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Rationalized charts for the method of fragments applied to confined seepage

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The Paper presents dimensionless charts for estimating seepage quantity and exit gradients in problems of confined flow. Based on the 'method of fragments' (Pavlovsky, 1935; Harr, 1962), the charts represent a simplified approach for obtaining the form factor Φ, which also allows for anisotropic soil permeability. The accuracy of the assumptions inherent in the method is assessed using finite elements. It is concluded that the four charts presented enable reliable estimates of flow rate and exit gradients to be made for a wide range of confined flow problems of practical interest.

Cet article présente des graphiques sans dimensions pour l'évaluation de la quantité de percolation et des gradients de sortie pour les problèmes d'écoulement confiné. Basées sur la méthode des fragments (Pavlovsky, 1935; Harr, 1962), ces graphiques représentent une méthode simplifiée pour obtenir le facteur de forms Φ, qui tient compte de la perméabilité isotrope du terrain. L'exactitude des assomptions inheréntes dans la méthode est évaluée par l'emploi d'éléments finis. On tire la conclusion que les quatre graphiques présentés permettent des évaluations justes du taux d'écoulement et des gradients de sortie pour une grande variété de problèmes d'écoulement confiné d'intérêt pratique.

NOTATION

- b width of apron
- h head loss across a fragment
- H total head loss
- i_E exit gradient
- k_H horizontal permeability
- k_V vertical permeability
- $\bar{k}$ equivalent isotropic permeability
- L width of fragment
- n number of fragments
- n_f number of flow channels
- n_d number of equipotential drops
- Q total flow rate
- R anisotropy factor
- s, s', s'' length of cut-off walls
- T depth of permeable layer
- x, y Cartesian co-ordinates
- Φ form factor
- $\boldsymbol{h}$ vector of nodal total head
- $\boldsymbol{q}$ vector of net nodal inflow/outflow
- $\boldsymbol{K}$ system 'stiffness' matrix

INTRODUCTION

The amount of seepage through the foundation soils of dams and water-retaining structures has always been an important design consideration for geotechnical engineers. Furthermore, when hydraulic gradients in the ground are significant, a knowledge of their magnitude is essential to guard against heave or piping, especially at the exit points where the flow direction opposes gravity.

For steady state conditions, the head within a soil mass with known boundary conditions is governed by Laplace's equation

$$k_H \frac{\partial^2 h}{\partial x^2} + k_V \frac{\partial^2 h}{\partial y^2} = 0 \tag{1}$$

and a wide variety of methods exist for its solution. The methods include analytical solutions using conformal mapping and complex variables (Pavlovsky, 1933; Harr & Deen, 1961; Polubarinova-Kochina, 1962; Harr, 1962; Verruijt, 1970), analogue methods (Aravin & Numerov, 1965) and stochastic analyses (Harr, 1977; Smith & Freeze, 1979). The best known method in soil mechanics, however, is flow net sketching. Flow nets are a powerful and versatile method in experienced hands, but they can be time consuming, and their accuracy is sometimes difficult to assess.

An alternative, simple, although approximate approach for solving problems of confined flow is called the method of fragments (Pavlovsky, 1935). This Paper shows how the method can be condensed into a series of charts, enabling rapid estimates of flow rates and exit gradients to be made for a variety of confined flow problems. The accuracy of the method is also assessed using finite element analyses, and it is shown that for the examples considered the results can be at least as accurate as a well-drawn flow net, and more easily obtained.

Discussion on this Paper closes on 1 October 1984. For further details see inside back cover.

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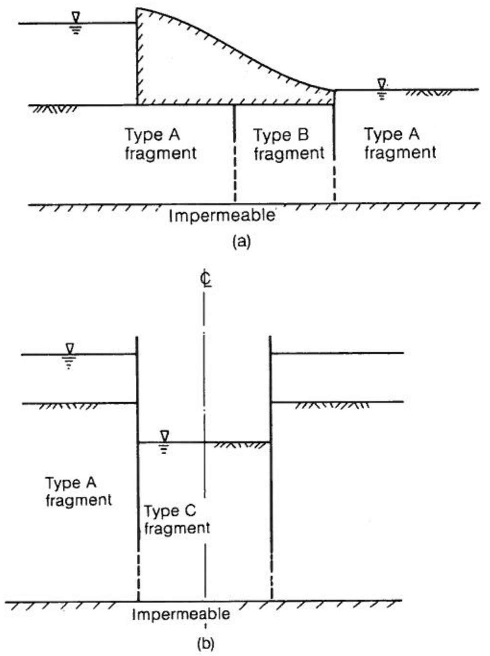

Fig. 1. Sub-division of flow regime into fragments

THE METHOD OF FRAGMENTS

The method of fragments forms the basis of the charts produced in this Paper. The method was first proposed by Pavlovsky in 1935 for the computation of seepage under hydraulic structures incorporating multiple piles. The method was publicized by Harr (1962), and more recently described in a standard soil mechanics text by Holtz & Kovacs (1981). The method is a powerful though approximate means of obtaining seepage quantities in confined flow problems. Although the method also has applications in free surface problems (Harr, 1977), the present discussion is confined to configurations in which the boundaries of the flow regime are known *a priori*.

