Selective Reduced Integration of Four-Node Plane Element in Closed Form

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Abstract: The exactly integrated four-node plane element is known to "lock" as elastic incompressible conditions are approached. The use of uniform reduced integration removes this problem but introduces another in the form of a zero-energy mode, which can result in "hourglass" deformations. A popular compromise is known as selective reduced integration (SRI), in which the shear contribution to stiffness is exactly integrated and the volumetric contribution is evaluated using reduced integration. This SRI approach is readily coded in a numerical algorithm but requires the evaluation of five stiffness matrix contributions per element. This paper describes a closed-form version of the SRI stiffness matrix that was generated with the help of computer algebra systems. It is shown that this "analytical" approach considerably reduces the central processing unit (CPU) time consumed during element integration.

INTRODUCTION

Finite-element matrices such as those for stiffness and mass are usually generated using Gaussian quadrature because it leads to convenient formulations in terms of local coordinates. This paper describes how the selective reduced integration stiffness matrix for a general four-node quadrilateral plane strain element can be generated in closed form with the help of computer algebra systems (CAS) and how this can lead to improvements in run times.

For an isotropic material, the plane strain stress-strain **D** matrix is

$$\mathbf{D} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}(1-\nu) & \nu & 0\\ \nu & (1-\nu) & 0\\ 0 & 0 & \dfrac{(1-2\nu)}{2}\end{bmatrix} \quad (1)$$

where E and ν denote Young's modulus and Poisson's ratio, respectively.

Selective reduced integration involves splitting the constitutive **D** matrix into volumetric and deviatoric factors as follows:

$$\mathbf{D} = \mathbf{D}^d + \mathbf{D}^v \quad (2)$$

The "problematic" part of the **D** matrix with respect to incompressibility lies in the denominator term $(1-2\nu)$ in (1), which becomes zero as $\nu \to 0.5$. In the SRI approach, this denominator term is hived off into the $\mathbf{D}^v$ matrix from (2). Two main partition types can be identified as follows (Mase 1970; Kidger and Smith 1992).

Partition I (Lamé's Parameters—μ, λ)

$$\mathbf{D}_1^d = \mu\begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{bmatrix}; \quad \mathbf{D}_1^v = \lambda\begin{bmatrix}1 & 1 & 0\\ 1 & 1 & 0\\ 0 & 0 & 0\end{bmatrix} \quad (3, 4)$$

where Lamé's parameters are defined as $\mu = E/[2(1+\nu)]$ and $\lambda = E\nu/[(1+\nu)(1-2\nu)]$.

This partition can also be expressed in the form

$$\mathbf{D}_1^d = G\begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{bmatrix}; \quad \mathbf{D}_1^v = \left(K - \frac{2G}{3}\right)\begin{bmatrix}1 & 1 & 0\\ 1 & 1 & 0\\ 0 & 0 & 0\end{bmatrix} \quad (5, 6)$$

where the shear and bulk moduli are defined $G = E/[2(1+\nu)]$ and $K = E/[3(1-2\nu)]$.

Partition II (Shear and Bulk Moduli—G, K)

$$D_2^d = \frac{G}{3}\begin{bmatrix}4 & -2 & 0\\ -2 & 4 & 0\\ 0 & 0 & 3\end{bmatrix}; \quad \mathbf{D}_2^v = K\begin{bmatrix}1 & 1 & 0\\ 1 & 1 & 0\\ 0 & 0 & 0\end{bmatrix} \quad (7, 8)$$

These two partitions can be combined linearly and generalized in the form

$$\mathbf{D}^d = (1-\theta)\mathbf{D}_1^d + \theta\mathbf{D}_2^d; \quad \mathbf{D}^v = (1-\theta)\mathbf{D}_1^v + \theta\mathbf{D}_2^v \quad (9, 10)$$

where θ varies in the range $0 \le \theta \le 1$.

The two main partition types can be considered special cases of the general formulation in which $\theta = 0$ gives partition I and $\theta = 1$ gives partition II. Other values of θ lead to an infinite number of alternative partitions that represent a different combination of the two main types. To illustrate the influence of different partitions of **D**, the following example is taken from Hughes (1987). A square four-node plane strain element is fixed on two adjacent sides, with a horizontal force P applied to the free corner. A computer algebra system has been used to compute the horizontal deflection of the corner for various element stiffness integration schemes and the results are shown in Table 1. The expressions show that as the material approaches incompressibility (i.e., $\nu \approx 0.5$), the FI case (fully integrated, 2 × 2) "locks" due to the presence of the term $(1-2\nu)$ in the numerator. The other cases, which use SRI or URI (uniformly reduced integration, 1 × 1), do not contain this term, and give finite values of the nodal displacement while maintaining the no-volume-change condition. This implies that the loaded node moves along a 45° line with the x- and y-displacements of the same magnitude.

