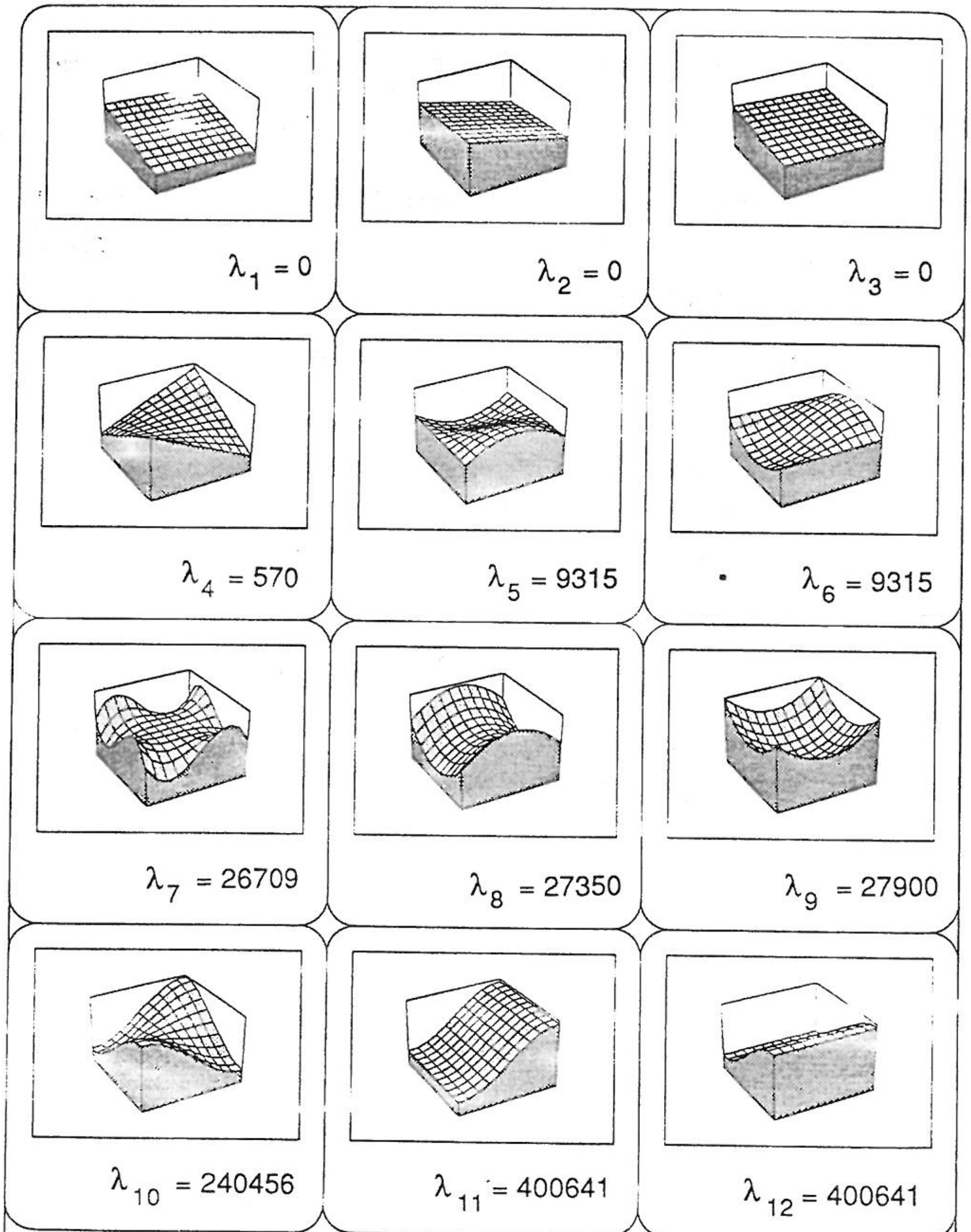
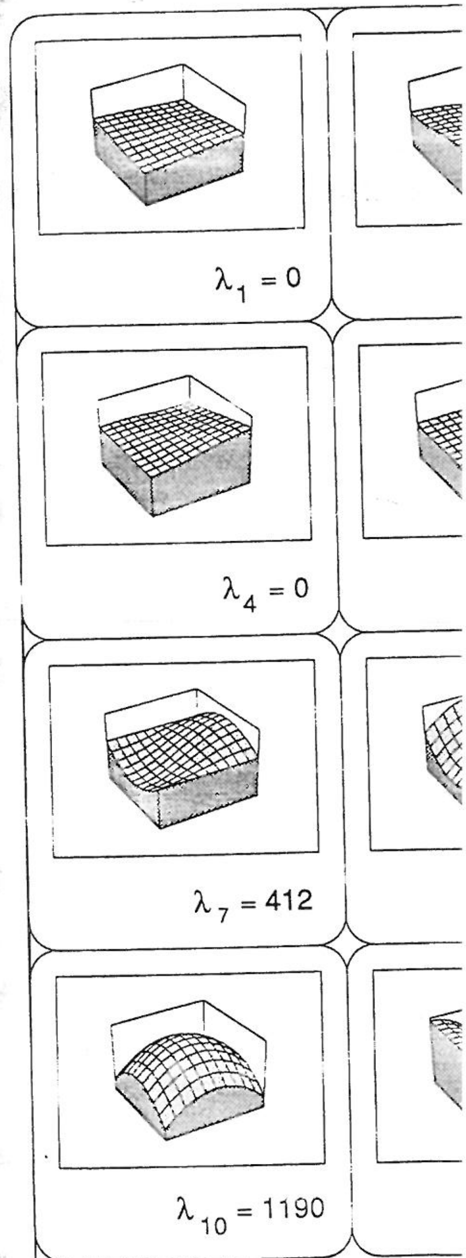
