Semi-Analytical Integration of the 8-Node Plane Element Stiffness Matrix Using Symbolic Computation

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The semi-analytical integration of an 8-node plane strain finite element stiffness matrix is presented in this work. The element is assumed to be super-parametric, having straight sides. Before carrying out the integration, the integral expressions are classified into several groups, thus avoiding duplication of calculations. Symbolic manipulation and integration is used to obtain the basic formulae to evaluate the stiffness matrix. Then, the resulting expressions are postprocessed, optimized, and simplified in order to reduce the computation time. *Maple* symbolic-manipulation software was used to generate the closed expressions and to develop the corresponding Fortran code. Comparisons between semi-analytical integration and numerical integration were made. It was demonstrated that semi-analytical integration required less CPU time than conventional numerical integration (using Gaussian-Legendre quadrature) to obtain the stiffness matrix. © 2005 Wiley Periodicals, Inc. Numer Methods Partial Differential Eq 22: 296–316, 2006

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I. INTRODUCTION

The finite element method and the boundary element method are techniques that require significant amounts of integration. This is usually achieved using numerical integration, which

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Fortran 95 subroutines developed using both semi-analytical and exact methodologies for a range of commonly encountered finite elements can be downloaded from the author's web site at: www.mines.edu/fs_home/vgriffit/analytical

can be a time-consuming process. In addition, numerical integration can also lead to inaccuracies in very distorted element geometries.

For some time now, researchers have been interested in both semi-analytical and analytical methods of integration of the stiffness, mass, and other element matrices required by these methods. Use of symbolic manipulation and integration of these matrices is also not new. Since the 1960s, symbolic computation techniques have been used to deal with very complicated expressions, such as those that can arise in analysis of problems in engineering and the physical sciences. The need to integrate complex and large expressions has motivated the development of symbolic software or Computer Algebra Systems (CAS) such as *Maple, Mathematica, Macsyma,* and *Derive.*

During the last 20 years, the development of computational codes based on symbolic manipulation has increased substantially in the mechanical analysis of continuous media. For example, the reader can refer to the works of Korncoff and Fenves [1], Korncoff [2], Noor and Andersen [3], Andersen and Noor [4], Jensen, Niordson [5], and Cardoso [6] among others.

For instance, the applications related to nonlinear dynamic analysis of plates and shells require very large CPU time to numerically integrate the stiffness matrices. Zienkiewicz et al. [7] present a semi-analytical integration of plates and shells, and report a reduction in CPU time of nearly 50%. Rengarajan et al. [8] analytically integrated a hybrid finite element for shell analysis and carried out comparisons between analytical and numerical integration. One of the advantages of hybrid finite elements is that the determinant of the Jacobian matrix does not appear in the denominator of the stiffness matrix expression as it does in conventional displacement based finite element formulations. Vlachoutsis [9] proposed analytical integration of a degenerated shell element, by decomposing the Jacobian matrix into two terms: those representing variables belonging to the middle surface of the shell and those related to the outside regions. Results indicated CPU saving of 9% and also demonstrated the greater accuracy of the analytical formulations.

Linear approximations for the Jacobian matrix of a shell element with thickness were treated using symbolic integration by Milford and Schnobrich [10]. Mahe et al. [11] considered a shell element, by integrating across the element thickness. Ledvinka [12] also analytically integrated a hybrid four-node element, while Crisfield [13] applied symbolic manipulation in the analysis of arches and shells.

In the field of the boundary element method (BEM) researchers have made a significant effort in the symbolic integration of the singular kernel that frequently appears in boundary integrals. The work of Gray [14] shows how to deal with hypersingular integrals, yielded by nontrivial Green functions. Chen et al. [15] presented the gradients computation at internal points in a 2D domain. They applied symbolic computation to evaluate singular integrals appearing in potential problems discretized by the BEM. Ye et al. [16] also discussed symbolic integration applied to BEM.

Returning to finite elements, Yagawa et al. [17] demonstrated CPU time savings of 15% using a combined technique involving both numerical and analytical integration of stiffness in plane elasticity. Bardel [18] showed that higher order polynomial expressions appearing in p-adaptive FEM can be obtained using symbolic integration, giving large CPU time savings. Mizukami [19] derived explicit integration formulas for a rectangular plane finite element with straight sides. Kikuchi [20] used the *Reduce* package to obtain explicit formulas for an isoparametric four-node finite element, showing accurate results for a distorted element. Rathod [21] presented analytical integration formulas for a four-node isoparametric finite element and showed that all the integration formulas could be obtained on the basis of four simple integrals.