The method is approximate because it relies on the assumption that the equipotentials at certain points within the soil mass are vertical. The zones between these 'vertical' equipotentials are called fragments, and because the fragments are often rectangular, with simple boundary conditions, solutions for the individual fragments can be found, and superposed to give the overall solution. Solutions for three different fragment types are presented later in the Paper in the form of dimensionless charts.

Figure 1(a) shows how the region beneath a concrete dam with two cut-off walls could be fragmented. By assuming vertical equipotentials wherever significant changes in boundary geometry occur, the problem is reduced to the superposition of three fragments. Fig. 1(b) shows a cross-section of parallel sheet pile walls. By assuming a vertical equipotential beneath each wall, and accounting for symmetry, the problem is reduced to the superposition of two fragments.

As with the flow net approach, the seepage quantity is calculated using

$$Q = \bar{k}H\frac{n_f}{n_d} \tag{2}$$

where Q is the volume flow rate, $\bar{k}$ is the soil permeability, H is the total head loss, n_f is the number of flow channels and n_d is the number of equipotential drops. Equation (2) can account for anisotropic soil permeability in the horizontal and vertical directions where

$$k_H \neq k_V \tag{3}$$

by considering a modified co-ordinate system in which all lengths in the horizontal direction are scaled by a factor $\sqrt{(k_V/k_H)}$. The modified permeability is then given by

$$\bar{k} = \sqrt{(k_H k_V)} \tag{4}$$

When dealing with anisotropic soil using flow nets, the physical boundaries of the problem must first be redrawn in the new co-ordinate system. After sketching a flow net according to the usual rules, the 'shape factor' n_f/n_d is obtained and the seepage quantity found from equations (2) and (4).

It may be noted that to avoid complication, the majority of flow nets include only a few flow channels ($n_f = 3$ or 4). Correspondingly, the number of equipotential drops (n_d) is usually assumed to be a whole number although the exact solution would rarely involve such a simple ratio of n_f/n_d.

In the Method of fragments, it is noted that n_f is constant in each fragment, but n_d varies from fragment to fragment in accordance with the proportion of total head lost in each. Thus for each fragment, a form factor Φ is obtained where for the ith fragment

$$\Phi_i = \frac{n_{d_i}}{n_f} \tag{5}$$

When the form factor for all the fragments has been obtained, the total seepage quantity is again given by equation (4) but in the form

$$Q = \frac{\bar{k}H}{\sum_{i=1}^{n}\Phi_i} \tag{6}$$

where n is the number of fragments.

The form factor is a function of the fragment type and its dimensions. For details of how it is obtained, the interested reader is referred to Harr (1962) who describes the transformation techniques which can lead to solution of Laplace's equation for certain boundary conditions.

The present Author's preference is for numerical techniques, and all subsequent work in this Paper has been developed and tested using finite element methods.

NUMERICAL SOLUTIONS PROCEDURE

The majority of work was performed using four-noded quadrilateral elements with a typical mesh shown in Fig. 2. Boundary conditions were particularly easy to implement; impermeable surfaces, which represent boundary streamlines, required no special treatment whereas boundaries on which the head was constant received prescribed values. For simplicity, these were made to vary from unity at the upstream face to zero downstream.

For details of the formulation, and listings of the programs used, the reader is referred to Smith (1982), but in all cases, the problem was reduced to the solution of simultaneous equations, thus

$$\boldsymbol{q} = \boldsymbol{K}\boldsymbol{h} \qquad (7)$$

where $\boldsymbol{h}$ is the vector of nodal head values, $\boldsymbol{q}$ is the vector of nodal net inflow/outflow and $\boldsymbol{K}$ is the system 'stiffness' matrix.

Prescribed values of nodal head were achieved by a 'stiff spring' technique, and the values of the head at all other points obtained by solving equation (7) using Gaussian elimination. When the vector $\boldsymbol{h}$ for all nodal points obtained, the vector $\boldsymbol{q}$ was obtained by multiplying $\boldsymbol{h}$ by the symmetric stiffness matrix $\boldsymbol{K}$. As the problem was one of steady state, the vector $\boldsymbol{q}$ was zero for all nodes except those at the up and downstream faces. By summing the nodal flow values for either of these faces, the total flow rate through the system was obtained.

Assuming that the soil was isotropic, with a permeability and a total head loss of unity, the form factor for a particular configuration was given by the reciprocal of the total flow rate through the system, thus from equation (6)

$$\Phi = 1/Q \qquad (8)$$

where Q is the total flow rate through the system. By varying the dimensions of a fragment, charts were developed which related the form factor to dimensionless groupings.

