Woodward, P. K. & Griffiths, D. V. (1998). *Géotechnique* **48**, No. 1, 137–141

TECHNICAL NOTE

Observations on the computation of the bearing capacity factor N_γ by finite elements

P. K. WOODWARD* and D. V. GRIFFITHS†

KEYWORDS: bearing capacity; failure; footings/foundations; numerical modelling and analysis; shear strength.

INTRODUCTION

It has been reported (Griffiths, 1982; Kay & Legein, 1994) that the bearing capacity factor N_γ, calculated from a finite element analysis, for a footing resting on the surface of cohesionless soil may be a function of the footing width B. The bearing capacity q_f is calculated from Terzaghi's equation (Terzaghi, 1943),

$$q_f = \frac{1}{2}\gamma B N_\gamma \tag{1}$$

where, in practice, N_γ is often calculated from one of the following equations:

$$N_\gamma = 1{\cdot}80(N_q - 1)\tan\phi \tag{2}$$

(Hansen, 1968)

$$N_\gamma = (N_q - 1)\tan(1{\cdot}4\phi) \tag{3}$$

(Meyerhof, 1963) and

$$N_q = \exp(\pi\tan\phi)\tan^2(45° + \phi/2) \tag{4}$$

The factor gives the contribution of the soil self-weight to the bearing capacity. In equations of this type no dependence on the footing width B is indicated.

Centrifuge tests by Hettler & Gudehus (1988) explained the dependence of N_γ on the width B by the pressure dependence of N_γ on the friction angle ϕ. It has also been suggested that progressive failure produces a grain size effect. Numerical predictions of bearing capacity on cohesionless soils have been published for many years (de Borst & Vermeer, 1984; Davidson & Chen, 1976; Griffiths, 1982; Kay & Legein, 1994; Manoharan & Dasgupta, 1995; Zienkiewicz *et al.*, 1978), often using a displacement-controlled footing. In displacement-controlled analyses an initial vertical stress state is assumed, usually the vertical distance of the stress point from the ground surface multiplied by the unit weight of the soil. The horizontal stresses are usually set by multiplying the vertical stresses by the earth pressure coefficient at rest, K_0. Failure is assumed to occur when the vertical stresses directly below the foundation level out, giving the bearing capacity of the foundation. N_γ can also be calculated by determining the equivalent nodal forces.

In this technical note, some observations on the computation of the bearing capacity factor N_γ are presented by performing numerical experiments using displacement control. The paper demonstrates that the same value of N_γ can be computed regardless of the footing width when the same mesh is used for each footing analysis, provided the footing width to the depth of the first row of elements is kept constant. For simplicity, a smooth footing was assumed. In the numerical experiments presented here, an elastic–perfectly plastic linear Mohr–Coulomb constitutive soil model was used, with a plane strain friction angle of $\phi = 25°$ and an angle of dilation of $\psi = 0°$ (non-associated). Using this type of constitutive model no pre-peak strain hardening, post-peak strain softening or pressure dependence of the friction angle ϕ is simulated. Displacements up to failure were not considered important and elastic constants of $E = 1{\cdot}0\text{E} \times 10^5$ kPa and $\nu = 0{\cdot}3$ were used. Eight-noded isoparametric quadrilateral elements were used with a 2×2 integration rule, $K_0 = 1{\cdot}0$ and $\gamma = 20{\cdot}0\ \text{kN/m}^3$. An elastic–viscoplastic algorithm was used for stress redistribution (e.g. Smith & Griffiths, 1988).

APPARENT EFFECT OF FOOTING WIDTH

In these analyses the dimensions of the mesh were kept constant (Fig. 1) and the footing width was varied, by simply increasing the number of nodes loaded (i.e. elements). Two arrangements were considered, an *even* arrangement, in which the footing edge finished at an element boundary (Fig. 2(b)), and an *odd* arrangement, in which the

Manuscript received 6 September 1996; revised manuscript accepted 19 November 1996.
Discussion on this technical note closes 1 May 1998; for further details see p. ii.
* Heriot-Watt University, Edinburgh.
† Colorado School of Mines, Golden.