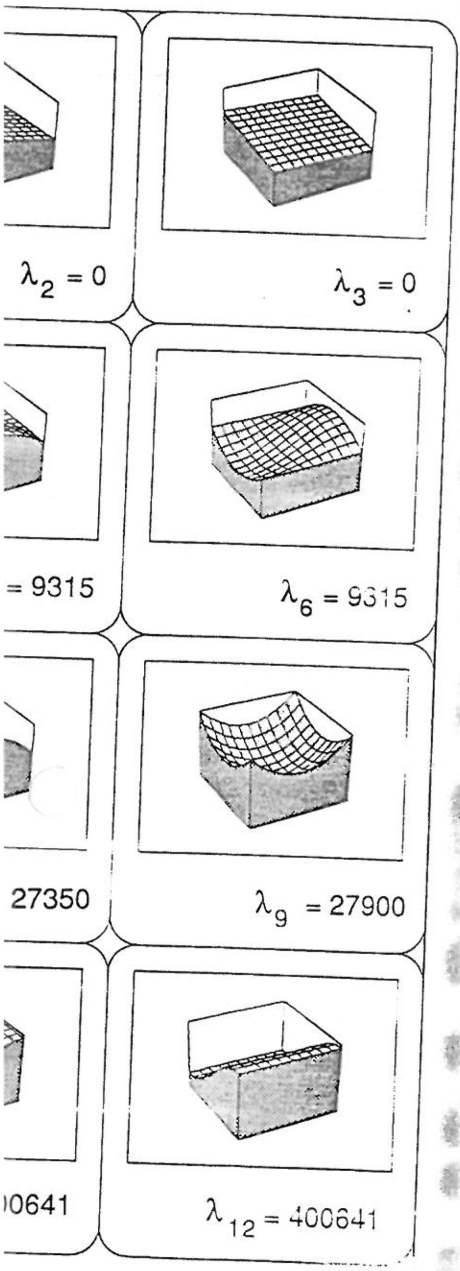