Exit gradients, which are also of great interest to the designer from a standpoint of heave and piping are also obtained in a straightforward way from the numerical output. Once the head has been calculated for all nodal points, the maximum exit gradient is found by dividing the head loss across the most critical finite element at the downstream end of the mesh by its vertical height (Fig. 3).

TREATMENT OF ANISOTROPY

If the permeability of the soil differs in the horizontal and vertical directions, this is dealt with directly in the charts by introducing the factor R where

$$R = \sqrt{(k_V/k_H)} \qquad (9)$$

As will be seen, any horizontal dimension which appears in the dimensionless groupings, such as b in fragment A or L in fragments B and C, is

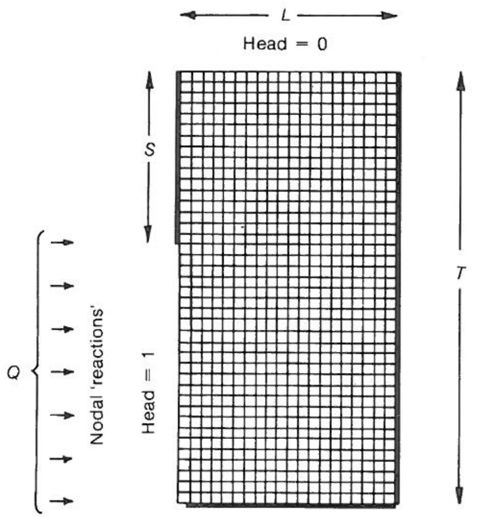

Fig. 2. Typical mesh for type C fragment

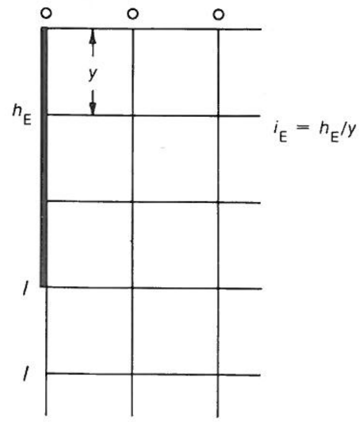

Fig. 3. Calculation of the exit gradient

first multiplied by the anisotropy factor R before entering the chart. Having obtained the form factor for all fragments, the method proceeds as before from equations (4) and (6).

DESCRIPTION OF FRAGMENT TYPES

Fragment A

This fragment (Fig. 4(a)) represents a semi-infinite domain of height T, with impermeable boundaries at its base and in the top corner as shown. The chart given in Fig. 5 enables the form factor to be obtained as a function of s/T and bR/T. This chart is based on the results of Pavlovsky, and is a rearrangement of a figure quoted by Polubarinova-Kochina (1962). The difference in the present work lies in the introduction of anisotropy and the ability to obtain the form factor directly.

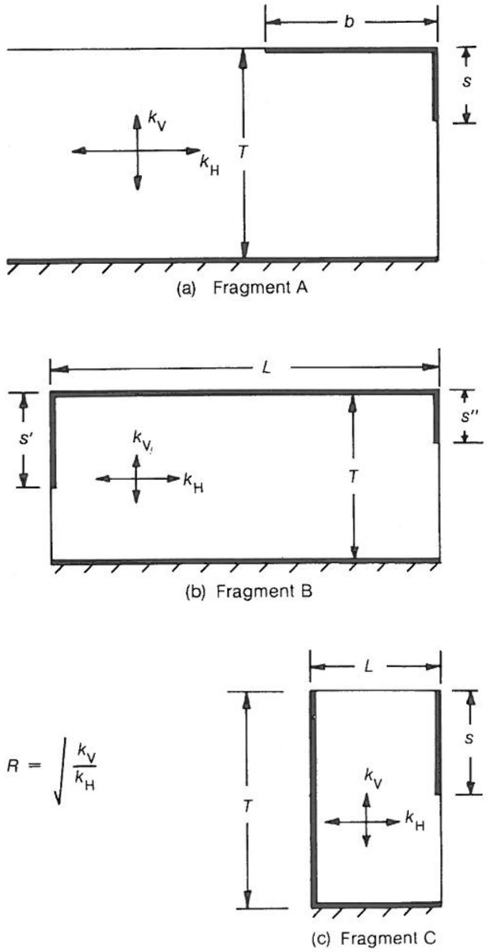

Fig. 4. Main fragment types: (a) fragment A; (b) fragment B; (c) fragment C

Finite element results were also obtained for this fragment in order to assess the accuracy of the program and the influence of element size. The values of Φ obtained numerically were always within a few percent of those in Fig. 5, although the solutions tended to diverge slightly for $s/T > 0{\cdot}9$.

