Probabilistic Foundation Settlement on Spatially Random Soil

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Abstract: By modeling soils as spatially random media, estimates of the reliability of foundations against serviceability limit state failure, in the form of excessive differential settlements, can be made. The soil's property of primary interest is its elastic modulus, E, which is represented here using a lognormal distribution and an isotropic correlation structure. Prediction of settlement below a foundation is then obtained using the finite element method. By generating and analyzing multiple realizations, the statistics and density functions of total and differential settlements are estimated. In this paper probabilistic measures of total settlement under a single spread footing and of differential settlement under a pair of spread footings using a two-dimensional model combined with Monte Carlo simulations are presented. For the cases considered, total settlement is found to be represented well by a lognormal distribution. Probabilities associated with differential settlement are conservatively estimated through the use of a normal distribution with parameters derived from the statistics of local averages of the elastic modulus field under each footing.

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Introduction

The settlement of structures founded on soil is a subject of considerable interest to practicing engineers since excessive settlements often lead to problems of serviceability. In particular, unless the total settlements themselves are particularly large, it is usually differential settlements which lead to unsightly cracks in facades and structural elements, possibly even to structural failure, especially in unreinforced masonry elements. Existing code requirements that limit differential settlements to satisfy serviceability limit states [see building codes ACI 318-89 (1989) or A23.3-M84 (1984)] specify maximum deflections ranging from $D/180$ to $D/480$, depending on the type of supported elements, where D is the center-to-center span of the structural element. In practice, differential settlements between footings are generally controlled, not by considering the differential settlement itself, but by controlling the total settlement predicted by analysis using an estimate of the soil elasticity. This approach is largely based on correlations between total settlements and differential settlements observed experimentally (see, e.g., the work of D'Appolonia et al. 1968) and leads to a limitation of 4–8 cm in total settlement under a footing as stipulated by the Canadian Foundation Engineering Manual, Part 2 (1978).

Because of the wide variety of soil types and possible loading conditions, experimental data on differential settlement of footings founded on soil are limited. With the aid of modern high-speed computers, it is now possible to probabilistically investigate differential settlements over a range of loading conditions and geometries. In this paper the findings of such a study are reported along with a reasonably simple, approximate approach to estimate probabilities associated with settlements. The case of a single footing, as shown in Fig. 1(a), is first considered and estimates of the probability density function governing total settlement of the footing as a function of footing width for various statistics of the underlying soil are given. Only the soil elasticity is considered to be spatially random. Uncertainties that arise from the model, test procedures and in the loads are not considered. In addition, the soil is assumed to be isotropic, that is, the correlation structure is assumed to be the same in both the horizontal and vertical directions where "correlation" refers to the correlation between two points in the soil mass. Although soils generally exhibit a stronger correlation in the horizontal direction, due to their layered nature, the degree of anisotropy is site specific. In that in this study we are attempting to establish the basic probabilistic behavior of settlement, anisotropy is left as a refinement in future research.

In foundation engineering, both immediate and consolidation settlements are traditionally computed using elastic theory. The elastic properties, E, that apply to either or both immediate and consolidation settlement are addressed here, since these are usually the most important components of settlement.

The footings are assumed to be founded on a soil layer underlain by bedrock. The assumption of an underlying bedrock can be relaxed if a suitable averaging region is used. Guidelines to this effect are suggested below. The results are generalized to allow the estimation of probabilities associated with settlements under footings in many practical cases.

In the second part of the paper the issue of differential settlements under a pair of footings, shown in Fig. 1(b), again for the particular case of footings founded on a soil layer underlain by bedrock is addressed. The mean and standard deviation of differential settlement are estimated as functions of the footing width for various input statistics of the underlying elastic modulus field. The probability distribution governing differential settlement is found to be conservatively estimated using a joint normal distribution with the correlation predicted using local averages of the elastic modulus field under the two footings.

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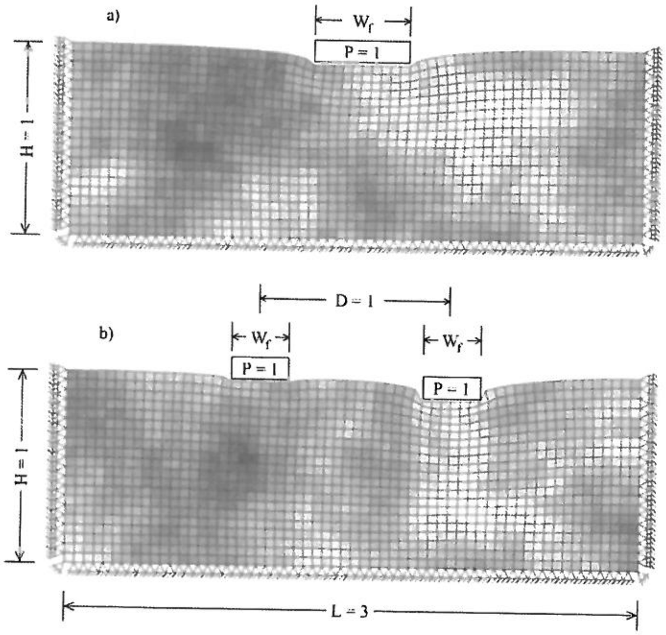

Fig. 1. Random field/FEM representation of (a) single footing and (b) two footings founded on soil layer

The physical problem is represented by use of a two-dimensional (2D) plane strain model following the work of Paice et al. (1996). If the footings extend a large distance in the out-of-plane direction, z, then the 2D elastic modulus field is interpreted either as an average over z or as having an infinite scale of fluctuation in the z direction. For footings of finite dimension, the 2D model is admittedly just an approximation. However, the approximation is considered reasonable since the elastic modulus field is averaged in the z direction in any case.

Random Field/Finite Element Model

As illustrated in Fig. 1, the soil mass is discretized into 60 four-noded quadrilateral elements in the horizontal direction by 20 elements in the vertical direction. Trials runs using 120×40 elements resulted in less than a 2.5% difference in settlements for the worst cases (narrowest footings), at a cost of more than 10 times the computing time, so the 60×20 discretization was considered adequate. The overall dimensions of the soil model are held fixed at $L=3$ in width by $H=1$ in height. No units will be used since the probabilistic properties of the soil domain are scaled by the correlation *scale of fluctuation* that will be discussed shortly. The left and right faces of the finite element model (FEM) are constrained against horizontal displacement but are free to slide vertically whereas the nodes on the bottom boundary are spatially fixed. The footing(s) are assumed to be rigid, to not undergo any rotations, and to have a rough interface with the underlying soil (no-slip boundary). A fixed load $P=1$ is applied to each footing. Since settlement varies linearly with load, the results are easily scaled to different values of P.

To investigate the effect of the footing width, the soil layer thickness, H, was held constant at 1.0 while the footing width was varied according to Table 1. Because the settlement problem is linear, the following results can be scaled to arbitrary footing widths and soil layer thicknesses. For example, the settlement of a footing of width $W_f'=2.0$ m on an $H'=20$ m thick soil layer with load $P'=1{,}000$ kN and elastic modulus $E'=60$ kN/m^2 corresponds to 0.06 times the settlement of a footing of width $W_f=0.1$ m on an $H=1.0$ m thick soil layer with $P=1$ kN and elastic modulus $E=1$ kN/m^2. The scaling factor from the case with primes to that without primes is $(P/P')(E'/E)(W_f/W_f')(H'/H)$.