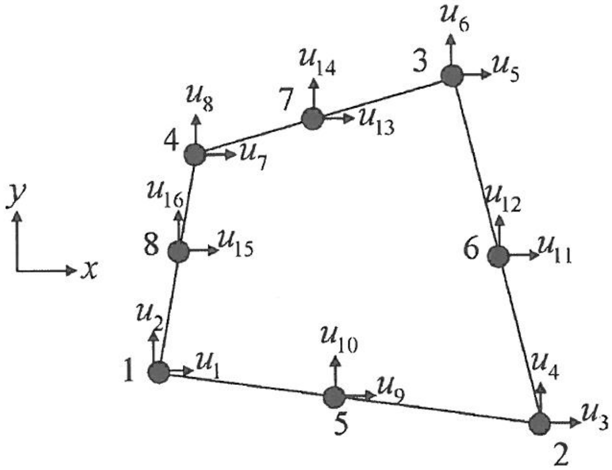

FIG. 1. Element node and freedom numbering.

In 1994, Griffiths [22] presented a general methodology based on coordinate transformations, to obtain the semi-analytical closed expression for the stiffness matrix of a four-node elastic plane finite element. This methodology was extrapolated to eight-node quadrilaterals by Cardoso [23]. The techniques developed by these authors were based on the semi-analytical integration of the expressions using 2 × 2 Gauss-Legendre integration points, leading to substantial CPU time savings. Similar formulations for four-node elements applied to steady seepage analysis have also been presented by Smith and Griffiths [23]. Using a different approach to Griffiths, Videla et al. [24] used the *Derive* package to generate closed-form analytical expressions for the integration of a four-node elastic plane strain finite element and showed important CPU savings. The formulas yielded by the symbolic software were then optimized to avoid unnecessary calculations.

In this article, a generalization of the Griffiths [22] methodology is presented in relation to an eight-node elastic plane strain finite element with straight sides (super-parametric). The technique allows easy computation of the stiffness matrix and substantially improves the CPU times as compared to conventional numerical integration. Accuracy and CPU times comparisons are reported, and the influence of geometric element distortion is considered.

Only eight-noded-plane finite elements with straight sides are considered in this work; however, the authors are working to develop the semi-analytical and analytical formulae for general curved-sides eight-noded elements. These expressions appear to be much more complex.

2. FORMULATION

Figure 1 shows the finite element configuration and node numbering. Two degrees of freedom are assumed at each node. A brief summary of the FEM formulation is now needed to set up the variables and constants meaning. The constitutive relation is

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}, \tag{1}$$

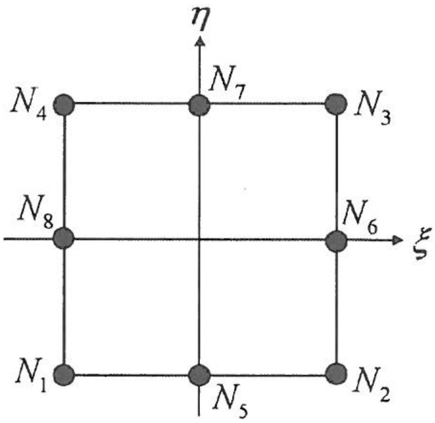

FIG. 2. Element in local coordinate space.

where

$$\boldsymbol{\sigma} = [\sigma_x \quad \sigma_y \quad \sigma_z]^t \tag{2}$$

$$\boldsymbol{\varepsilon} = [\varepsilon_x \quad \varepsilon_y \quad \varepsilon_z]^t. \tag{3}$$

For an isotropic material, the elastic matrix is

$$\mathbf{D} = \begin{bmatrix} E_1 & E_2 & 0 \\ E_2 & E_1 & 0 \\ 0 & 0 & G \end{bmatrix}, \tag{4}$$

being the shear modulus $G = E/[2(1 + v)]$.
For plane stress we have

$$E_1 = \frac{E}{1 - v^2}, \qquad E_2 = vE_1 \tag{5}$$

and for plane deformation

$$E_1 = \frac{E(1 - v)}{(1 + v)(1 - 2v)} \qquad E_2 = \frac{vE_1}{(1 - v)}. \tag{6}$$

After the well-known coordinate transformation, the element in the Gauss-plane is shown in Fig. 2. The shape functions to interpolate the geometry and the displacements fields are

$$N_1 = \frac{-1}{4}(1 - \eta)(1 - \xi)(1 + \eta + \xi) \qquad N_2 = \frac{-1}{4}(1 - \eta)(1 + \xi)(1 - \xi + \eta)$$

$$N_3 = \frac{-1}{4}(1 + \eta)(1 + \xi)(1 - \xi - \eta) \qquad N_4 = \frac{-1}{4}(1 + \eta)(1 - \xi)(1 + \xi - \eta)$$

$$N_5 = \tfrac{1}{2}(1-\eta)(1-\xi^2) \qquad N_6 = \tfrac{1}{2}(1-\eta^2)(1+\xi)$$

$$N_7 = \tfrac{1}{2}(1+\eta)(1-\xi^2) \qquad N_8 = \tfrac{1}{2}(1-\eta^2)(1-\xi), \tag{7}$$

where the coordinate transformation between the Cartesian plane and the Gauss-plane is given by

$$x = \sum_{i=1}^{8} N_i(\xi, \eta)x_i, \qquad y = \sum_{i=1}^{8} N_i(\xi, \eta)y_i. \tag{8}$$