Consider a general four-node quadrilateral element with

TABLE 1. Influence of D Partition and Integration

Integration (1)	δ_H (2)	$\delta_H(\nu \approx 0.5)$ (3)
FI	$[96(3-4\nu)(1+\nu)(1-2\nu)P]/[(9-16\nu)(15-16\nu)E]$	0
SRI (μ, λ) ($\theta = 0$)	$[16(2-3\nu)(1+\nu)P]/[3(5-6\nu)E]$	$2P/E$
SRI (G, K) ($\theta = 1$)	$[144(17-25\nu)(1+\nu)P]/[25(43-50\nu)E]$	$2.16P/E$
URI	$[(3-4\nu)(1+\nu)P]/[(1-\nu)E]$	$3P/E$

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nodal coordinates (numbered in a clockwise sense) given by (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), and the elastic properties E and ν. The 8×8 stiffness matrix of the element is customarily obtained using Gauss-Legendre quadrature with two integrating points in each of the two local coordinate directions [see e.g. Smith and Griffiths (1988)].

Analytical expressions for the exactly integrated stiffness matrix of a rectangular four-node element have been published by Hacker and Schreyer (1989) and for a general quadrilateral by Griffiths (1994). Analytical integration formulas for linear isoparametric elements have also been presented by Babu and Pinder (1984) and Rathod (1988). This paper extends the earlier work by Griffiths to show how the SRI stiffness matrix of a general quadrilateral element can be expressed in closed form by expanding and simplifying the terms that are generated by the numerical integration summation. The computer algebra system Maple (Harper 1989) has been used to help generate the expressions.

Computer algebra systems have considerable potential in the area of finite-element software generation. In particular, Bettess and Bettess (1987) and Barbier et al. (1990) showed how the computer algebra system REDUCE (Rayna 1987) could be used to automatically generate shape functions for any finite element. The systems usually have the additional facility of being able to generate output in the FORTRAN programming language; thus complex algebraic expressions can be coded without the usual risk of typographical errors.

FORMULATION

For the present development, we assume that the stress conditions are those of two-dimensional plane strain elasticity, hence the stress/strain relationship is given by

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon} \qquad (11)$$

where

$$\boldsymbol{\sigma} = [\sigma_x \sigma_y \tau_{xy}]^T; \quad \boldsymbol{\varepsilon} = [\varepsilon_x \varepsilon_y \gamma_{xy}]^T \qquad (12, 13)$$

The element stiffness relationship is given by

$$\mathbf{ku} = \mathbf{f} \qquad (14)$$

where the stiffness matrix $\mathbf{k}$ to be formed using SRI can be written as the sum of two terms [see e.g. Hughes (1987)]

$$\mathbf{k} = \int_{V^e} \mathbf{B}^T\mathbf{D}^d\mathbf{B}\, d(\text{vol}) + \int_{V^e} \mathbf{B}^T\mathbf{D}^v\mathbf{B}\, d(\text{vol}) \qquad (15)$$

in which the deviatoric term is integrated exactly, and the volumetric term approximately using "reduced" (1×1) integration.

The strain/displacement matrix $\mathbf{B}$ is given by

$$\mathbf{B} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 & \frac{\partial N_4}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} & 0 & \frac{\partial N_4}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial y} & \frac{\partial N_4}{\partial x} \end{bmatrix} \qquad (16)$$

It may be noted that this expression for $\mathbf{B}$ assumes that the x- and y-nodal displacements are numbered alternately at each node, thus

$$\mathbf{u} = [u_{x1}\ u_{y1}\ u_{x2}\ u_{y2}\ u_{x3}\ u_{y3}\ u_{x4}\ u_{y4}]^T \qquad (17a)$$

$$\mathbf{f} = [f_{x1}\ f_{y1}\ f_{x2}\ f_{y2}\ f_{x3}\ f_{y3}\ f_{x4}\ f_{y4}]^T \qquad (17b)$$