Table 1. Input Parameters Varied in this Study while Keeping $H=1$, $D=1$, $P=1$, $\mu_E=1$, and $\nu=0.25$ Constant

Parameter	Values considered
σ_E	0.1, 0.5, 1.0, 2.0, 4.0
$\theta_{\ln E}$	0.01, 0.05, 0.1, 0.3, 0.5, 0.7, 1.0, 2.0, 5.0, 10.0, 50.0
W_f	0.1, 0.2, 0.5, 1.0 (single footing)
	0.1, 0.3, 0.5 (two footings)

In the two footing case, the distance between footing centers was held constant at 1.0, while the footing widths (assumed to be equal) were varied. Footings of width greater than 0.5 were not considered since this situation approaches that of a strip footing (the footings would be joined when $W_f=1.0$). The soil has two properties of interest to the settlement problem: these are the (effective) elastic modulus, $E(\mathbf{x})$, and the Poisson ratio, $\nu(\mathbf{x})$, where $\mathbf{x}$ is the spatial position. Only the elastic modulus is considered to be a spatially random soil property. The Poisson ratio is believed to have smaller relative spatial variability and only second order importance to settlement statistics. It is held fixed at 0.25 over the entire soil mass for all simulations.

Fig. 1 shows a gray-scale representation of a possible realization of the elastic modulus field, along with the finite element mesh. Lighter areas denote smaller values of $E(\mathbf{x})$ so that the elastic modulus field shown in Fig. 1(b) corresponds to a higher elastic modulus under the left footing than under the right; this leads to the substantial differential settlement, indicated by the deformed mesh. This is just one possible realization of the elastic modulus field; the next realization could just as easily show the opposite trend.

The elastic modulus field is assumed to follow a lognormal distribution so that $\ln(E)$ is a Gaussian (normal) random field with mean $\mu_{\ln E}$ and variance $\sigma^2_{\ln E}$. The choice of a lognormal distribution is motivated by the fact that the elastic modulus is strictly non-negative, a property of the lognormal distribution (but not the normal), while still having a simple relationship with the normal distribution. A Markovian spatial correlation function, which gives the correlation coefficient between log-elastic modulus values at points separated by distance τ is used

$$\rho_{\ln E}(\tau)=\exp\left\{-\frac{2|\tau|}{\theta_{\ln E}}\right\} \tag{1}$$

in which $\tau=\mathbf{x}-\mathbf{x}'$ is the vector between spatial points $\mathbf{x}$ and $\mathbf{x}'$, and $|\tau|$ is the absolute length of this vector (the lag distance). The correlation function decay rate is governed by the so-called scale of fluctuation, $\theta_{\ln E}$, which, loosely speaking, is the distance over which log-elastic moduli are significantly correlated [when the separation distance $|\tau|$ is greater than $\theta_{\ln E}$, the correlation between $\ln E(\mathbf{x})$ and $\ln E(\mathbf{x}')$ is less than 14%].

The assumption of isotropy is, admittedly, somewhat restrictive. In principle, the methodology presented in the following is easily extended to anisotropic fields although the accuracy of the proposed distribution parameter estimates would then need to be verified. For both the single footing and the two footing problems, however, it is the horizontal scale of fluctuation that is more important. As will be seen, the settlement variance and covariance depend on the statistics of a local average of the log–elastic modulus field under the footing. If the vertical scale of fluctuation is less than the horizontal, this can be handled simply by reducing the vertical averaging dimension H to $H(\theta_{\ln E_h}/\theta_{\ln E_v})$. For very deep soil layers, the averaging depth H should probably be restricted to no more than about $10W_f$ since the stress under a footing falls off approximately according to $W_f/(W_f+H)$.

In practice, one approach by which to estimate $\theta_{\ln E}$ involves collecting elastic modulus data from a series of locations in space, estimating the correlations between the log data as a function of the separation distance, and then fitting Eq. (1) to the estimated correlations. See, e.g., Asaoka and Grivas (1982), Marsily (1985), Soulié et al. (1990), Ravi (1992), De Groot and Baecher (1993), Chiasson et al. (1995), and Fenton (1999) for further information on the characterization of the spatial variability of soil properties. The estimation of $\theta_{\ln E}$ is not a simple problem since it tends to depend on the distance over which it is estimated. For example, sampling soil properties every 5 cm over 2 m will likely yield an estimated $\theta_{\ln E}$ of about 20 cm, while sampling every 1 km over 1000 km will likely yield an estimate of about 200 km. This is because soil properties vary at many scales: looked at closely, soil can change significantly within a few meters relative to the few meters considered. However, soils are formed by weathering and glacial actions which can span thousands of kilometers, yielding soils which have much in common over large distances. Thus, soils can conceptually have lingering correlations over entire continents (even planets).

This lingering correlation in the spatial variability of soils implies that scales of fluctuation estimated in the literature should not just be used blindly. One should attempt to select a scale which has been estimated on a similar soil *over a domain of similar size* to the site being characterized. In addition, the level of detrending used to estimate the reported scale of fluctuation must be matched at the site being characterized. For example, if a scale of fluctuation, as reported in the literature, were estimated from data with a quadratic trend removed, then sufficient data must be gathered at the site being characterized to allow a quadratic trend to be fitted to the site data. The estimated scale of fluctuation then applies to the residual random variation around the trend. To facilitate this, researchers providing estimates of variance and scale in the literature should report estimates with the trend removed, including the trend itself, and estimates without the trend removed. The latter will typically yield significantly larger estimated variance and scales, giving a truer sense of the actual soil variability.

In the case of two footings, the use of a scale of fluctuation equal to D is conservative in that it yields differential settlement variances which are at, or close to, their maxima, as will be seen. In some cases, however, setting $\theta_{\ln E}=D$ may be unreasonably conservative. If sufficient site sampling has been carried out to estimate the mean and variance of the soil properties at the site, then a significantly reduced scale is warranted. The literature should then be consulted to find a similar site on which a spatial statistical analysis has been carried out and an estimated scale reported.

In the case of a single footing, taking $\theta_{\ln E}$ as large is conservative; in fact, the assumption that E is (lognormally distributed and) spatially constant leads to the largest variability (across realizations) in footing settlement. Thus, traditional approaches to randomness in footing settlement, using a single random variable to characterize E, are conservative, settlement will generally be less than predicted.

Throughout, the mean elastic modulus, μ_E, is held fixed at 1.0. Since settlement varies linearly with the soil elastic modulus, it is always possible to scale the settlement statistics to the actual mean elastic modulus. The standard deviation of the elastic modulus is varied from 0.1 to 4.0 to investigate the effects of elastic modulus variability on settlement variability. The parameters of the transformed ln(E) Gaussian random field may be obtained from the relations

$$\sigma_{\ln E}^2=\ln(1+\sigma_E^2/\mu_E^2) \tag{2a}$$

$$\mu_{\ln E}=\ln(\mu_E)-\tfrac{1}{2}\sigma_{\ln E}^2 \tag{2b}$$

from which it can be seen that the variance of ln(E), $\sigma_{\ln E}^2$, varies from 0.01 to 2.83 in this study [note also that the mean of ln(E) depends on both μ_E and σ_E].