The stiffness matrix terms of an element are obtained as follows:

$$K_{ij} = \int_A B_i^t D B_j dxdy, \tag{9}$$

being the displacement-deformation matrix B:

$$B_i = \begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 \\ 0 & \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} \end{bmatrix}. \tag{10}$$

The coordinate transformation leads to

$$K_{ij} = \int_{-1}^{1}\int_{-1}^{1} B_i^t(\xi, \eta) D B(\xi, \eta) |\mathbf{J}| d\xi d\eta, \tag{11}$$

where $|\mathbf{J}|$ is the determinant of the Jacobian matrix:

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}. \tag{12}$$

By using the notation: $|\mathbf{J}| = H$, $T_i = \partial N_i/\partial x$, and $S_i = \partial N_i/\partial y$ one obtains

$$\mathbf{B}_i^t \mathbf{D} \mathbf{B}_j H = \begin{bmatrix} E_1 T_i T_j + G S_i S_j & E_2 T_i S_j + G S_i T_j \\ E_2 S_i T_j + G T_i S_j & E_1 S_i S_j + G T_i T_j \end{bmatrix} \frac{1}{H}. \tag{13}$$

As a consequence, the stiffness matrix terms of an 8-noded-plane finite element in the local coordinate space are giving by

$$\begin{bmatrix} \int_{-1}^{1}\int_{-1}^{1}\frac{1}{H}(E_1T_iT_j + GS_iS_j)d\xi d\eta & \int_{-1}^{1}\int_{-1}^{1}\frac{1}{H}(E_2T_iS_j + GS_iT_j)d\xi d\eta \\ \int_{-1}^{1}\int_{-1}^{1}\frac{1}{H}(E_2S_iT_j + GT_iS_j)d\xi d\eta & \int_{-1}^{1}\int_{-1}^{1}\frac{1}{H}(E_1S_iS_j + GT_iT_j)d\xi d\eta \end{bmatrix}. \tag{14}$$

2.1. "Parent" Terms of the Stiffness Matrix

The stiffness matrix terms will be obtained using a 2×2 Gauss-Legendre integration rule. Because the matrix is symmetric and order 16×16, only 136 elements are needed. These elements are classified into 10 groups (A, B, C, D, E, F, G, H, I, J), according to the *adjacency* (which is a key-factor in this methodology from Griffiths [22]) between the degrees of freedom (DOF) of the element nodes, as shown in Table I. Figure 3 displays the various adjacency relationships between the nodes. Parallel and orthogonal freedoms are treated separately as shown in Table I.

All the stiffness matrix terms are of the form

$$K_{ij} = 9\left[\frac{A_1(E^*s_1 + Gs_2) + f_1(E^*s_3 + Gs_4)}{A_1^2 - 3f_1^2} + \frac{A_1(E^*t_1 + Gt_2) + f_2(E^*t_3 + Gt_4)}{A_1^2 - 3f_2^2}\right], \tag{15}$$

where $E^* = E_1$ for groups A, C, E, G and I; $E^* = E_2$ for groups B, D, F, H and J, whereas the functions A_1, f_1, and f_2 are as follows:

$$A_1 := \tfrac{9}{8}\{(y_2 - y_4)(x_1 - x_3) + (y_3 - y_1)(x_2 - x_4)\} \tag{16}$$

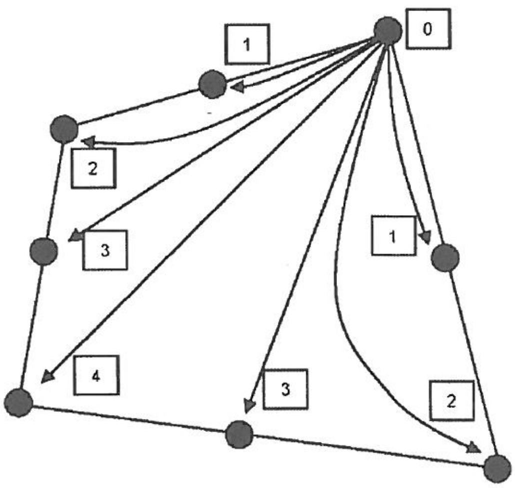

FIG. 3. Adjacency around a typical node.

TABLE I. Stiffness matrix classification into 10 groups.