It should be mentioned that when modelling semi-infinite domains using finite element methods, consideration must be given to the effects of boundary proximity on the major variables under consideration. One possibility is to incorporate a row of infinite finite elements (Chow & Smith, 1981), but the criterion used in the present work (Lo, 1983) was that if $LR/T \geqslant 2$, the solution essentially corresponded to the semi-infinite case.

Fragment B

Fragment B (Fig. 4(b)) represents an enclosed, rectangular domain of width L and height T with impermeable boundaries above and below,

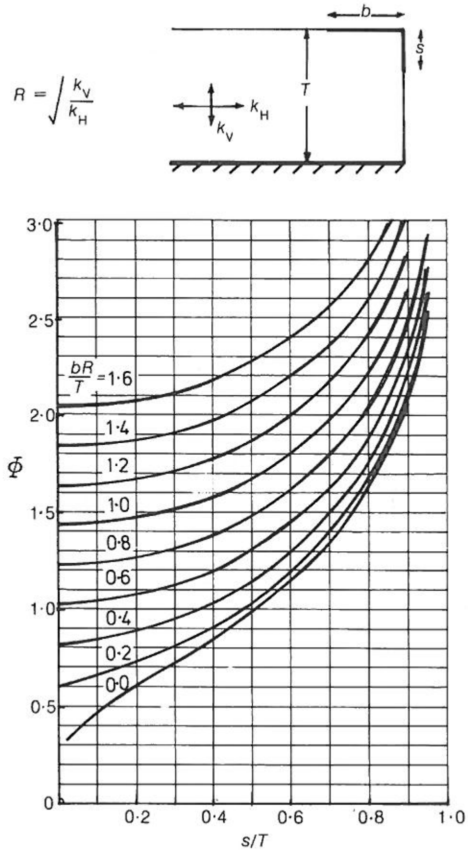

Fig. 5. Form factor for fragment A (rearranged from Polubarinova-Kochina, 1962)

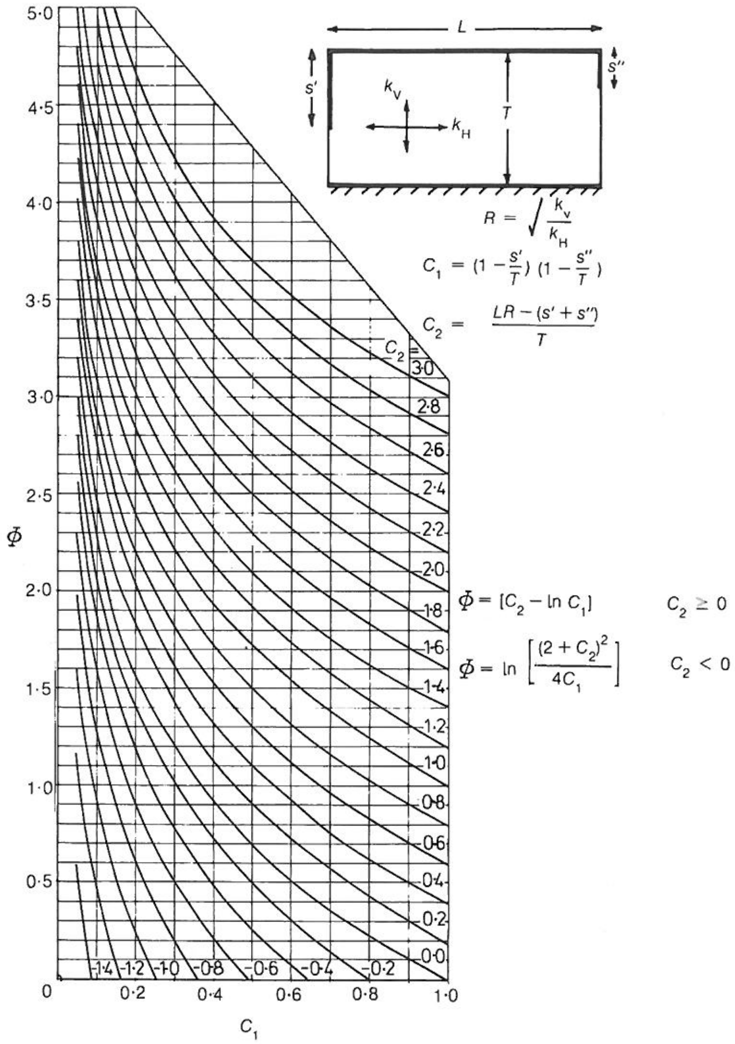

Fig. 6. Form factor for fragment B (condensed from Harr, 1962)

and cut-off walls at the sides of height s' and s''. This fragment has applications for flow under dams such as that in Fig. 1(a). The chart given in Fig. 6 enables the form factor to be obtained as a function of dimensionless numbers C_1 and C_2 where

$$C_1 = \left(1 - \frac{s'}{T}\right)\left(1 - \frac{s''}{T}\right)$$
$$C_2 = \frac{LR - (s' + s'')}{T} \quad (10)$$