To investigate the effect of the scale of fluctuation, $\theta_{\ln E}$, on the settlement statistics, $\theta_{\ln E}$ is varied from 0.01 (i.e., very much smaller than the soil model size) to 50.0 (i.e., substantially bigger than the soil model size) and up to 200 in the two footing case. In the limit as $\theta_{\ln E}\to 0$, the elastic modulus field becomes a white noise field, with E values at any two distinct points independent. In terms of the finite elements themselves, values of $\theta_{\ln E}$ smaller than the elements result in a set of elements which are largely independent (increasingly independent as $\theta_{\ln E}$ decreases). Because of the averaging effect of the details of the elastic modulus field under a footing, the settlement in the limiting case of $\theta_{\ln E}\to 0$ is expected to approach that obtained in the deterministic case, with $E=\mu_E$ everywhere, and has vanishing variance for finite $\sigma_{\ln E}^2$.

By similar reasoning the differential settlement [as shown in Fig. 1(b)] as $\theta_{\ln E}\to 0$ is expected to go to zero. At the other extreme, as $\theta_{\ln E}\to\infty$, the elastic modulus field becomes the same everywhere. In this case, the settlement statistics are expected to approach those obtained using a single lognormally distributed random variable, E, to model the soil, $E(\mathbf{x})=E$. That is, since the settlement, δ, under a footing founded on a soil layer with uniform (but random) elastic modulus E is given by

$$\delta=\frac{\delta_{\text{det}}\mu_E}{E} \tag{3}$$

where δ_{det}="deterministic" settlement obtained from a single finite element analysis (or an appropriate approximate calculation) of the problem using $E=\mu_E$ everywhere. In this case as $\theta_{\ln E}\to\infty$ the settlement assumes a lognormal distribution with parameters

$$\mu_{\ln\delta}=\ln(\delta_{\text{det}})+\ln(\mu_E)-\mu_{\ln E}=\ln(\delta_{\text{det}})+\tfrac{1}{2}\sigma_{\ln E}^2 \tag{4a}$$

$$\sigma_{\ln\delta}=\sigma_{\ln E} \tag{4b}$$

where Eq. (2b) was used in Eq. (4a). Also since, in this case, the settlement under the two footings of Fig. 1(b) becomes equal, the differential settlement becomes zero. Thus, the differential settlement is expected to approach zero at both very small and at very large scales of fluctuation. The Monte Carlo approach adopted here involves the simulation of a realization of the elastic modulus field and subsequent finite element analysis (e.g., Smith and Griffiths 1998) of that realization to yield a realization of the footing settlement(s). Repeating the process over an ensemble of realizations generates a set of possible settlements which can be plotted in the form of a histogram and from which distribution parameters can be estimated. In this study, 5000 realizations are

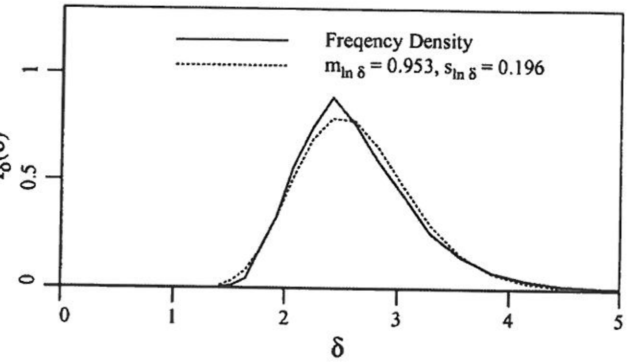

Fig. 2. Typical frequency density plot and fitted lognormal distribution of settlement under single footing

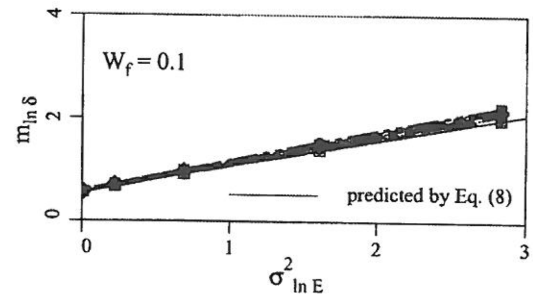

Fig. 3. Estimated mean of log settlement

performed for each input parameter set (σ_E, $\theta_{\ln E}$, and W_f). If it can be assumed that log settlement is approximately normally distributed (which is seen later to be a reasonable assumption and is consistent with the distribution selected for E) and that $m_{\ln\delta}$ and $s^2_{\ln\delta}$ are the estimators of the mean and variance of log settlement, respectively, then the standard deviations of these estimators obtained from 5000 realizations are given by

$$\sigma_{m_{\ln\delta}} \simeq s_{\ln\delta}/\sqrt{n} = 0.014 s_{\ln\delta} \quad (5a)$$

$$\sigma_{s^2_{\ln\delta}} \simeq \sqrt{\frac{2}{n-1}} s^2_{\ln\delta} = 0.02 s^2_{\ln\delta} \quad (5b)$$

so that the estimator "errors" are negligible compared to the estimated variance (i.e., about 1 or 2% of the estimated standard deviation).

Realizations of the log–elastic modulus field are produced using the two-dimensional local average subdivision (LAS) technique (Fenton and Vanmarcke 1990; Fenton 1994). The elastic modulus value assigned to the ith element is

$$E(\mathbf{x}_i) = \exp[\mu_{\ln E} + \sigma_{\ln E} G(\mathbf{x}_i)] \quad (6)$$

where $G(\mathbf{x}_i)$ = local average over the element centered at $\mathbf{x}_i$ of a zero mean, unit variance Gaussian random field.

Single Footing Case

A typical histogram of the settlement under a single footing, estimated by 5,000 realizations, is shown in Fig. 2 for $W_f = 0.1$, $\sigma_E/\mu_E = 1$, and $\theta_{\ln E} = 0.1$. With the requirement that settlement be non-negative, the shape of the histogram suggests a lognormal distribution, which was adopted in this study [see also Eqs. (4)]. The histogram is normalized to produce a frequency density plot, in which a straight line is drawn between the interval midpoints.

Superimposed on the histogram is a fitted log–normal distribution with parameters given by $m_{\ln\delta}$ and $s_{\ln\delta}$ in the line key. At least visually, the fit appears reasonable. In fact, this is one of the worst cases, of all the 220 parameter sets given in Table 1; a chi-square goodness-of-fit test yields a p value of 8×10^{-10}. Large p values, up to 1.0, support the lognormal hypothesis, so this small value suggests that the data do not follow a lognormal distribution. Unfortunately, when the size of the sample is large ($n = 5{,}000$ in this case) goodness-of-fit tests are quite sensitive to the "smoothness" of the histogram. They perhaps correctly indicate that the true distribution is not exactly as hypothesized, but provide little information about the *reasonableness* of the assumed distribution. As can be seen from Fig. 2, the lognormal distribution certainly appears reasonable.

Over the entire set of simulations performed for each parameter of interest (W_f, σ_E, and $\theta_{\ln E}$), 80% of the fits have p values that exceed 5% and only 5% have p values of less than 0.0001. This means that the lognormal distribution is generally a close approximation to the distribution of the simulated settlement data, typically at least as good as seen in Fig. 2.