Group	Terms	Description	Adjacency
A	$K_{1,1}$, $K_{2,2}$, $K_{3,3}$, $K_{4,4}$, $K_{5,5}$, $K_{6,6}$, $K_{7,7}$, $K_{8,8}$, $K_{9,9}$, $K_{10,10}$, $K_{11,11}$, $K_{12,12}$, $K_{13,13}$, $K_{14,14}$, $K_{15,15}$, $K_{16,16}$	Parallel DOF in the same node	0
B	$K_{1,2}$, $K_{3,4}$, $K_{5,6}$, $K_{7,8}$, $K_{9,10}$, $K_{11,12}$, $K_{13,14}$, $K_{15,16}$	Orthogonal DOF at the same node	0
C	$K_{1,3}$, $K_{3,5}$, $K_{5,7}$, $K_{1,7}$, $K_{9,11}$, $K_{11,13}$, $K_{13,15}$, $K_{9,15}$, $K_{2,4}$, $K_{4,6}$, $K_{6,8}$, $K_{2,8}$, $K_{10,12}$, $K_{12,14}$, $K_{14,16}$, $K_{10,16}$	Parallel DOF of two nodes with one intermediate node between them	2
D	$K_{2,3}$, $K_{3,6}$, $K_{6,7}$, $K_{2,7}$, $K_{10,11}$, $K_{11,14}$, $K_{14,15}$, $K_{10,15}$, $K_{4,5}$, $K_{5,8}$, $K_{1,8}$, $K_{1,4}$, $K_{12,13}$, $K_{13,16}$, $K_{9,16}$, $K_{9,12}$	Orthogonal DOF of two nodes with one intermediate node between them	2
E	$K_{1,5}$, $K_{9,13}$, $K_{3,7}$, $K_{11,15}$, $K_{2,6}$, $K_{4,8}$, $K_{10,14}$, $K_{12,16}$	Parallel DOF at opposite nodes	4
F	$K_{1,6}$, $K_{9,14}$, $K_{3,8}$, $K_{11,16}$, $K_{2,5}$, $K_{10,13}$, $K_{4,7}$, $K_{12,15}$	Orthogonal DOF at opposite nodes	4
G	$K_{1,9}$, $K_{3,9}$, $K_{3,11}$, $K_{5,11}$, $K_{5,13}$, $K_{7,13}$, $K_{7,15}$, $K_{1,15}$, $K_{2,10}$, $K_{4,10}$, $K_{4,12}$, $K_{6,12}$, $K_{6,14}$, $K_{8,14}$, $K_{8,16}$, $K_{2,16}$	Parallel DOF at adjacent nodes	1
H	$K_{1,10}$, $K_{3,10}$, $K_{3,12}$, $K_{5,12}$, $K_{5,14}$, $K_{7,14}$, $K_{7,16}$, $K_{1,16}$, $K_{2,9}$, $K_{4,9}$, $K_{4,11}$, $K_{6,11}$, $K_{6,13}$, $K_{8,13}$, $K_{8,15}$, $K_{2,15}$	Orthogonal DOF at adjacent nodes	1
I	$K_{1,11}$, $K_{7,11}$, $K_{7,9}$, $K_{5,9}$, $K_{5,15}$, $K_{3,15}$, $K_{3,13}$, $K_{1,13}$, $K_{2,12}$, $K_{8,12}$, $K_{8,10}$, $K_{6,10}$, $K_{6,16}$, $K_{4,16}$, $K_{4,14}$, $K_{2,14}$	Parallel DOF at two nodes with two intermediate nodes between them	3
J	$K_{2,11}$, $K_{8,11}$, $K_{8,9}$, $K_{6,9}$, $K_{6,15}$, $K_{4,15}$, $K_{4,13}$, $K_{2,13}$, $K_{1,12}$, $K_{7,12}$, $K_{7,10}$, $K_{5,10}$, $K_{5,16}$, $K_{3,16}$, $K_{3,14}$, $K_{1,14}$	Orthogonal DOF at two nodes with two intermediate nodes between them	3

$$
\begin{aligned}
f_1 :=& \tfrac{9}{8}(y_4 - y_2)(x_1 + x_3) + \tfrac{1}{4}\{y_4x_4 - x_2y_2\} + \tfrac{1}{2}\{y_4x_2 - y_2x_4\} + \tfrac{5}{8}\{(y_1 - y_4)(x_1 + x_2) + (y_3 - y_4) \\
&\times (x_2 + x_3) + (y_2 - y_3)(x_4 + x_3) + (y_2 - y_1)(x_1 + x_4)\} + \tfrac{1}{8}\{(y_2 - y_1)(x_2 + x_3) + (y_1 - y_4) \\
&\times (x_4 + x_3) + (y_3 - y_4)(x_1 + x_4) + (x_4 - x_2)(y_3 + y_1) + (y_2 - y_3)(x_1 + x_2)\}
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
f_2 :=& \tfrac{9}{8}(x_4 + x_2)(y_1 - y_3) + \tfrac{1}{4}\{y_3x_3 - x_1y_1\} + \tfrac{1}{2}\{y_3x_1 - x_3y_1\} + \tfrac{5}{8}\{(x_1 + x_2)(y_2 - y_3) + (x_3 + x_2) \\
&\times (y_2 - y_1) + (x_3 + x_4)(y_1 - y_4) + (x_1 + x_4)(y_3 - y_4)\} - \tfrac{1}{8}\{(x_1 + x_2)(y_4 - y_1) + (x_3 + x_2) \\
&\times (y_3 - y_4) + (x_1 - x_3)(y_2 + y_4) + (x_1 + x_4)(y_2 - y_1)\}
\end{aligned} \tag{18}
$$

On the other hand, functions s_1, s_2, s_3, s_4, t_1, t_2, t_3, and t_4 depend on the nodal coordinates as well as on the group. These functions are provided below. Now, having the "Parent" terms of each group, the rest of the terms in the corresponding group (see Table I) can be generated using one of the four transformations based on nodal coordinates, as shown in Table II.