The results quoted are a condensed form of work originally performed by Pavlovsky on the seven variations shown in Fig. 7.

$$\Phi = \frac{L}{T} \quad (11a)$$

$$\Phi = \ln\left(1 + \frac{L}{a}\right) \quad (11b)$$

$$\Phi = \ln\left(1 + \frac{s}{a}\right) + \frac{L - s}{T} \quad (11c)$$

$$\Phi = 2\ln\left(1 + \frac{L}{2a}\right) \quad (11d)$$

$$\Phi = 2\ln\left(1 + \frac{s}{a}\right) + \frac{L - 2s}{T} \quad (11e)$$

$$\Phi = \ln\left[\left(1 + \frac{L + (s' - s'')}{2a'}\right)\left(1 + \frac{L - (s' - s'')}{2a''}\right)\right] \quad (11f)$$

$$\Phi = \ln\left[\left(1 + \frac{s'}{a'}\right)\left(1 + \frac{s''}{a''}\right)\right] + \frac{L - (s' + s'')}{T} \quad (11g)$$

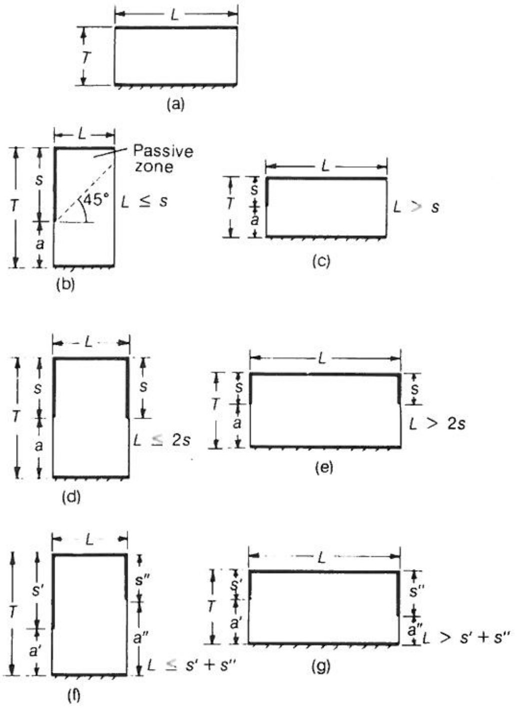

Fig. 7. Special cases of fragment B

The derivations obtained were quoted by Harr (1962), and are listed here as equations 11(a)–(g). With the exception of equation (11a), all the expressions are approximate due to Pavlovsky's assumption, based on analogue experiments, that negligible flow occurs above a line inclined at 45° and passing through the base of the cut-off walls (Fig. 7(b)). Subsequent finite element checks (Lo, 1983), without involving any such assumption, have agreed closely with Pavlovsky's expressions.

By making the substitutions of equation (10), it can be shown that all of equation (11) can be reduced to

$$\Phi = C_2 - \ln(C_1) \qquad C_2 \geqslant 0$$
$$\Phi = \ln\left[\frac{(2+C_2)^2}{4C_1}\right] \qquad C_2 < 0 \tag{12}$$

enabling the seven variations to be condensed in the single chart of Fig. 6.

Fragment C

Fragment C (Fig. 4(c)) represents an enclosed, rectangular domain of height T and width L, with impermeable boundaries at the bottom and on one side, with a cut-off wall of height s on the other. The fragment has applications in the study of flow beneath a double-wall sheet-pile cofferdam (Fig. 1(b)), in which the depth of each wall is equal. Symmetry in this case dictates that the centre-line of the cofferdam acts as an impermeable boundary. The flow under such structures can thus be found by superposition of type A and C fragments (Fig. 1(b)). The chart in Fig. 8 was computed using a similar mesh to that in Fig. 2 and gives the form factor as a function of s/T and LR/T. In all cases the mesh was made sufficiently fine to eliminate significant mesh-effects.

Although fragments A and C are different in character, it may be noted that when bR/T is put equal to zero in fragment A, the problem is identical to that when $LR/T = \infty$ in fragment C.

EXIT GRADIENTS

No study of confined seepage would be complete without a knowledge of the exit gradient (i_E) that is likely to occur. The most critical point is usually where the uppermost streamline emerges, because it is here that the equipotentials in a flow net would be at their closest packing. Analytical solutions for a single sheet pile (Harr, 1962) and for a double-wall system in an infinitely deep stratum (Harr & Deen, 1961) are available, and the former is shown in Fig. 9 for the double-wall system with $LR/T = \infty$. Solutions for the exit gradient in type C fragments for other values of LR/T have also been computed and are included in this figure. In order to render the figure dimensionless, the vertical axis corresponds to values of $i_E s/h$ where h is the head lost across the fragment in which the exit gradient is required. The head loss across any individual fragment can be found by simple proportioning of the form factors. For example if the head loss in the jth fragment is required, and H is the total head lost over n fragments, then

$$\frac{h_j}{H} = \frac{\Phi_j}{\sum_{i=1}^{n} \Phi_i} \tag{13}$$

A large safety factor against piping is usually required in the design of water-retaining structures, and Harr (1962) suggests that it should be no less than 4. Although the exit gradient will generally provide the most significant design criterion, Terzaghi (1943) has also suggested a procedure for checking against the possibility of heave at depth.