Accepting the lognormal distribution as a reasonable fit to the simulation results, the next task is to estimate the parameters of the fitted lognormal distributions as functions of the input parameters (W_f, σ_E, and $\theta_{\ln E}$). The lognormal distribution

$$f_\delta(x) = \frac{1}{\sqrt{2\pi}\sigma_{\ln\delta} x}\exp\left[-\frac{1}{2}\left(\frac{\ln x - \mu_{\ln\delta}}{\sigma_{\ln\delta}}\right)^2\right], \quad 0 \le x < \infty \quad (7)$$

has two parameters, $\mu_{\ln\delta}$ and $\sigma_{\ln\delta}$. Fig. 3 shows how the estimator of $\mu_{\ln\delta}$, $m_{\ln\delta}$, varies with $\sigma_{\ln E}$ for $W_f = 0.1$. All scales of fluctuation are drawn in the plot, but they are not labeled individually since they lie so close together. This observation implies that the mean log settlement is largely independent of the scale of fluctuation, $\theta_{\ln E}$. This is as expected since the scale of fluctuation does not affect the mean of a local average of a normally distributed process. Fig. 3 suggests that the mean of log settlement can be closely estimated by a straight line of the form [suggested by Eq. (4a)],

$$\mu_{\ln\delta} = \ln(\delta_{\text{det}}) + \tfrac{1}{2}\sigma^2_{\ln E} \quad (8)$$

Eq. (8) is also shown in Fig. 3 and it can be seen that the agreement is very good. Even closer results were found for the other footing widths.

Estimates of the standard deviation of log settlement, $s_{\ln\delta}$, are plotted in Fig. 4 (by symbols) for the smallest and largest footing widths. Intermediate footing widths give similar results. In all cases, $s_{\ln\delta}$ increases to $\sigma_{\ln E}$ as $\theta_{\ln E}$ increases. The reduction in variance as $\theta_{\ln E}$ decreases is due to the local averaging variance reduction of the log–elastic modulus field under the footing (for smaller $\theta_{\ln E}$, there are more "independent" random field values, so the variance reduces faster under averaging; see Vanmarcke 1984, for more details on local averaging theory).

Following the above reasoning, and assuming that local averaging of the area under the footing accounts for all of the reduction in variance seen in Fig. 4, the standard deviation of log settlement is

$$\sigma_{\ln\delta} = \sqrt{\gamma(W_f, H)}\sigma_{\ln E} \quad (9)$$

where $\gamma(W_f, H)$ = so-called variance function (Vanmarcke 1984), which depends on the averaging region, $W_f \times H$, as well as on the

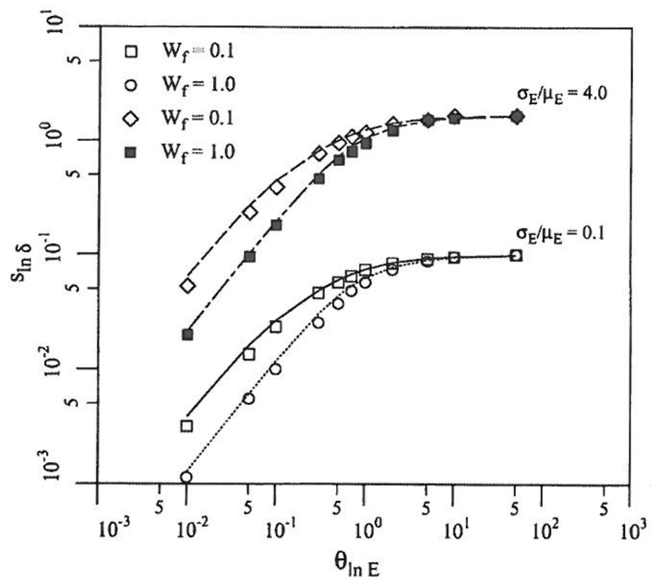

Fig. 4. Comparison of simulated sample standard deviation of log settlement, shown by symbols, with theoretical estimate via Eq. (9), shown by lines

scale, $\theta_{\ln E}$. Since $\sigma^2_{\ln E}$ is constant for each value of σ_E/μ_E [see Eq. (2*a*)], Fig. 4 is essentially a plot of the variance function, $\gamma(W_f, H)$, illustrating how the variance of a local average decreases as the scale of fluctuation decreases.

Specifically, the variance function gives the amount that the log-elastic modulus variance is reduced when its random field is averaged over a region of $W_f \times H$. The dependence of the variance function on H is apparently only valid for the geometries considered; if the footing is founded on a much deeper soil mass, one would not expect to average over the entire depth due to the reduction of stress with depth. As suggested in the "Random Field/Finite Element Model," H should be limited to no more than about $10W_f$. If in doubt, taking H to be relatively small (even zero) yields a conservative estimate of the settlement distribution, having large variability. This is equivalent to taking $\theta_{\ln E}$ as large, as noted previously in the "Random Field/Finite Element Model." In practice, however, values of the normalized averaging area $W_f H/\theta^2_{\ln E}$ greater than about 5 yield values of $\sigma_{\ln \delta}$ less than about 15% of $\sigma_{\ln E}$ so changes in H above this level have only a minor effect on the overall reduction in variance.

The variance function that corresponds to the isotropic Markov correlation function [Eq. (1)] is approximated by

$$\gamma(d_1, d_2) = \tfrac{1}{2}[\gamma(d_1)\gamma(d_2|d_1) + \gamma(d_2)\gamma(d_1|d_2)] \tag{10}$$

where

$$\gamma(d_i) = \left[1 + \left(\frac{d_i}{\theta_{\ln E}}\right)^{3/2}\right]^{-(2/3)}, \quad \gamma(d_i|d_j) = \left[1 + \left(\frac{d_i}{R_j}\right)^{3/2}\right]^{-(2/3)} \tag{11a}$$

$$R_j = \theta_{\ln E}\left\{\frac{\pi}{2} + \left(1 - \frac{\pi}{2}\right)\exp\left[-\left(\frac{d_j}{(\pi/2)\theta_{\ln E}}\right)^2\right]\right\} \tag{11b}$$

in which d_i = dimensions of the averaging region (in this case, $d_1 = W_f$ and $d_2 = H$). Predictions of $\sigma_{\ln \delta}$ using Eq. (9) are superimposed in Fig. 4 by lines. The agreement is remarkable. Intermediate cases show similar, if not better, agreement with the predictions.

An alternative physical interpretation of Eqs. (8) and (9) can be made by generalizing the relationship given by Eq. (3) to the following form:

$$\delta = \frac{\delta_{\det}\mu_E}{E_g} \tag{12}$$

where E_g = geometric mean of the elastic modulus values over the region of influence,

$$E_g = \exp\left(\frac{1}{W_f H}\int_0^H \int_0^{W_f} \ln E(x,y)\,dx\,dy\right) \tag{13}$$

Taking the logarithm of Eq. (12) and then computing its mean and variance leads to Eqs. (8)–Eq. (4*a*), and Eq. (9). The geometric mean is dominated by small values of the elastic modulus, which means that the total settlement is dominated by low elastic modulus regions that underly the footing, as would be expected.