TABLE II. Transformations based on nodal coordinates.

Node	Transformation 1	Transformation 2	Transformation 3	Transformation 4
(x_1, y_1)	(x_4, y_4)	(x_4, y_4)	(y_4, x_4)	(y_1, x_1)
(x_2, y_2)	(x_1, y_1)	(x_3, y_3)	(y_3, x_3)	(y_2, x_2)
(x_3, y_3)	(x_2, y_2)	(x_2, y_2)	(y_2, x_2)	(y_3, x_3)
(x_4, y_4)	(x_3, y_3)	(x_1, y_1)	(y_1, x_1)	(y_4, x_4)

2.1.1. Group A

Using equation 15, the "Parent" term $k_{1,1}$ is obtained. Then, the diagonal terms of corner nodes can be obtained by applying the transformations in Table II. $k_{1,1} \xrightarrow{T_1} k_{3,3} \xrightarrow{T_1} k_{5,5} \xrightarrow{T_1} k_{7,7} \xrightarrow{T_3} k_{8,8} \xrightarrow{T_1} k_{2,2} \xrightarrow{T_1} k_{4,4} \xrightarrow{T_1} k_{6,6}$, with

$$
\begin{aligned}
s_1 &:= \tfrac{1}{8}(y_2 - y_4)^2 \\
s_2 &:= \tfrac{1}{8}(x_2 - x_4)^2 \\
s_3 &:= \tfrac{3}{16}(y_2 - y_4)^2 \\
s_4 &:= \tfrac{3}{16}(x_2 - x_4)^2
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
t_1 &:= \{\tfrac{1}{432}\}((y_2 + y_1)^2 + (y_2 - y_4)^2 + (y_1 + (y_4)^2 - 4y_3(y_2 + y_4 - 2y_3 + 2y_1)) \\
t_2 &:= \tfrac{1}{864}(4x_3 - 2x_1 - x_4 - x_2)^2 + \tfrac{1}{288}(x_2 - x_4)^2 \\
t_3 &:= \tfrac{1}{144}(4y_3 - 2y_1 - y_2 - y_4)(y_2 - y_4) \\
t_4 &:= \tfrac{1}{144}(4x_3 - 2x_1 - x_4 - x_2)(x_2 - x_4)
\end{aligned}
\tag{20}
$$

Now, with the "Parent" term $K_{9,9}$, we can generate the diagonal terms of the midside nodes:

$$k_{9,9} \xrightarrow{T_1} k_{11,11} \xrightarrow{T_1} k_{13,13} \xrightarrow{T_1} k_{15,15} \xrightarrow{T_3} k_{14,14} \xrightarrow{T_1} k_{16,16} \xrightarrow{T_1} k_{10,10} \xrightarrow{T_1} k_{12,12},$$

with

$$
\begin{aligned}
s_1 &:= \tfrac{1}{216}(-5y_1 + 5y_4 - y_2 + y_3)^2 + \tfrac{1}{72}(3y_1 - y_2 - y_4 - y_3)^2 \\
s_2 &:= \tfrac{1}{216}(5x_4 - 5x_1 + x_3 - x_2)^2 + \tfrac{1}{72}(3x_1 - x_2 - x_4 - x_3)^2
\end{aligned}
$$

$$s_3 := \frac{1}{36}(y_3 + 3y_4)^2 + \frac{1}{9}y_4(y_2 - y_4) - \frac{1}{36}(y_2 + y_1)^2 + \frac{1}{9}y_1(4y_1 - 2y_3 - 5y_4)$$

$$s_4 := \frac{1}{36}(-3x_1 + x_2 + x_4 + x_3)(5x_4 - 5x_1 + x_3 - x_2) \tag{21}$$

$$t_1 := \{\tfrac{1}{54}\}((y_2 - y_1)^2 + (y_4 + 2y_3)^2 + 3(y_3 - 2y_2)^2 - y_3(5y_2 + y_1) + y_4(y_1 - 7y_2))$$

$$t_2 := \frac{1}{216}(x_4 - x_1 + 5x_3 - 5x_2)^2 + \frac{1}{72}(x_1 - 3x_2 + x_4 + x_3)^2$$

$$t_3 := \frac{1}{36}(y_1 - y_4 + 5y_2 - 5y_3)(y_3 + y_1 - 3y_2 + y_4)$$

$$t_4 := -\frac{1}{36}(x_4 - x_1 + 5x_3 - 5x_2)(x_1 - 3x_2 + x_4 + x_3). \tag{22}$$