The Ns are the shape functions in local coordinates, which for this four-node quadrilateral element are given as

$$N_1 = (1/4)(1 - \xi)(1 - \eta); \quad N_2 = (1/4)(1 - \xi)(1 + \eta) \qquad (18a,b)$$

$$N_3 = (1/4)(1 + \xi)(1 + \eta); \quad N_4 = (1/4)(1 + \xi)(1 - \eta) \qquad (18c,d)$$

As the element is isoparametric, the relationship between local and global coordinate systems is given by

$$x = N_1x_1 + N_2x_2 + N_3x_3 + N_4x_4 \qquad (19a)$$

$$y = N_1y_1 + N_2y_2 + N_3y_3 + N_4y_4 \qquad (19b)$$

The numerically integrated element stiffness matrix can be expressed as

$$\mathbf{k} = \sum_{i=1}^{2}\sum_{j=1}^{2} w_{ij}(\det\mathbf{J})_{ij}(\mathbf{B}^T\mathbf{D}^d\mathbf{B})_{ij} + w_c(\det\mathbf{J})_c(\mathbf{B}^T\mathbf{D}^v\mathbf{B})_c \qquad (20)$$

where the subscripts i and j index the four integrating points used for the exact integration contribution; and the subscript c indexes the fifth integrating point at the element centroid used for the reduced integration contribution.

For the two-point formula, the weighting coefficients $w_{ij} = 1$ for all values of i and j, and the integrating points are located at $\xi, \eta = \pm 1/\sqrt{3}$ in local coordinates. For the one-point formula, $w_c = 4$ and the integrating point is located at $\xi = \eta = 0$.

The scalar $(\det\mathbf{J})$ is the determinant of the Jacobian matrix evaluated at the integrating point, where

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \qquad (21)$$

Properties of Stiffness Matrix

Before evaluating the terms of the stiffness matrix, some observations can be made about its structure, as this will lead to a reduction in computational effort.

The matrix will be symmetrical, so only those terms on and below the main diagonal will need to be evaluated. These terms are given in (22)

$$\mathbf{k} = \begin{bmatrix} \mathbf{k}_{11} & & & & & & & \\ \mathbf{k}_{21} & k_{22} & & & & & & \\ \mathbf{k}_{31} & k_{32} & k_{33} & & & & & \\ \mathbf{k}_{41} & k_{42} & k_{43} & k_{44} & & & & \\ \mathbf{k}_{51} & k_{52} & k_{53} & k_{54} & k_{55} & & & \\ \mathbf{k}_{61} & k_{62} & k_{63} & k_{64} & k_{65} & k_{66} & & \\ \mathbf{k}_{71} & k_{72} & k_{73} & k_{74} & k_{75} & k_{76} & k_{77} & \\ \mathbf{k}_{81} & k_{82} & k_{83} & k_{84} & k_{85} & k_{86} & k_{87} & k_{88} \end{bmatrix} \qquad (22)$$

The terms can also split in six different groups as given in Table 2. Within each group, the bold term can be considered to be the "parent" member, from which all the others can

TABLE 2. Types of Stiffness Terms in Four-Node Element

Group (1)	Description (2)	Terms (3)
A	diagonals	$\mathbf{k}_{11}k_{33}k_{55}k_{77}k_{22}k_{44}k_{66}k_{88}$
B	orthogonal freedoms at the same node	$\mathbf{k}_{21}k_{43}k_{65}k_{87}$
C	parallel freedoms at adjacent nodes	$\mathbf{k}_{31}k_{53}k_{75}k_{71}k_{86}k_{64}k_{42}k_{82}$
D	orthogonal freedoms at adjacent nodes	$\mathbf{k}_{41}k_{63}k_{85}k_{72}k_{32}k_{54}k_{76}k_{81}$
E	parallel freedoms at opposite nodes	$\mathbf{k}_{51}k_{73}k_{84}k_{62}$
F	orthogonal freedoms at opposite nodes	$\mathbf{k}_{61}k_{83}k_{52}k_{74}$

be obtained by a simple rearrangement of the nodal coordinates.

The distribution of the different groups in a matrix layout is

$$\mathbf{k} = \begin{bmatrix} \mathbf{A} & & & & & & & \\ \mathbf{B} & A & & & & & & \\ \mathbf{C} & D & A & & & & & \\ \mathbf{D} & C & B & A & & & & \\ \mathbf{E} & F & C & D & A & & & \\ \mathbf{F} & E & D & C & B & A & & \\ C & D & E & F & C & D & A & \\ D & C & F & E & D & C & B & A \end{bmatrix} \quad (23)$$

Parent Terms of Stiffness Matrix

The parent member from each group in Table 2 is given in this section (Cardoso 1994), followed by a description of the nodal coordinate transformations necessary to retrieve all the other members of that group.