UPLIFT PRESSURES

In flow-net solutions, the excess head at any point can be estimated from the positions at which the equipotentials intersect the impermeable boundaries of the structure. This usually

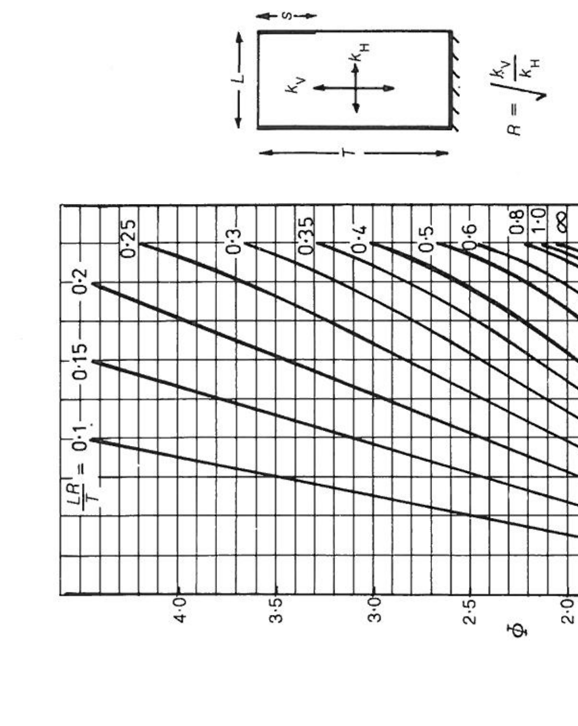

Fig. 8. Form factor for fragment C (from finite element solution)

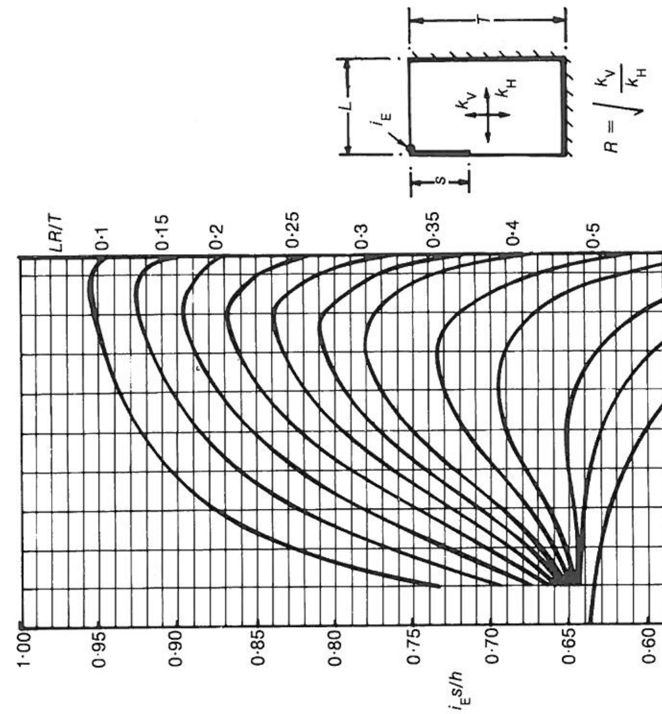

Fig. 9. Exit gradients for fragment C (from finite element solution)

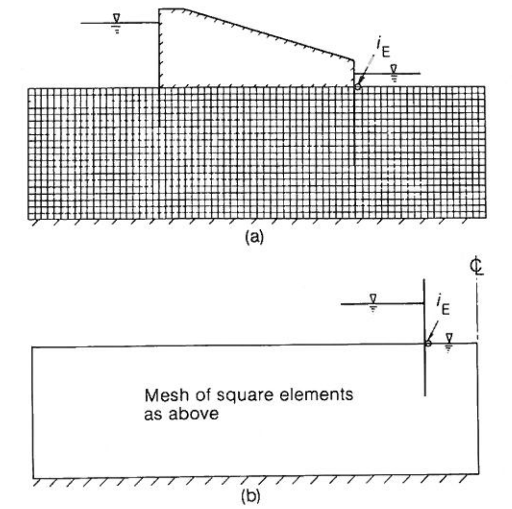

Fig. 10. Finite element meshes for full analyses: (a) dam with upstream and downstream cut-off walls; (b) double-wall sheet pile cofferdam

indicates a gradual and fairly uniform loss of head as water moves downstream along the uppermost streamline. In the method of fragments, a linear loss of head is assumed within each fragment. The rate of head loss equals the head loss in the fragment from equation (13), divided by the length of the uppermost impermeable surface.