2.1.2. Group B

With the "Parent" term $K_{1,2}$, the terms related to corner nodes are generated:

$$k_{1,2} \xrightarrow{T_1} k_{3,4} \xrightarrow{T_1} k_{5,6} \xrightarrow{T_1} k_{7,8},$$

with

$$s_1 := 0$$

$$s_2 := -\frac{1}{8}(y_2 - y_4)(x_2 - x_4)$$

$$s_3 := 0$$

$$s_4 := -\frac{3}{16}(y_2 - y_4)(x_2 - x_4) \tag{23}$$

$$t_1 := 0$$

$$t_2 := \{\tfrac{1}{432}\}((-2y_1 - y_2 - y_4 + 4y_3)x_1 + (-2y_2 + 2y_3 + y_1 - y_4)x_2 + 2(-4y_3 + 2y_1 + y_2 + y_4)x_3 + (y_2 - y_1 + 2y_3 - 2y_4)x_4)$$

$$t_3 := 0$$

$$t_4 := \{\tfrac{1}{144}\}((y_2 - y_4)(x_1 - 2x_3) + (y_2 + y_1 - 2y_3)x_2 + (-y_1 - y_4 + 2y_3)x_4). \tag{24}$$

The rest of the terms of this group are calculated from $K_{9,10}$ as follows:

$$k_{9,10} \xrightarrow{T_1} k_{11,12} \xrightarrow{T_1} k_{13,14} \xrightarrow{T_1} k_{15,16},$$

with

$$s_1 := (5(y_1 - y_4)(2x_2 - x_4) + (y_2 - y_3)(2x_2 - x_4) + (2y_2 - 2y_1 - y_3 + y_4)x_3 + (2y_3 - 7y_1 + y_2 + 4y_4)x_1)\{\tfrac{1}{216}\}$$

$$s_2 := \{\tfrac{1}{108}\}((-26x_1 + 2x_2 + 17x_4 + 7x_3)y_1 + (-y_3 + y_4 - 2y_2)x_2 + (-y_2 - 4y_4 - 2y_3)x_3 + (-14y_4 + y_2 - 4y_3)x_4 + (2y_2 + 17y_4 + 7y_3)x_1)$$

$$s_3 := \{\tfrac{1}{72}\}((3y_4 + y_3 - 4y_1)x_1 + 2(-y_3 - y_4 - y_2 + 3y_1)x_2 + (y_2 + y_1 - 2y_4)x_3 + (-3y_1 + y_3 + y_2 + y_4)x_4)$$

$$s_4 := \{\tfrac{1}{36}\}((-15y_1 + 10y_4 + 4y_3 + y_2)x_1 + (y_2 + y_1 - 2y_4)x_2 + (-3y_4 + 4y_1 - y_3)x_3 + (-2y_2 - 5y_4 - 3y_3 + 10y_1)x_4) \quad (25)$$

$$t_1 := \{\tfrac{1}{24}\}(-2y_3 + y_2 + y_1)(x_1 - x_3)$$

$$t_2 := \{\tfrac{1}{108}\}((y_3 + 2y_2 - 2y_1 - y_4)x_1 + (2y_1 + 17y_3 + 7y_4 - 26y_2)x_2 + (17y_2 - 14y_3 - 4y_4 + y_1)x_3 + (-y_1 - 4y_3 + 7y_2 - 2y_4)x_4)$$

$$t_3 := \{\tfrac{1}{24}\}(-2y_2 + 2y_1 - y_3 + y_4)(x_1 - x_3)$$

$$t_4 := \{\tfrac{1}{36}\}((2y_3 - y_2 - y_1)x_1 + (-10y_3 - 4y_4 + 15y_2 - y_1)x_2 + (2y_1 + 5y_3 + 3y_4 - 10y_2)x_3 + (y_4 - 4y_2 + 3y_3)x_4). \quad (26)$$

2.1.3. Group C

Using the "Parent" term $K_{1,3}$, the following terms are generated:

$$k_{1,3} \xrightarrow{T_1} k_{3,5} \xrightarrow{T_1} k_{5,7} \xrightarrow{T_1} k_{1,7} \xrightarrow{T_3} k_{6,8} \xrightarrow{T_1} k_{2,8} \xrightarrow{T_1} k_{2,4} \xrightarrow{T_1} k_{4,6},$$

with

$$s_1 := \{\tfrac{1}{48}\}((y_2 - y_4)y_1 - 3y_4y_2 + 4y_4^2 + y_2^2)$$

$$s_2 := \{\tfrac{1}{48}\}((x_1 - x_3)x_1 - 3x_4x_2 + 4x_4^2 + x_2^2)$$

$$s_3 := \{\tfrac{1}{48}\}(2(y_2 - y_4)y_1 + y_2^2 + (-y_3 - 3y_4)y_2 + y_4y_3 + 4y_4^2)$$

$$s_4 := \{\tfrac{1}{48}\}(2(x_2 - x_4)x_1 + x_2^2 + (-x_3 - 3x_4)x_2 + x_4x_3 + 4x_4^2) \tag{27}$$

$$t_1 := \{\tfrac{1}{48}\}(y_1^2 + (y_2 - 3y_3)y_1 - y_3y_2 + 4y_3^2)$$

$$t_2 := \{\tfrac{1}{48}\}(x_1^2 + (-3x_3 + x_2)x_1 - x_2x_3 + 4x_3^2)$$

$$t_3 := \{\tfrac{1}{48}\}(-y_1(y_1 - 3y_3 - y_4 + 2y_2) + y_3(2y_2 - 2y_3 - y_4))$$

$$t_4 := \{\tfrac{1}{48}\}(-x_1(x_1 - 3x_3 - x_4 + 2x_2) + x_3(-2x_3 + 2x_2 - x_4)). \tag{28}$$