Single Footing Example

Consider a single footing of width $W_f = 2.0$ m to be founded on a soil layer of depth 10.0 m that will support a load of P = 1,000 kN. Suppose also that samples taken at the site have allowed estimates of the elastic modulus mean and standard deviation at the site to be 40 and 40 MPa, respectively. In a similar way, test results on a regular array at this or at a similar site have resulted in an estimated scale of fluctuation of $\theta_{\ln E} = 3.0$ m. Assume also that the Poisson ratio is 0.25.

The results from the previous section can be used as follows to estimate the probability that the settlement under the footing will exceed 0.10 m.

1. A finite element analysis of the given problem with soil elastic modulus everywhere equal to $\mu_E = 40$ MPa gives a deterministic settlement of $\delta_{\det} = 0.03531$ m.
2. Compute the variance of the log-elastic modulus from Eq. (2*a*), $\sigma^2_{\ln E} = \ln(2) = 0.69315$, so that $\sigma_{\ln E} = 0.83256$.
3. Compute the mean of log settlement from Eq. (8), $\mu_{\ln \delta} = \ln(\delta_{\det}) + 0.5\sigma^2_{\ln E} = -3.3437 + 0.5(0.69315) = -2.9971$.
4. Compute the standard deviation of log settlement using Eqs. (9)–(11), $\sigma_{\ln \delta} = \sqrt{\gamma(W_f, H)}\sigma_{\ln E} = \sqrt{0.22458}(0.83256) = 0.39455$.

   As an aside, for $\mu_{\ln \delta} = -2.9971$ and 0.39455, the corresponding settlement mean and standard deviations are $\mu_\delta = \exp(\mu_{\ln \delta} + \frac{1}{2}\sigma^2_{\ln \delta}) = 0.0540$ m and $\sigma_\delta = \mu_\delta\sqrt{e^{\sigma^2_{\ln \delta}} - 1} = 0.0222$ m, respectively. A trial run of 5,000 realizations for this problem gives $m_\delta = 0.0562$ and $s_\delta = 0.0201$ for relative differences of 3.9 and 10.4%, respectively. The estimated relative standard error on m_δ is approximately 0.5% for 5,000 realizations.
5. Compute the desired probability using the lognormal distribution, $P(\delta > 0.10) = 1 - \Phi(1.7603) = 0.0392$, where $\Phi(\cdot)$ is the standard normal cumulative distribution.

A simulation run for this problem yielded 160 samples of 5,000 having settlement greater than 0.10 m. This gives a simulation based estimate of the above probability of 0.032.

Two Footing Case

Having established, with reasonable confidence, the distribution associated with settlement under a single footing founded on a soil layer, attention can now be turned to the more difficult problem of finding a suitable distribution with which to model differential settlement between footings. Analytically, if δ_1 is the settlement under the left footing shown in Fig. 1 and δ_2 is the settlement of the right footing, then according to the results of the "Single Footing Case," δ_1 and δ_2 will be jointly lognormally distributed random variables,

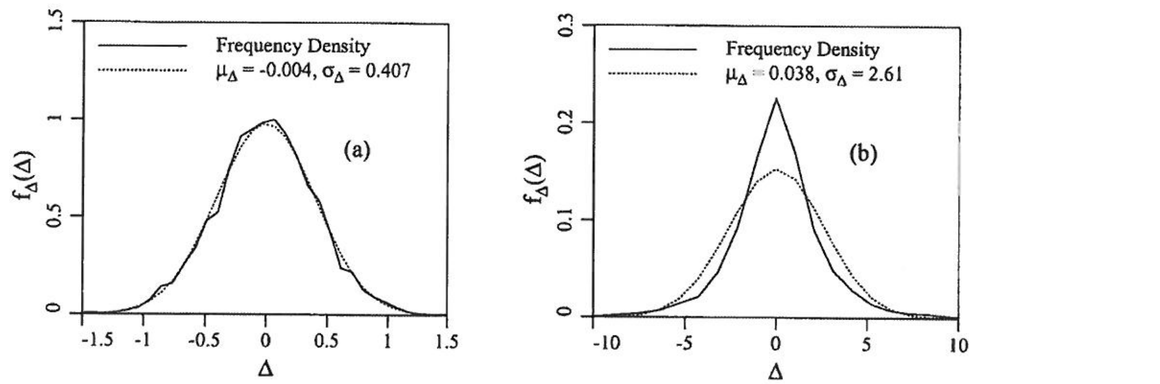

Fig. 5. Frequency density and fitted distribution for differential settlement under two footings of equal size with $\theta_{\ln E}/D=0.05$ in (a) and 1.0 in (b)

$$f_{\delta_1\delta_2}(x,y)=\frac{1}{2\pi\sigma^2_{\ln\delta}rxy}\exp\left(-\frac{1}{2r^2}(\Psi_x^2-\rho_{\ln\delta}\Psi_x\Psi_y+\Psi_y^2)\right),\quad x\geq 0,\quad y\geq 0 \tag{14}$$

where $\Psi_x=(\ln x-\mu_{\ln\delta})/\sigma_{\ln\delta}$, $\Psi_y=(\ln y-\mu_{\ln\delta})/\sigma_{\ln\delta}$, $r^2=1-\rho^2_{\ln\delta}$, and $\rho_{\ln\delta}$ is the correlation coefficient between the log settlement of the two footings. It is assumed in the above that δ_1 and δ_2 have the same mean and variance, which, for the symmetric conditions shown in Fig. 1(b), is a reasonable assumption.

If the differential settlement between footings is defined by $\Delta=\delta_1-\delta_2$ then the mean of Δ is zero if the elastic modulus field is statistically stationary. As indicated by Fenton (1999), stationarity is a mathematical assumption that in practice depends on the level of knowledge that one has about the site. If a trend in the effective elastic modulus is known to exist at the site, then the following results can be still be used by computing the deterministic differential settlement using the mean "trend" values in a deterministic analysis, then computing the probability of an *additional* differential settlement using the equations that follow. In this case the following probabilistic analysis would be performed with the trend removed from the elastic modulus field.

The exact distribution governing the differential settlement, assuming that Eq. (14) holds, is given by

$$f_\Delta(x)=\begin{cases}\displaystyle\int_0^\infty f_{\delta_1\delta_2}(x+y,y)\,dy & \text{if } x\geq 0\\ \displaystyle\int_{-x}^\infty f_{\delta_1\delta_2}(x+y,y)\,dy & \text{if } x<0\end{cases} \tag{15}$$

which can be evaluated numerically, but which has no analytical solution so far as the authors are aware. This distribution is the subject of continuing research. In the following a normal approximation to the distribution of Δ will be investigated.

Fig. 5 shows two typical frequency density plots of differential settlement between two footings of equal size ($W_f/D=0.1$) with superimposed fitted normal distributions, where the fit was obtained by directly estimating the mean and standard deviations from the simulation. The normal distribution appears to be a reasonable fit in Fig. 5(a). Since a lognormal distribution begins to look very much like a normal distribution when $\sigma_{\ln\delta}/\mu_{\ln\delta}$ is small, then for small $\sigma_{\ln\delta}/\mu_{\ln\delta}$ both δ_1 and δ_2 will be approximately normally distributed. For small $\theta_{\ln E}$, therefore, since this leads to small $\sigma_{\ln\delta}$, the difference $(\delta_1-\delta_2)$ will be very nearly normally distributed, as seen in Fig. 5(a). For larger scales of fluctuation (and/or smaller D), the histogram of differential settlements becomes narrower than the normal distribution, as seen in Fig. 5(b). What is less obvious in Fig. 5(b) is that the histogram has much longer tails than those predicted by the normal distribution. These long tails lead to a variance estimate which is larger than that dictated by the center region of the histogram. Although the variance could be artificially reduced so that the fit is better near the origin, the result would be a significant underestimate of the probability of large differential settlements. This issue will be discussed at more length, shortly, when differential settlement probabilities are considered. Both plots are for $\sigma_E/\mu_E=1.0$ and are typical of other coefficients of variation (COV).