All parent terms are of the form

$$k_{i1} = (G/2)\{[A_2(v_d s_1 + s_2) + f_1(v_d s_3 + s_4)]/(3A_2^2 - f_1^2)$$
$$+ [A_2(v_d t_1 + t_2) + f_2(v_d t_3 + t_4)]/(3A_2^2 - f_2^2)\} + C_v v_v \quad (24)$$

where $i = 1, 2, \ldots, 6$; and

$$A_2 = (x_4 - x_2)(y_3 - y_1) - (x_3 - x_1)(y_4 - y_2)$$
$$= \text{twice the area of the element} \quad (25)$$

$$f_1 = (x_1 + x_3)(y_4 - y_2) - (y_1 + y_3)(x_4 - x_2)$$
$$- 2(x_2 y_4 - x_4 y_2) \quad (26)$$

$$f_2 = (y_2 + y_4)(x_3 - x_1) - (x_2 + x_4)(y_3 - y_1)$$
$$- 2(x_3 y_1 - x_1 y_3) \quad (27)$$

$$C_v = [3K - 2G(1 - \theta)]/12A_2 \quad (28)$$

The values of v_d and v_v, and the functions s_1, s_2, s_3, s_4, t_1, t_2, t_3, and t_4, which depend on the nodal coordinates, are given for each of the parent terms in the next six sections.

Having obtained the parent member of each group, the other members of each group from Table 2 can be obtained using one of the three nodal coordinate transformations listed here.

In Table 3, the symbol "$\Leftarrow$" means "is overwritten by." It should be noted that the transformations always apply to the s and t functions; however, transformation types 1 and 2 also apply to the f functions. None of the transformations should be applied to A_2, which is always twice the area of the element.

Group A: k_{11}

$$v_d = 2(3 - \theta)/3; \quad v_v = s_1 \quad (29, 30)$$

where

$$s_1 = 2(y_4 - y_2)^2; \quad s_2 = 2(x_4 - x_2)^2 \quad (31, 32)$$
$$s_3 = -s_1/2; \quad s_4 = -s_2/2 \quad (33, 34)$$
$$t_1 = (y_2 - y_3)^2 + (y_3 - y_4)^2 + (y_4 - y_2)^2 \quad (35)$$
$$t_2 = (x_2 - x_3)^2 + (x_3 - x_4)^2 + (x_4 - x_2)^2 \quad (36)$$
$$t_3 = (y_4 - y_3)^2 - (y_3 - y_2)^2 \quad (37)$$
$$t_4 = (x_4 - x_3)^2 - (x_3 - x_2)^2 \quad (38)$$

(See Table 4.)

Group B: k_{21}

$$v_d = -2\theta/3; \quad v_v = s_1 \quad (39, 40)$$

where

$$s_1 = 2(x_2 - x_4)(y_4 - y_2); \quad s_2 = s_1 \quad (41, 42)$$
$$s_3 = -s_1/2; \quad s_4 = s_3 \quad (43, 44)$$
$$t_1 = x_2(y_4 - 2y_2 + y_3) + x_3(y_2 - 2y_3 + y_4)$$
$$+ x_4(y_2 - 2y_4 + y_3) \quad (45)$$
$$t_2 = t_1; \quad t_3 = x_2(y_2 - y_3) + x_3(y_4 - y_2) + x_4(y_3 - y_4) \quad (46, 47)$$
$$t_4 = t_3 \quad (48)$$

(See Table 5.)

Group C: k_{31}

$$v_d = 2(3 - \theta)/3; \quad v_v = 2(t_1 + t_3) \quad (49, 50)$$

where

$$s_1 = (y_4 - y_2)(2y_1 - y_3 - y_4) \quad (51)$$
$$s_2 = (x_4 - x_2)(2x_1 - x_3 - x_4) \quad (52)$$
$$s_3 = (y_4 - y_2)(y_4 - y_1); \quad s_4 = (x_4 - x_2)(x_4 - x_1) \quad (53, 54)$$
$$t_1 = (y_3 - y_1)(2y_2 - y_3 - y_4) \quad (55)$$