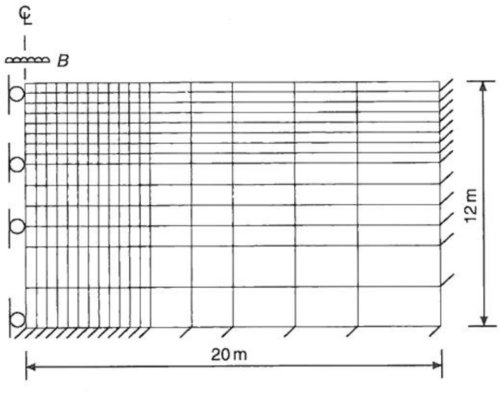

Fig. 1. Finite element mesh

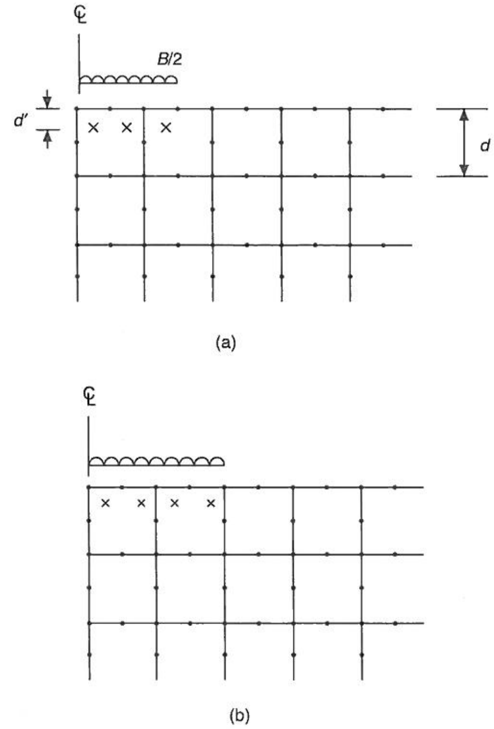

Fig. 2. (a) Odd arrangement; (b) even arrangement

footing edge finished at the mid-point of an element (Fig. 2(a)).

Figure 3 shows the results of increasing the footing width (i.e. the number of Gauss points under the footing) on the calculation of N_γ and N_q. Equation (4) gives $N_q = 10{\cdot}7$, which compares reasonably well with the finite element prediction of $N_q^{FE} = 10{\cdot}3$ (the procedure used to calculate N_q^{FE} is given by Griffiths (1982)). For smooth footings, Bolton & Lau (1993) predict a value of $N_\gamma = 3{\cdot}51$, whereas from the figure it appears that N_γ is a function of the footing width B (albeit relatively small in the even analysis). However, the figure also shows that this reduction may be due to the changing number of Gauss points used to average the non-linear stress distribution under the footing. There is also a large discrepancy between the odd and even arrangements for nodal loading and stress averaging. The results of the N_q analysis indicate that an even arrangement is better, as a constant value of N_q is predicted regardless of footing size. The results suggest that the calculation of N_γ may in fact be mesh-dependent.

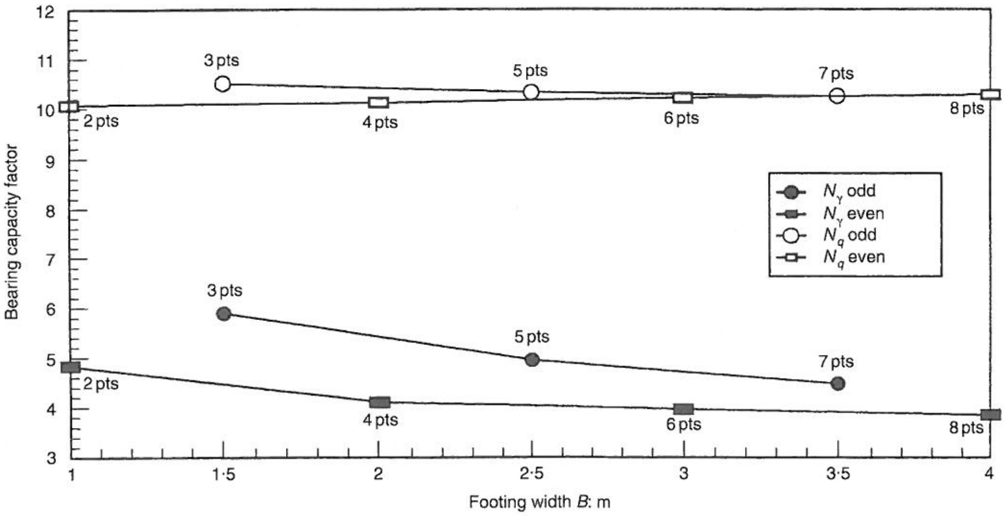

Fig. 3. Effect of footing width B and number of Gauss points (pts) on N_q and N_γ

Figure 4 shows the effect of increasing the number of Gauss points (i.e. elements) using a constant footing width of $B = 1{\cdot}5$ m. The odd arrangement clearly should not be used, since a 'constant' value of N_γ is not predicted when the number of Gauss points is increased (although it tends towards the even result as the number of Gauss points increases). In all of the subsequent analyses an even arrangement is used.