Assuming that $\Delta=\delta_1-\delta_2$ is at least approximately normally distributed, and that δ_1 and δ_2 are identically and lognormally distributed with correlation coefficient ρ_δ, then differential settlement has parameters of

$$\mu_\Delta=0,\quad \sigma_\Delta^2=2(1-\rho_\delta)\sigma_\delta^2 \tag{16}$$

Note that when $\theta_{\ln E}$ approaches zero, the settlement variance σ_δ^2 also approaches zero. When $\theta_{\ln E}$ becomes very large, the correlation coefficient between settlements under the two footings approaches one. Thus, Eqs. (16) are in agreement with the expectation that differential settlements will disappear for both very small and very large values of $\theta_{\ln E}$.

Since local averaging of the log–elastic modulus field under the footing was found to be useful in predicting the variance of log settlement, it seems reasonable to suggest that the covariance between log settlements under a pair of footings will be predicted well by the covariance between local averages of the log–elastic modulus field under each footing. For footings of equal size, the covariance between local averages of the log–elastic modulus field under two footings separated by distance D is given by

$$C_{\ln\delta}=\frac{\sigma^2_{\ln E}}{W_f^2H^2}\int_0^H\int_0^{W_f}\int_0^H\int_D^{D+W_f}\rho_{\ln E}(x_1-x_2,y_1-y_2)\,dx_2\,dy_2\,dx_1\,dy_1 \tag{17}$$

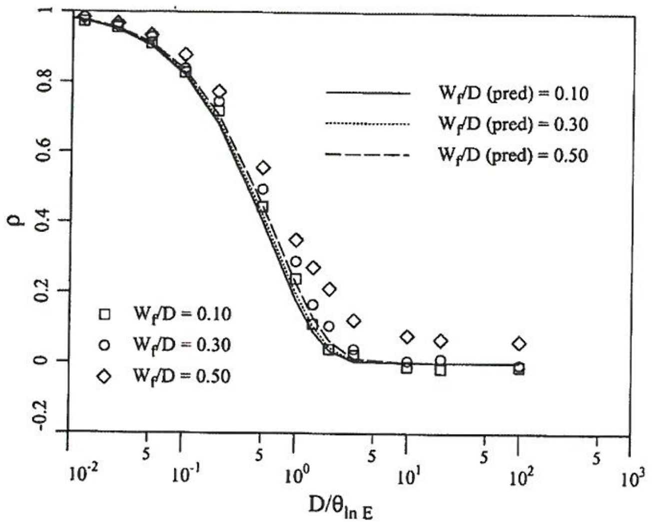

Fig. 6. Predicted and sample correlation coefficients between footing settlements for various relative separation distances between the footings and for $\sigma_E/\mu_E = 1$

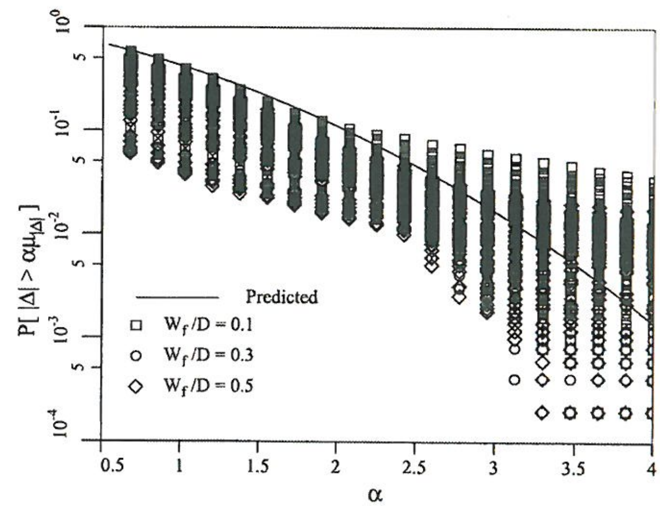

Fig. 8. Simulation based estimates of $P(|\Delta|>\alpha\mu_{|\Delta|})$ for all cases compared to that predicted by Eqs. (19) and (20)

which can be evaluated reasonably accurately using a three-point Gaussian quadrature if $\rho_{\ln E}$ is smooth, like it is in Eq. (1). See the Appendix for details.

The correlation coefficient between settlements can now be obtained by transforming back from log space,

$$\rho_\delta = \frac{\exp(C_{\ln \delta}) - 1}{\exp(\sigma^2_{\ln \delta}) - 1} \tag{18}$$

where $\sigma_{\ln \delta}$ is given by Eq. (9). The agreement between the correlation coefficient predicted by Eq. (18) and the correlation coefficient estimated from the simulations is shown in Fig. 6. In order to extend the curve up to correlation coefficients close to one, four additional scales of fluctuation were considered (now a total of 15 scales is considered), all the way up to $\theta_{\ln E} = 200$. The general trends between prediction and simulation results are the same although the simulations show more correlation for larger footing widths than was predicted by the above theory. For larger footing widths there is a physical interaction between the footings, where the stress under one footing begins to add to the stress under the other footing, so the true correlation is expected to be larger than that predicted purely on the basis of local averaging.

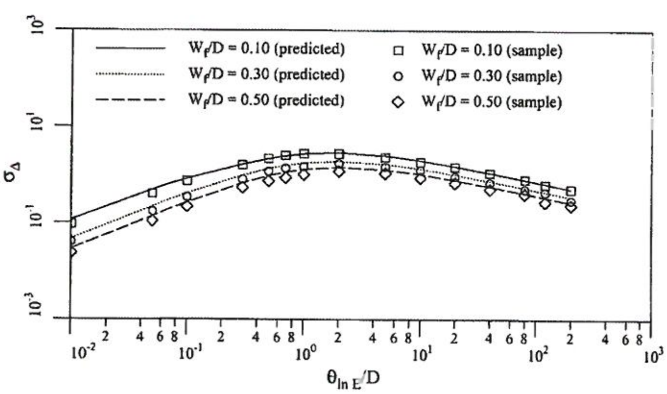

Fig. 7. Predicted and sample standard deviations of differential settlement for $\sigma_E/\mu_E = 1$

The correlation predicted by Eq. (18), however, is at least conservative in that smaller correlations lead to larger probabilities of differential settlement.

Fig. 7 shows the estimated standard deviation of Δ as a function of $\theta_{\ln E}/D$ for three footing widths and for $\sigma_E/\mu_E = 1$. Other values of σ_E/μ_E are similar. Superimposed on the sample standard deviations (shown as symbols) are the predicted standard deviations using Eq. (16) (shown by solid or dashed lines). The agreement is very good over the entire range of scales.