The other terms in group C are generated from $K_{9,11}$ as follows:

$$k_{9,11} \xrightarrow{T_1} k_{11,13} \xrightarrow{T_1} k_{13,15} \xrightarrow{T_1} k_{9,15} \xrightarrow{T_3} k_{12,14} \xrightarrow{T_1} k_{14,16} \xrightarrow{T_1} k_{10,16} \xrightarrow{T_1} k_{10,12},$$

with

$$s_1 := \{\tfrac{1}{108}\}(y_1(2y_1 - 2y_3 - 7y_4 + 5y_2) + (5y_3 - 8y_4 - y_2)y_2 + y_3(2y_3 - 7y_4) + 11y_4^2)$$

$$s_2 := \{\tfrac{1}{108}\}(x_1(2x_1 - 2x_3 - 7x_4 + 5x_2) + (5x_3 - 8x_4 - x_2)x_2 + x_3(2x_3 - 7x_4) + 11x_4^2)$$

$$s_3 := \{\tfrac{1}{36}\}(y_1(y_1 + 5y_2 - 7y_4) + y_3(7y_4 - 5y_2 - y_3))$$

$$s_4 := \{\tfrac{1}{36}\}(x_1(x_1 + 5x_2 - 7x_4) + x_3(7x_4 - 5x_2 - x_3)) \tag{29}$$

$$t_1 := \{\tfrac{1}{108}\}\{y_1(y_1 - 5y_4 - 16y_3 + 19y_2) + y_2(-26y_2 + 19y_3 + 14y_4) + y_4(-2y_4 - 5y_3) + y_3^2\}$$

$$t_2 := \{\tfrac{1}{108}\}\{x_1(x_1 - 5x_4 - 16x_3 + 19x_2) + x_2(-26x_2 + 14x_4 + 19x_3) + x_4(-2x_4 - 5x_3) + x_3^2\}$$

$$t_3 := \{\tfrac{1}{36}\}(y_1(2y_1 + 4y_3 + 3y_4 - 11y_2) + y_2(15y_2 - 8y_4 - 11y_3) + y_4(y_4 + 3y_3) + 4y_3^2)$$

$$t_4 := \{\tfrac{1}{36}\}(x_1(2x_1 + 4x_3 + 3x_4 - 11x_2) + x_2(15x_2 - 8x_4 - 11x_3) + x_4(x_4 + 3x_3) + 4x_3^2). \tag{30}$$

2.1.4. Group D

From the "Parent" term $K_{1,4}$ the following terms are generated:

$$k_{1,4} \xrightarrow{T_1} k_{3,6} \xrightarrow{T_1} k_{5,8} \xrightarrow{T_1} k_{2,7} \xrightarrow{T_4} k_{2,3} \xrightarrow{T_1} k_{1,8} \xrightarrow{T_1} k_{6,7} \xrightarrow{T_1} k_{4,5},$$

with

$$s_1 := \tfrac{1}{48}(y_2 - y_4)(x_1 + x_2 - 2x_4)$$

$$s_2 := \tfrac{1}{48}(x_2 - x_4)(-y_2 - y_1 + 2y_4)$$

$$s_3 := -\tfrac{1}{48}(y_2 - y_4)(-2x_4 + 2x_1 - x_3 + x_2)$$

$$s_4 := -\tfrac{1}{48}(x_2 - x_4)(2y_1 - 2y_4 + y_2 - y_3) \tag{31}$$

$$t_1 := \tfrac{1}{48}(x_1 - x_3)(2y_3 - y_2 - y_1)$$

$$t_2 := -\tfrac{1}{48}(y_3 - y_1)(2x_3 - x_1 - x_2)$$

$$t_3 := \tfrac{1}{48}(x_1 - x_3)(y_1 - y_4 + 2y_2 - 2y_3)$$

$$t_4 := -\tfrac{1}{48}(y_3 - y_1)(-x_4 + x_1 - 2x_3 + 2x_2). \tag{32}$$

The terms related with midside nodes of group D are generated from $K_{9,12}$:

$$k_{9,12} \xrightarrow{T_1} k_{11,14} \xrightarrow{T_1} k_{13,16} \xrightarrow{T_1} k_{10,15} \xrightarrow{T_3} k_{12,13} \xrightarrow{T_1} k_{14,15} \xrightarrow{T_1} k_{9,16} \xrightarrow{T_1} k_{10,11},$$

with

2.1.6. Group F

The "Parent" term $K_{1,6}$ is used to calculate the terms:

$$k_{1,6} \xrightarrow{T_1} k_{3,8} \xrightarrow{T_1} k_{2,5} \xrightarrow{T_1} k_{4,7},$$