TABLE 3. Transformations

Transformation (1)	Terms affected (2)
(*a*) Type 1	
$(x_1, y_1) \Leftarrow (x_2, y_2)$ $(x_2, y_2) \Leftarrow (x_3, y_3)$ $(x_3, y_3) \Leftarrow (x_4, y_4)$ $(x_4, y_4) \Leftarrow (x_1, y_1)$	f_1, f_2 s_1, s_2, s_3, s_4 t_1, t_2, t_3, t_4
(*b*) Type 2	
$(x_1, y_1) \Leftarrow (y_3, x_3)$ $(x_2, y_2) \Leftarrow (y_2, x_2)$ $(x_3, y_3) \Leftarrow (y_1, x_1)$ $(x_4, y_4) \Leftarrow (y_4, x_4)$	f_1, f_2 s_1, s_2, s_3, s_4 t_1, t_2, t_3, t_4
(*c*) Type 3	
$(x_1, y_1) \Leftarrow (y_1, x_1)$ $(x_2, y_2) \Leftarrow (y_2, x_2)$ $(x_3, y_3) \Leftarrow (y_3, x_3)$ $(x_4, y_4) \Leftarrow (y_4, x_4)$	s_1, s_2, s_3, s_4 t_1, t_2, t_3, t_4

TABLE 4. Group A

To compute (1)	From (2)	Use transformation type (3)
k_{33}	k_{11}	1
k_{55}	k_{33}	1
k_{77}	k_{55}	1
k_{88}	k_{77}	2
k_{22}	k_{88}	1
k_{44}	k_{22}	1
k_{66}	k_{44}	1

TABLE 5. Group B

To compute (1)	From (2)	Use transformation type (3)
k_{43}	k_{21}	1
k_{65}	k_{43}	1
k_{87}	k_{65}	1

$$t_2 = (x_3 - x_1)(2x_2 - x_3 - x_4) \quad (56)$$

$$t_3 = (y_3 - y_1)(y_3 - y_2); \quad t_4 = (x_3 - x_1)(x_3 - x_2) \quad (57, 58)$$

(See Table 6.)

Group D: k_{41}

$$v_d = -2\theta/3; \quad v_v = 2(t_1 + t_3) \quad (59, 60)$$

where

$$s_1 = (x_3 - x_1)(y_4 - y_2) + (x_4 - x_1)(y_4 - y_2) \quad (61)$$

$$s_2 = (y_3 - y_1)(x_4 - x_2) + (y_4 - y_1)(x_4 - x_2) \quad (62)$$

$$s_3 = (x_4 - x_1)(y_2 - y_4); \quad s_4 = (y_4 - y_1)(x_2 - x_4) \quad (63, 64)$$

$$t_1 = (x_3 - x_1)(y_4 - y_2) + (x_3 - x_1)(y_3 - y_2) \quad (65)$$

$$t_2 = (y_3 - y_1)(x_4 - x_2) + (y_3 - y_1)(x_3 - x_2) \quad (66)$$

$$t_3 = (x_3 - x_1)(y_2 - y_3); \quad t_4 = (y_3 - y_1)(x_2 - x_3) \quad (67, 68)$$

(See Table 7.)

Group E: k_{51}

$$v_d = 2(3 - \theta)/3; \quad v_v = 2s_1 \quad (69, 70)$$

where

$$s_1 = -(y_4 - y_2)^2; \quad s_2 = -(x_4 - x_2)^2 \quad (71, 72)$$

$$s_3 = 0; \quad s_4 = 0 \quad (73, 74)$$

TABLE 6. Group C

To compute (1)	From (2)	Use transformation type (3)
k_{53}	k_{31}	1
k_{75}	k_{53}	1
k_{71}	k_{75}	1
k_{86}	k_{71}	2
k_{82}	k_{86}	1
k_{42}	k_{82}	1
k_{64}	k_{42}	1

TABLE 7. Group D

To compute (1)	From (2)	Use transformation type (3)
k_{63}	k_{41}	1
k_{85}	k_{63}	1
k_{72}	k_{85}	1
k_{32}	k_{41}	3
k_{54}	k_{32}	1
k_{76}	k_{54}	1
k_{81}	k_{76}	1

TABLE 8. Group E

To compute (1)	From (2)	Use transformation type (3)
k_{73}	k_{51}	1
k_{84}	k_{73}	2
k_{62}	k_{84}	1

TABLE 9. Group F

To compute (1)	From (2)	Use transformation type (3)
k_{83}	k_{61}	1
k_{52}	k_{61}	3
k_{74}	k_{52}	1