Figure 4

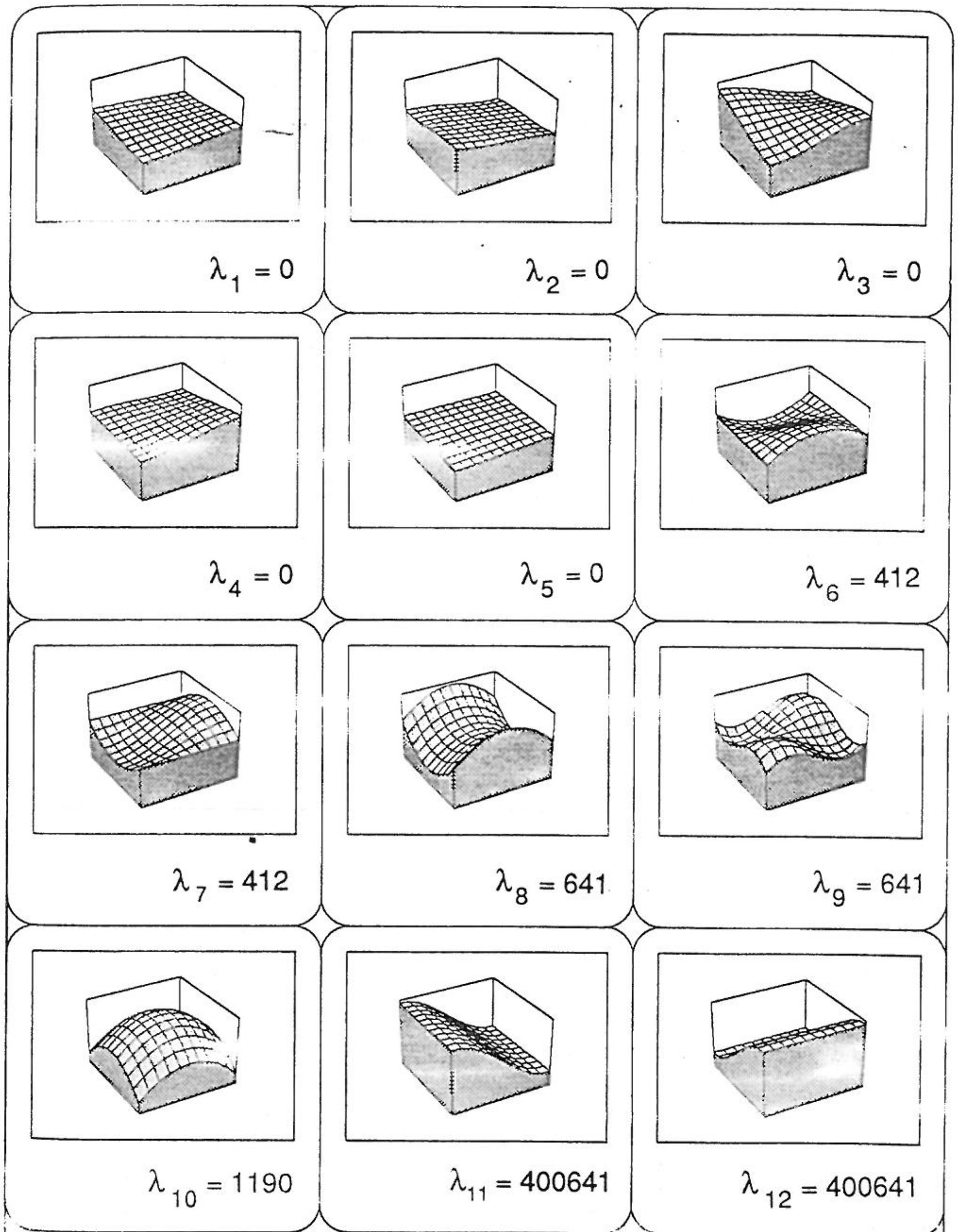
Eigenvalues and Mode Shapes,
Full Integration



Eigenvalue
Reduc



le Shapes,
on



<u>Figure 5</u>

Eigenvalues and Mode Shapes,
Reduced Integration

anywhere within the mesh.

Such constraint may be provided by a built-in or simply supported boundary condition, or where there are an even number of nodes between symmetrically similar piles. This last condition was demonstrated by the 2x2 square pile group, and is explained with the aid of fig. 6. In each of fig. 6 a, b and c, the two piles must displace by the same amount due to symmetry. It is obvious that all the elements in fig. 6a and fig. 6c are able to form the twist mode while still allowing the piles to displace by equal amounts. From fig. 6b it can be seen that if the elements form this twist mode, the piles would be forced to displace by different amounts. Thus in this case the twist mode is unable to form, and the plate elements perform adequately.

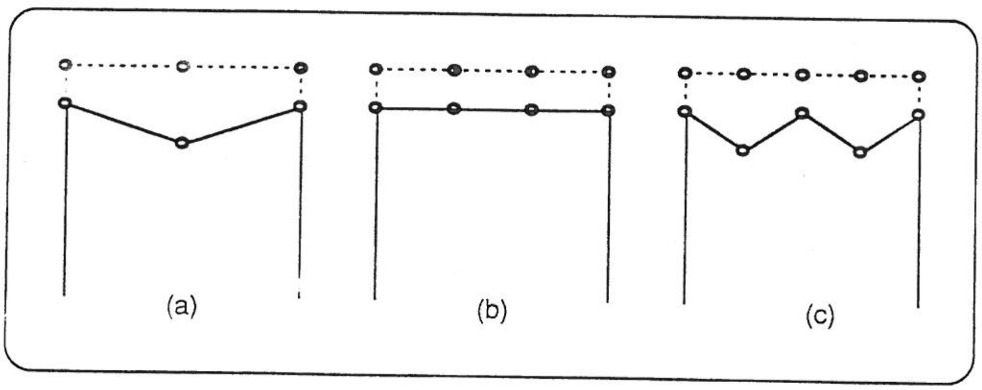


Figure 6

All elements 'twist' by the same amount. Thus, an even number of nodes between symmetrically similar piles prevents formation of this spurious zero-energy mode