To test the capability of the assumed distribution to accurately estimate probabilities, the probability that the absolute value of Δ exceeds some threshold is compared to empirical probabilities derived from simulation. For generality, thresholds of $\alpha\mu_{|\Delta|}$ will be used, where $\mu_{|\Delta|}$ is the mean absolute differential settlement, which, if Δ is normally distributed, is given by

$$\mu_{|\Delta|} = \sqrt{\frac{2}{\pi}}\sigma_\Delta \tag{19}$$

Note that this relationship indicates that the mean absolute differential settlement is directly related to the standard deviation of Δ, which in turn is related to the correlation between the elastic moduli under the footings and the variability of the elastic moduli. In particular, this means that the mean absolute differential settlement is a function of just δ_{det}, $\sigma^2_{\ln E}$, and $\theta_{\ln E}$, increasing with δ_{det} and $\sigma^2_{\ln E}$, and reaching a maximum when $\theta_{\ln E}/D$ is near 1.0 (see Fig. 7).

Fig. 8 shows a plot of the probability

$$P(|\Delta|>\alpha\mu_{|\Delta|}) = 2\Phi\left(\frac{-\alpha\mu_{|\Delta|} - \mu_\Delta}{\sigma_\Delta}\right) = 2\Phi\left(-\alpha\sqrt{\frac{2}{\pi}}\right) \tag{20}$$

for α varying from 0.5 to 4.0, shown by a solid line. The symbols show empirical probabilities that $|\Delta|$ is greater than $\alpha\mu_{|\Delta|}$ obtained via simulation (5,000 realizations) for the three footing widths, 15 scales of fluctuation, and five elastic modulus COVs (thus, each column of symbols contains 225 points, 75 for each footing width).

It can be seen that the predicted probability is in very good agreement with average simulation results for large differential settlements, while being conservative (higher probabilities of exceedance) at lower differential settlements.

The normal distribution is considered to be a reasonable approximation for differential settlement in at least two ways: first of all it is a *conservative* approximation, that is, it *overestimates* the probability of differential settlement for the bulk of the data. Second, it is a *consistent* approximation in that it improves as the scale of fluctuation decreases, by virtue of the fact that the difference, $\delta_1 - \delta_2$, approaches a normal distribution. Since the estimated scale of fluctuation decreases as a site is more thoroughly investigated and trends are removed, the normal distribution then becomes more accurate as more is known about the site. Conversely, if little is known about the site, the normal distribution properly reflects inherent uncertainty by generally predicting larger differential settlements.

Two Footing Example

Consider two footings each of width $W_f = 2.0$ m separated by $D = 10$ m center to center. They are founded on a soil layer of 10 m depth and each supports a load of $P = 1{,}000$ kN. Assume also that $\mu_E = 40$ MPa, $\sigma_E = 40$ MPa, $\theta_{\ln E} = 1.0$ m, and the Poisson ratio is 0.25. If the footings support a floor or beam that is not attached to elements likely to be damaged by large deflection, then differential settlement is limited to $D/360 = 2.8$ cm. What is the probability that $|\Delta| > 2.8$ cm? The approach to the solution is outlined as follows (see the previous example for some of the earlier details; note, however, that the scale of fluctuation has changed in this example).

1. A deterministic finite element analysis of this problem gives $\delta_{det} = 0.03578$ under each footing (this number is only slightly different than that found in the single footing case due to interactions between the two footings). For $\sigma^2_{\ln E} = 0.69315$, the log-settlement statistics under either footing are $\mu_{\ln \delta} = \ln(\delta_{det}) + \frac{1}{2}\sigma^2_{\ln E} = -2.9838$ and $\sigma_{\ln \delta} = \sqrt{\gamma(W_f, H)\sigma^2_{\ln E}} = \sqrt{(0.055776)(0.69315)} = 0.19662$.
2. To calculate $C_{\ln \delta}$, a short program written to implement the approach presented in the Appendix gives $C_{\ln \delta} = 3.1356 \times 10^{-7}$.
3. In terms of actual settlement under each footing, the mean, standard deviation, and correlation coefficient are $\mu_\delta = \exp(\mu_{\ln \delta} + \frac{1}{2}\sigma^2_{\ln \delta}) = 0.051587$, $\sigma_\delta = \mu_\delta \sqrt{e^{\sigma^2_{\ln \delta}} - 1} = 0.010242$, and $\rho_\delta = (e^{C_{\ln \delta}} - 1)/(e^{\sigma^2_{\ln \delta}} - 1) = 7.9547 \times 10^{-6}$, respectively. A 5000 realization simulation run for this problem gave estimates of settlement mean and standard deviation of 0.0530 and 0.0081, respectively, and an estimated correlation coefficient of -0.014 (where the negative correlation coefficient estimate is clearly due to bias in the classical estimator, see Fenton (1999a) for a discussion of this issue).
4. The differential settlement, Δ, has parameters $\mu_\Delta = 0$ and $\sigma^2_\Delta = 2(1 - 7.9547 \times 10^{-6})(0.010242)^2 = 0.0002098$ and the mean absolute differential settlement in this case is predicted to be $\mu_{|\Delta|} = \sqrt{2(0.0002098)/\pi} = 0.011$. The simulation run estimated the mean absolute differential settlement to be 0.009.
5. The desired probability is predicted to be $P(|\Delta| > 0.028) = 2\Phi(-0.028/\sqrt{0.0002098}) = 2\Phi(-1.933) = 0.0532$. The empirical estimate of this probability from the simulation run is 0.0204.

The normal distribution approximation to Δ somewhat overestimates the probability that $|\Delta|$ will exceed $D/360$. This is, therefore, a conservative estimate. From a design point of view, if the probability derived in step 5 is deemed unacceptable, one solution is to widen the footing. This will result in a rapid decrease in $P(|\Delta| > 0.028)$ in the case given above. In particular, if W_f is increased to 3.0 m, the empirical estimate of $P(|\Delta| > 0.028)$ reduces by more than a factor of 10 to 0.0016.

The distribution of absolute differential settlement is, however, highly dependent on the scale of fluctuation, primarily through the calculation of $\sigma_{\ln \delta}$. As discussed earlier, the scale of fluctuation is a quantity that is very difficult to estimate and one that is poorly understood for real soils, particularly in the horizontal direction. If $\theta_{\ln E}$ is increased to 10.0 m in the above example, the empirical estimate of $P(|\Delta| > 0.028)$ increases dramatically to 0.44. From a design point of view, the problem is compounded since, for such a large scale of fluctuation, $P(|\Delta| > 0.028)$ now decreases very slowly as the footing width is increased (holding the load constant). For example, a footing width of 5.0 m, with $\theta_{\ln E} = 10.0$ m, has $P(|\Delta| > 0.028) = 0.21$. Thus, establishing the scale of fluctuation in the horizontal direction is a critical issue in differential settlement limit state design, and one which needs much more work.

Conclusions

On the basis of this simulation study, the following observations can be made.