TABLE 10. Timing for Element Stiffness Matrix Computations

N (1)	CPU Time (seconds): Analytical (2)	CPU Time (seconds): Numerical (3)	Speed-up (4)
(*a*) PC486			
100	0.16×10^0	0.28×10^0	1.7
1,000	0.11×10^1	0.31×10^1	2.8
10,000	0.11×10^2	0.31×10^2	2.9
(*b*) Sun SPARClassic			
100	0.90×10^{-1}	0.24×10^0	2.7
1,000	0.92×10^0	0.26×10^1	2.8
10,000	0.87×10^1	0.24×10^2	2.8
(*c*) DEC 3000 Model 4000			
100	0.49×10^{-2}	0.19×10^{-1}	3.8
1,000	0.50×10^1	0.19×10^0	3.8
10,000	0.50×10^0	0.19×10^1	3.8
(*d*) Amdahl Vector Processor VP1200			
100	0.47×10^{-2}	0.25×10^{-1}	5.5
1,000	0.44×10^1	0.25×10^0	5.7
10,000	0.44×10^0	0.25×10^1	5.7

$$t_1 = (y_3 + y_1)(y_4 + y_2) - 2(y_4 - y_2)^2 - 2(y_1y_3 + y_2y_4) \quad (75)$$

$$t_2 = (x_3 + x_1)(x_4 + x_2) - 2(x_4 - x_2)^2 - 2(x_1x_3 + x_2x_4) \quad (76)$$

$$t_3 = (y_4 - y_2)(y_1 - y_2 + y_3 - y_4) \quad (77)$$

$$t_4 = (x_4 - x_2)(x_1 - x_2 + x_3 - x_4) \quad (78)$$

(See Table 8.)

Group F: k_{61}

$$v_d = -2\theta/3; \quad v_v = 2s_1 \quad (79, 80)$$

where

$$s_1 = (x_4 - x_2)(y_4 - y_2); \quad s_2 = s_1 \quad (81, 82)$$

$$s_3 = 0; \quad s_4 = 0 \quad (83, 84)$$

$$t_1 = (x_4 - x_2)(y_4 - y_2) + (x_2 - x_1)(y_2 - y_3) + (x_4 - x_1)(y_4 - y_3) \quad (85)$$

$$t_2 = (y_4 - y_2)(x_4 - x_2) + (y_2 - y_1)(x_2 - x_3) + (y_4 - y_1)(x_4 - x_3) \quad (86)$$

$$t_3 = (x_2 - x_1)(y_3 - y_2) + (x_4 - x_1)(y_4 - y_3) \quad (87)$$

$$t_4 = (y_2 - y_1)(x_3 - x_2) + (y_4 - y_1)(x_4 - x_3) \quad (88)$$

(See Table 9.)

Subroutine Timings

The expressions described in the previous section were entered into a Fortran subroutine. This enabled an efficiency comparison to be made with more-conventional methods of forming the stiffness matrix that use the Gaussian quadrature formulation given in (20).

The test involved N repeated calculations of the element stiffness matrix of a general quadrilateral element. Table 10 shows the central processing unit (CPU) time used by both the analytical approach as described in this paper, and the conventional numerical integration approach. The speed-up ratio is also given in all cases. The timing comparisons were performed on several different machines as indicated. In each case θ was set equal to 0.5.

CONCLUDING REMARKS

The stiffness matrix of a general plane strain four-node quadrilateral finite element using selective reduced integration (SRI) has been presented in closed form. The algebraic expressions were obtained with the help of a computer algebra system, and based on the summation of the five terms that would be required to integrate the element stiffness numerically. A general approach to the deviatoric/volumetric split of the constitutive **D** matrix was presented based on a linear combination of the two main partitions. The influence of the scalar quantity θ on computed results of boundary-value problems, especially in elastoplasticity may be a fruitful area of research in the future.

A comparison of the processing speed of the analytical and numerical approaches indicated a speed-up in CPU time of between 2.8 and 5.7 depending on the computer.

It should be noted that no attempt was made in the present work to optimize either the analytical or the numerical approaches. Undoubtedly, further improvements could be made to both algorithms; however, it is believed that the analytical approach will always run faster due to the extensive use of assignment statements and the simplifications that are possible through canceling of terms not occurring in the numerical formulation.

It is recognized that the savings in time described herein in relation to the element stiffness matrix formulation will often constitute a modest proportion of the total computing time as compared with the equation solution phase of the calculation.

The FORTRAN subroutine that produces the four-node element stiffness matrix described in this paper is available from D. V. Griffiths on request.

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