Concl

The use of selective reduced int
foundation improves the computed di
zero-energy modes. In order to supp
boundary restraint conditions must b
particular example, and for the loadi
energy mode will not occur provided

1. There are an even number of p
4x4 piles, etc.).

2. There are an even number of p

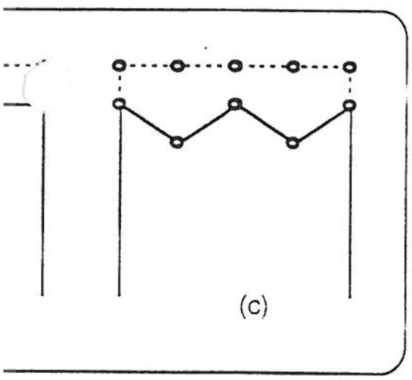
With these conditions satisfied
two adjacent mid-edge piles must di
vertical constraint required.

Under other conditions, the pi
provide sufficient vertical constraint
mode, and hence great care must be
application.

1. Poulos, H.G. and Davis, E.H.
Pile Foundation Analysis and D
John Wiley and Son
1980
pp 109,110.

2. Smith, I.M. and Griffiths, D.V
Programming the Finite Elemen
John Wiley and Sons
1988

t-in or simply supported boundary
nodes between symmetrically
ated by the 2x2 square pile group,
of fig. 6 a, b and c, the two piles
etry. It is obvious that all the
he twist mode while still allowing
ig. 6b it can be seen that if the
e forced to displace by different
nable to form, and the plate



e amount. Thus,
symmetrically similar
ous zero-energy mode

Concluding Remarks

The use of selective reduced integration in this example of a piled raft foundation improves the computed displacements, but introduced two spurious zero-energy modes. In order to suppress the formation of these modes, certain boundary restraint conditions must be met. It has been shown that in this particular example, and for the loading conditions under consideration, a zero energy mode will not occur provided that:

1. There are an even number of piles along each side of the pile group (e.g. 2x2, 4x4 piles, etc.).

2. There are an even number of plate nodes between piles.

With these conditions satisfied, the symmetry of the problem dictates that the two adjacent mid-edge piles must displace by the same amount, thus providing the vertical constraint required.

Under other conditions, the piles to which the plate has been attached do not provide sufficient vertical constraint to prevent the formation of a zero-energy mode, and hence great care must be taken when using this element for such an application.

References

1. Poulos, H.G. and Davis, E.H.
Pile Foundation Analysis and Design
John Wiley and Son
1980
pp 109,110.

2. Smith, I.M. and Griffiths, D.V.
Programming the Finite Element Method, 2nd edition
John Wiley and Sons
1988

pp 365-371.

3. Poulos, H.G. and Davis, E.H.
Pile Foundation Analysis and Design
John Wiley and Sons
1980
p 126, Table 6.4a.

4. Hughes, T.J.R.
The Finite Element Method
Prentice-Hall
1987
Chapter 5, pp 310-382.

5. Hou-Cheng Huang
Static and Dynamic Analysis of Plates and Shells
Springer-Verlag
1989
Chapter 3, 25-43.

6. Timoshenko, S. and Woinowsky-Krieger, S.
Theory of Plates and Shells, 2nd edition
1959
McGraw-Hill Book Company.

Received 26 January 1991; revised version received 29 May 1991; accepted 30 May 1991

Date and Venue	Confere
4-9 August 1991 Montreal Canada	2nd CANMET/AC Conference on Concrete
9-13 September 1991 Munich Germany	1st European Conference
10-12 September 1991 Beijing China	6th Internati on Ground Fr
16-20 September 1991 Aachen Germany	Internationa Rock Mechani
23-26 September 1991 Karlsruhe Germany	5th Internat on Soil Dyna Earthquake E
0-6 December 1991 Atlanta, Georgia USA	Conference Temperature Modelling: Application