EFFECT OF MESH AND FOOTING SIZE

In these analyses the width of the footing was varied by multiplying the overall dimensions (i.e. the horizontal and vertical position of all the nodes) by the same 'mesh multiplication factor'. In this way the footing width could be varied, but the mesh density for each analysis kept constant and the (footing width)/(first element depth) ratio B/d (Fig. 2) kept constant.

Table 1 clearly shows that N_γ is not a function of the width of the footing, using this soil model, as a constant value is predicted, provided the B/d ratio is kept constant. The dependence of N_γ on the B/d ratio is examined more closely in the analyses below. To determine changes in N_γ with footing width, more sophisticated constitutive soil models must be used that can simulate the pressure dependence of ϕ.

EFFECT OF FIRST ELEMENT DEPTH d ON N_γ

In these analyses footing widths of $B = 1{\cdot}5$ m and 3·0 m were used. The depth of the first row of elements directly under each footing was varied to give equal B/d ratios.

Table 1. Effect of mesh size on N_γ using a constant B/d ratio

Mesh multiplication factor	Modified width B: m	B/d	N_γ
0·1	0·15	3	4·30
0·2	0·3	3	4·31
1	1·5	3	4·30
2	3·0	3	4·31
10	15	3	4·30
20	30	3	4·31

Table 2 shows that the calculation of N_γ is a function of the depth of the first row of elements (i.e. the vertical distance from the footing to the first row of Gauss points d'). By calculating N_γ on the basis of the vertical stress at the first row of Gauss points, the stress includes the weight of the soil above in the computation. By averaging the calculated stresses at the Gauss points we are approximating a footing situated at the Gauss point depth d'.

REANALYSIS OF RESULTS

In Tables 3 and 4 the component of the surcharge at the first Gauss point depth d' is subtracted from the averaged vertical stresses at the Gauss points to determine N_γ. A conservative approach is adopted where $\gamma d' N_q$ is subtracted from q_f. As before, for equal B/d ratios, modified

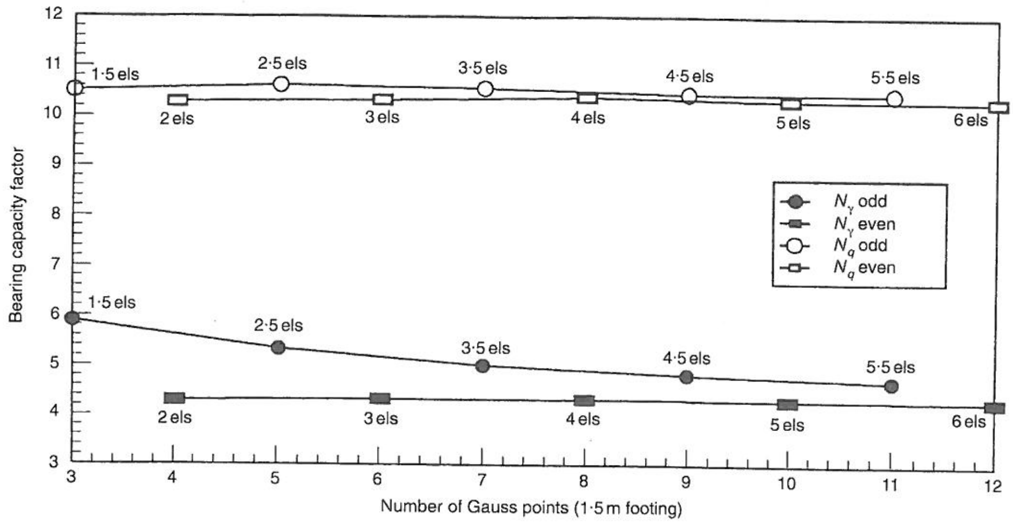

Fig. 4. Effect of number of Gauss points on N_γ and N_q for a footing width $B = 1{\cdot}5$ m (els = number of elements)

Table 2. Effect of B/d ratio on N_γ

$B = 1.5$ m (4 Gauss points)			$B = 3.0$ m (6 Gauss points)		
d: m	B/d	N_γ	d	B/d	N_γ
0·75	2	4·93	1·5	2	4·87
0·5	3	4·30	1·0	3	4·31
0·375	4	4·16	0·75	4	4·15
0·3	5	4·04	0·6	5	4·02