The settlement under a footing founded on a spatially random elastic modulus field of finite depth overlying bedrock is represented well by a lognormal distribution with parameters $\mu_{\ln \delta}$ and $\sigma^2_{\ln \delta}$ if E is also lognormally distributed. The first parameter, $\mu_{\ln \delta}$, is dependent on the mean and variance of the underlying log–elastic modulus field and may be closely approximated by considering limiting values of $\theta_{\ln E}$. One of the primary contributions of this paper is the observation that the second parameter, $\sigma^2_{\ln \delta}$, is approximated very well by the variance of a local average of the log–elastic modulus field in the region directly under the footing. This conclusion is motivated by the observation that settlement is inversely proportional to the geometric mean of the elastic modulus field and gives the prediction of $\sigma^2_{\ln \delta}$ some generality that can be extended beyond the actual range of simulation results considered herein. For very deep soil layers that underly the footing, it is recommended that the depth of the averaging region not exceed about 10 times its width due to reduction of stress with depth. Once the statistics of the settlement, $\mu_{\ln \delta}$ and $\sigma^2_{\ln \delta}$, have been computed, using Eqs. (8)–(10), the estimation of probabilities associated with settlement involves little more than referring to a standard normal distribution table.

The differential settlement follows a more complicated distribution than that of settlement itself [see Eq. (15)]. This is seen also in the differential settlement histograms which tend to be quite erratic, with their narrow peaks and long tails, particularly at large $\theta_{\ln E}/D$ ratios. Although the difference between two lognormally distributed random variables is not normally distributed, the normal approximation has nevertheless been found to be reasonable, giving conservative estimates of probability over the bulk of the distribution. For a more accurate estimation of probability relating to differential settlement, where it can be assumed that footing settlement is lognormally distributed, Eq. (15) should be numerically integrated. However, both the simplified normal approximation and the numerical integration of Eq. (15) depend upon a reasonable estimate of the covariance between footing settlements. Another important contribution of this paper is that this covariance is closely (and conservatively) estimated using the covariance between local averages of the log–elastic modulus field under the two footings. Discrepancies between the covariance predicted on this basis and in simulation results are due to interactions between the footings when they are closely spaced—such interactions lead to higher correlations than predicted by local average theory, which leads to smaller differential settlements than predicted in practice. This is conservative. The recommendations regarding the maximum averaging depth made for the single footing case would also apply here.

Example calculations were provided above to illustrate how the findings of the paper may be used. These calculations are reasonably simple for calculations by hand [except for the numerical integration in Eq. (17)] and are also easily programmed. They allow probability estimates with regard to settlement and differential settlement, which in turn allows the estimation of the risk associated with this particular limit state for a structure.

A critical issue in the risk assessment of differential settlement that is unresolved is the estimation of the scale of fluctuation, $\theta_{\ln E}$, since it significantly affects the differential settlement distribution. A tentative recommendation is to use a scale of fluctuation which is some fraction of the distance between footings, say, $D/10$. There is, at this time, little justification for such a recommendation, aside from the fact that scales that approach D or bigger yield differential settlements which are felt to be unrealistic in a practical sense and, for example, not observed in the work of D'Appolonia et al. (1968). Research into this problem is ongoing.

Appendix

The numerical computation of Eq. (17) can be accomplished reasonably accurately and efficiently using a three-point Gauss integration scheme. The four-fold integration can be written as a two-fold sum if the correlation function is quadrant symmetric [i.e., $\rho_{\ln E}(x,y)=\rho_{\ln E}(-x,y)=\rho_{\ln E}(x,-y)=\rho_{\ln E}(-x,-y)$], as is Eq. (1),

$$C_{\ln\delta}=\frac{\sigma^2_{\ln E}}{16}\sum_{i=1}^{3} w_i[(1+z_i)P_i+(1-z_i)Q_i] \tag{21}$$

where

$$P_i=\sum_{j=1}^{3} w_j[(1+z_j)\rho_{\ln E}(x_{i1},y_{j1})+(1-z_j)\rho_{\ln E}(x_{i1},y_{j2})] \tag{22a}$$

$$Q_i=\sum_{j=1}^{3} w_j[(1+z_j)\rho_{\ln E}(x_{i2},y_{j1})+(1-z_j)\rho_{\ln E}(x_{i2},y_{j2})] \tag{22b}$$

$$x_{i1}=\tfrac{1}{2}(z_i-1)W_f+D \tag{22c}$$

$$x_{i2}=\tfrac{1}{2}(z_i+1)W_f+D \tag{22d}$$

$$y_{j1}=\tfrac{1}{2}(z_j-1)H \tag{22e}$$

$$y_{j2}=\tfrac{1}{2}(z_j+1)H \tag{22f}$$

and where the weights, w_i, and Gauss points, z_i, are as follows:

$$w_1=5/9 \quad w_2=8/9 \quad w_3=5/9 \tag{23a}$$

$$z_1=-\sqrt{3/5} \quad z_2=0 \quad z_3=\sqrt{3/5}. \tag{23b}$$

Notation

The following symbols are used in this paper:

- $C_{\ln\delta}$ = covariance between log settlements under the two footings;
- D = center-to-center distance between footings;
- E = elastic modulus;
- E_g = elastic modulus geometric mean;
- f_Δ = differential settlement probability density function;
- f_δ = settlement probability density function;
- $f_{\delta_1\delta_2}$ = joint settlement probability density function;
- $G(\mathbf{x})$ = standard normal (Gaussian) random field
- H = overall depth of the soil layer;
- L = overall width of the soil model;
- $m_{\ln\delta}$ = estimated mean of log settlement via simulation;
- m_δ = estimated mean of footing settlement via simulation;
- P = applied footing load;
- R_j = effective scale of fluctuation used in γ calculations;
- s_δ = estimated standard deviation of footing settlement via simulation;
- $s_{\ln\delta}$ = estimated standard deviation of log settlement via simulation;
- W_f = footing width;
- $\mathbf{x}$ = spatial coordinate or position;
- γ = variance function (reduction in variance due to local averaging);
- Δ = differential settlement between footings;
- δ = footing settlement, positive downward;
- δ_{det} = footing settlement when $E=\mu_E$ everywhere;
- $\theta_{\ln E}$ = isotropic scale of fluctuation of the log–elastic modulus field;
- $\theta_{\ln E_h}$ = horizontal scale of fluctuation of the log–elastic modulus field;
- $\theta_{\ln E_v}$ = vertical scale of fluctuation of the log–elastic modulus field;
- μ_E = mean elastic modulus;
- $\mu_{\ln E}$ = mean of the log–elastic modulus;
- $\mu_{\ln\delta}$ = mean of log settlement;
- μ_Δ = mean differential footing settlement;
- $\mu_{|\Delta|}$ = mean absolute differential footing settlement;
- μ_δ = mean footing settlement;
- $\rho_{\ln E}$ = correlation coefficient between $\ln(E)$ at two points;
- $\rho_{\ln\delta}$ = correlation coefficient between log-footing settlements;
- ρ_δ = correlation coefficient between footing settlements;
- σ_E = standard deviation of the elastic modulus;
- $\sigma_{\ln E}$ = standard deviation of the log–elastic modulus;
- $\sigma_{\ln\delta}$ = standard deviation of log settlement;
- σ_Δ = standard deviation of differential settlement;
- σ_δ = standard deviation of footing settlement;
- τ = lag distance, equal to $|\boldsymbol{\tau}|$;
- $\boldsymbol{\tau}$ = spatial lag vector;
- ν = Poisson ratio; and
- Φ = standard normal cumulative distribution function.

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