ASSESSMENT OF ACCURACY

The accuracy of the method depends almost exclusively on the validity of the assumption that equipotentials can be considered vertical at certain locations. A study of published flow nets such as those of Cedergren (1977) or Lambe & Whitman (1969) would indicate that the assumption is a reasonable one. A potential user of the method of fragments might also consider how accurately a flow net could be drawn in the time available for the problem under consideration. A well-drawn flow net will approach the exact solution, whereas the method of fragments

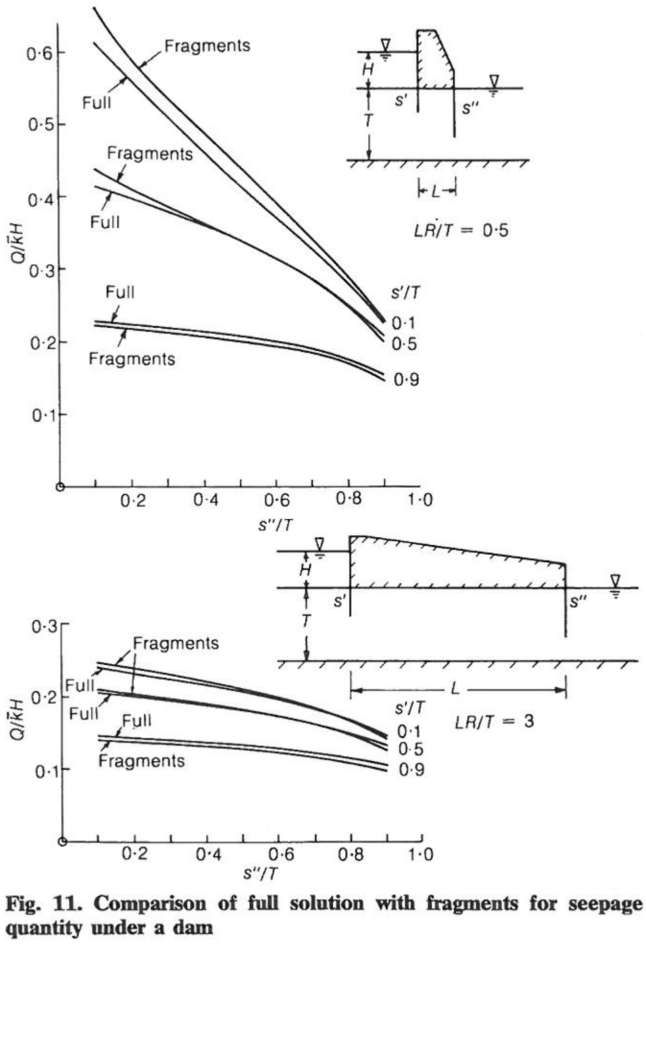

Fig. 11. Comparison of full solution with fragments for seepage quantity under a dam

will always be approximate unless the problem exhibits considerable symmetry.

In order to check the accuracy of the charts provided, full numerical solutions were obtained for the two configurations shown in Fig. 10. The object of this exercise was to compute the total rate Q and the exit gradient i_E for the full problem without making any assumptions regarding the equipotentials. These results could then be compared with the fragments solutions using charts in which the assumption is built in.

The programs were set up in such a way that the depth of embedment and positioning of the cut-off walls could be altered by a simple change of data.

The comparisons are given in Figs 11–14 and show a generally good agreement between the fragments solution and the full solution.

In the case of flow under a dam with cut-off walls, two widths of dam were considered (Fig. 11). The worst case occurred with the narrow dam, when both cut-off walls were short relative to the depth of the permeable layer. In this instance, the method of fragments tended to overestimate the seepage quantity obtained by the full solution, but the error was always less than 10 %. As the depth of cut-off walls was increased, however, the solutions from both approaches converged to give almost identical values. For the wider dam, the method of fragments appeared to give good predictions of flow rate for all combination of cut-off wall lengths.

When considering the exit gradients under the dam a similar picture emerged. The method of fragments tended to give conservative estimates of the exit gradient for the narrow dam with short cut-off walls, but almost exact correspondance for greater depths of embedment. As before, the predictions for the wider dam were generally good.

For the case of the double-wall cofferdam, the assumption of a vertical equipotential beneath the pile appeared to be very reasonable. Analysing only half the problem due to symmetry, three different pile spacings were considered together with a variety of cut-off depths. In all cases (Figs 13, 14), the results from the full finite element analyses and the charts were almost indistinguishable.