Table 3. Modified calculation of N_γ for $B = 1.5$ m

d	B/d	N_γ	Stress at failure q_f: kPa	$\gamma d' N_q$ $N_q^{FE} = 10.3$	Corrected N_γ
0·75	2	4·93	73·91	32·55	2·76
0·5	3	4·30	64·60	21·77	2·86
0·375	4	4·16	62·41	16·27	3·08
0·3	5	4·04	60·60	12·98	3·17

Table 4. Modified calculation of N_γ for $B = 3.0$ m

d	B/d	N_γ	Stress at failure q_f: kPa	$\gamma d' N_q$ $N_q^{FE} = 10.3$	Corrected N_γ
1·5	2	4·87	146·07	65·30	2·69
1·0	3	4·31	129·24	43·46	2·86
0·75	4	4·15	124·56	32·55	3·07
0·6	5	4·02	120·63	26·16	3·15
0·5	6	3·96	118·87	21·84	3·23
0·375	8	3·86	115·87	16·27	3·32
0·3	10	3·81	114·38	12·98	3·38

values of N_γ are very similar for the two different widths. In Table 4 when $B/d = 10$, q_f has almost levelled out. Using this method, there is only a 3·7% difference in N_γ when $B/d = 10$ compared to Bolton & Lau (1993). Column 2 in Table 5 shows corrected values of N_γ for different friction angles.

CONCLUDING REMARKS

Much debate has focused on the value of the bearing capacity factor N_γ calculated using finite elements. This technical note shows that, for the simple footing considered, if an elastic–perfectly plastic Mohr–Coulomb constitutive soil model is used there is no effect of the footing width on N_γ. However, a sufficient number of Gauss points (in these studies 6) must be used directly under the footing when averaging the stresses and the B/d ratio kept the same. An allowance should be made for the surcharge component of the bearing capacity at the Gauss point depth. 'Odd' arrangements, in which nodal loadings finish mid-point along an element, should be avoided.

REFERENCES

Bolton, M. D. & Lau, C. K. (1993). Vertical bearing capacity factors for circular and strip footings on Mohr-Coulomb soil. *Can. Geotech. J.* **30**, 1024–1033.

Table 5. Predictions of N_γ for smooth footings

ϕ	Corrected N_γ	Hansen & Christiensen (1969) N_γ	Bolton & Lau (1993) N_γ	Manoharan & Dasgupta (1995) N_γ
10	0·3		0·3	≅0·7
15	0·7	0·7	0·7	
20	1·5	1·6	1·6	≅2·1
25	3·4	3·5	3·5	≅4·4
30	7·6	7·5	7·7	≅9·1

Davidson, H. L. & Chen, W. F. (1976). Non-linear analyses in soil and solid mechanics. In *Numerical methods in geomechanics*, pp. 205–218. New York: American Society of Civil Engineers.

de Borst, R. & Vermeer, P. A. (1984). Possibilities and limitations of finite elements for limit analysis. *Géotechnique* **34**, No. 2, 199–210.

Griffiths, D. V. (1982). Computation of bearing capacity factors using finite elements. *Géotechnique* **32**, No. 3, 195–202.

Hansen, J. B. (1968). *A revised extended formula for bearing capacity*. Danish Geotechnical Institute Bulletin, No. 28.

Hansen, B. & Christiensen, N. H. (1969). Discussion of theoretical bearing capacity of very shallow footings. *J. Soil Mech. Fdns Div. Am. Soc. Civ. Engrs* **95**, 1568–1572.

Hettler, A. & Gudehus, G. (1988). Influence of the foundation width on the bearing capacity factor. *Soil and Fdns, Japan Soc. Soil Mech. Fdns Engng* **28**, No. 4, 81–92.

Kay, S. & Legein, J. J. D. (1994). Size effects by finite elements. *Proc. 8th Int. Conf. Comp. Meth. Adv. Geom., Morgantown, West Virginia, USA*, Vol. III, pp. 2343–2347.

Manoharan, N. & Dasgupta, S. P. (1995). Bearing capacity of surface footings by finite elements. *Comput. Struct.* **54**, No. 4, 563–586

Meyerhof, G. G. (1963). Some recent research on the bearing capacity of foundations. *Can. Geotech. J.* **1**, No. 1, 16–26.

Smith, I. M. & Griffiths, D. V. (1988). *Programming the finite element method*, 2nd edn, chapter 6. Chichester: Wiley.

Terzaghi, K. (1943). *Theoretical soil mechanics*, chapter 8. New York: Wiley.

Zienkiewicz, O. C., Norris, V. A., Winnicki, L. A., Naylor, D. J. & Lewis, R. W. (1978). A unified approach to the soil mechanics problems of offshore foundations. In *Numerical methods in offshore engineering*, pp. 361–411. New York: Wiley.