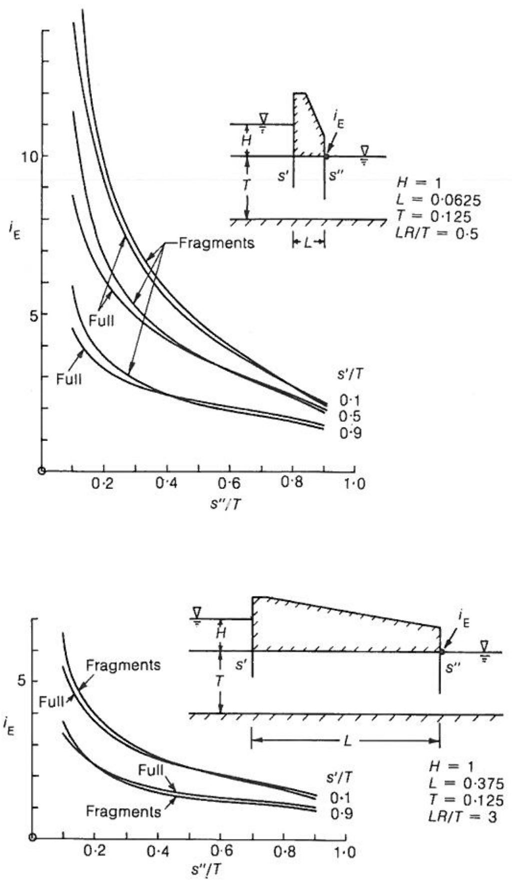

Fig. 12. Comparison of full solution with fragments for exit gradients downstream of a dam (not to scale)

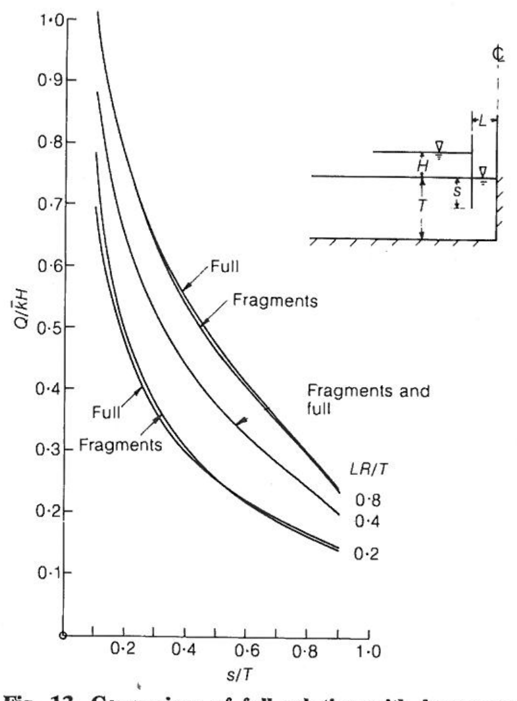

Fig. 13. Comparison of full solution with fragments for seepage quantity under a double-wall cofferdam

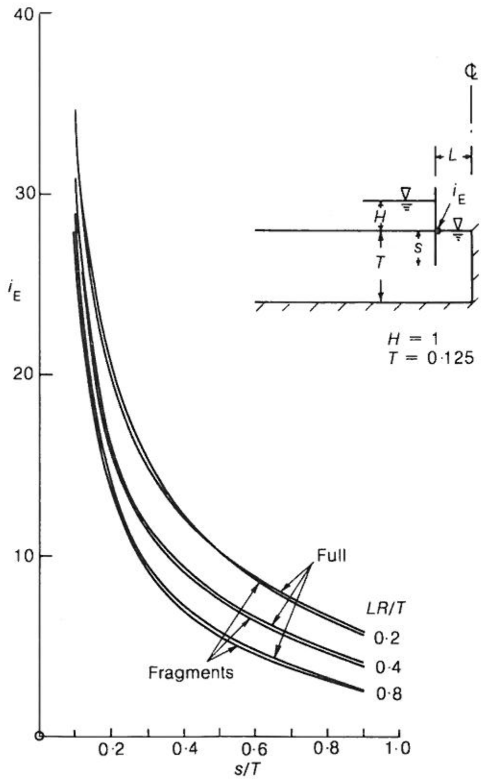

Fig. 14. Comparison of full solution with fragments for exit gradients in a double-wall cofferdam (not to scale)

CONCLUSIONS

Design charts for estimating seepage quantities and exit gradients using the method of fragments for confined flow problems have been presented. The results are based on a combination of published analytical solutions and numerical analyses performed by the Author. The presentation enables many different fragment types to be considered with the aid of rationalized dimensionless charts in which anisotropic soil properties can be accounted for directly.

The fundamental assumption in the method, that the equipotentials are vertical at certain locations, has been assessed by performing some analyses of full seepage problems using finite elements. At worst, the comparison between the method of fragments and the full solution was adequate and conservative. For the majority of cases considered, however, the agreement was good and the vertical equipotential assumption appeared jsutified.

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