with

$$s_1 := \{\tfrac{-1}{16}\}(y_2 - y_4)(x_2 - x_4)$$

$$s_2 := \{\tfrac{-1}{16}\}(y_2 - y_4)(x_2 - x_4)$$

$$s_3 := 0$$

$$s_4 := 0 \tag{39}$$

$$t_1 := \{\tfrac{1}{432}\}((-4y_1 - 2y_2 - 2y_4 + 8y_3)x_1 + (2y_2 - 2y_3 + y_1 - y_4)x_2 + (-4y_3 + y_2 + y_4 + 2y_1)x_3 + (-y_2 + y_1 - y_3 + y_4)x_4)$$

$$t_2 := \{\tfrac{1}{432}\}((-4y_1 + y_2 + y_4 + 2y_3)x_1 + (2y_2 - 2y_1 + y_3 - y_4)x_2 + 2(-2y_3 - y_2 - y_4 + 4y_1)x_3 + (-2y_1 + 2y_4 - y_2 + y_3)x_4)$$

$$t_3 := \{\tfrac{1}{144}\}(-(y_2 - y_4)(x_1 - x_3) - y_2 - y_1 + 2y_3(x_2 - x_4))$$

$$t_4 := \{\tfrac{1}{144}\}(-(y_2 - y_4)x_1 + (-y_2 - y_3 + 2y_1)x_2 + 2(y_2 - y_4)x_3 + (-2y_1 + y_4 + y_3)x_4). \tag{40}$$

Now, terms associated to mid-side nodes are generated from $K_{9,14}$:

$$k_{9,14} \xrightarrow{T_1} k_{11,16} \xrightarrow{T_1} k_{10,13} \xrightarrow{T_1} k_{12,15},$$

with

$$s_1 := \{\tfrac{1}{108}\}((7y_4 + 2y_1 - y_3 - 8y_2)x_1 + (7y_3 - 4y_2 - 2y_1 - y_4)x_2 + (y_2 - y_1 + 2y_3 - 2y_4)x_3 + (-4y_4 - 8y_3 + 11y_2 + y_1)x_4)$$

$$s_2 := \{\tfrac{1}{108}\}((-y_3 + y_4 - 2y_2 + 2y_1)x_1 + (y_3 - 4y_2 - 8y_1 + 11y_4)x_2 + (-y_1 + 2y_3 + 7y_2 - 8y_4)x_3 + (-4y_4 - 2y_3 - y_2 + 7y_1)x_4)$$

$$s_3 := \{\tfrac{1}{36}\}((-5y_2 + y_1 + 4y_4)x_1 + (-4y_3 + 3y_2 + y_1)x_2 + (2y_2 - y_4 - y_3)x_3 + (5y_3 - 2y_1 - 3y_4)x_4)$$

$$s_4 := \{\tfrac{1}{36}\}((y_2 + y_1 - 2y_4)x_1 + (2y_3 + 3y_2 - 5y_1)x_2 + (-4y_2 + 5y_4 - y_3)x_3 + (-y_3 + 4y_1 - 3y_4)x_4) \quad (41)$$

$$t_1 := \{\tfrac{1}{108}\}((7y_4 - 4y_1 - y_3 - 2y_2)x_1 + (7y_3 + 2y_2 - 8y_1 - y_4)x_2 + (11y_1 - 4y_3 + y_2 - 8y_4)x_3 + (-2y_3 + 2y_4 - y_2 + y_1)x_4)$$

$$t_2 := \{\tfrac{1}{108}\}((y_4 - 4y_1 + 11y_3 - 8y_2)x_1 + (y_3 - y_4 + 2y_2 - 2y_1)x_2 + (-y_1 - 4y_3 + 7y_2 - 2y_4)x_3 + (2y_4 - 8y_3 - y_2 + 7y_1)x_4)$$

$$t_3 := \{\tfrac{1}{36}\}((-y_2 - 3y_1 + 4y_4)x_1 + (-4y_3 - y_2 + 5y_1)x_2 + (2y_2 - 5y_4 + 3y_3)x_3 + (y_3 - 2y_1 + y_4)x_4)$$

$$t_4 := \{\tfrac{1}{36}\}((5y_2 - 3y_1 - 2y_4)x_1 + (2y_3 - y_2 - y_1)x_2 + (-4y_2 + y_4 + 3y_3)x_3 + (-5y_3 + 4y_1 + y_4)x_4) \quad (42)$$

2.1.7. Group G

All the terms of this group are generated with only one "Parent" term ($K_{7,15}$):

$$k_{7,15} \xrightarrow{T_1} k_{1,9} \xrightarrow{T_1} k_{3,11} \xrightarrow{T_1} k_{5,13} \xrightarrow{T_3} k_{4,10} \xrightarrow{T_1} k_{6,12} \xrightarrow{T_1} k_{8,14} \xrightarrow{T_1} k_{2,16}$$

and

$$k_{7,15} \xrightarrow{T_2} k_{7,13} \xrightarrow{T_1} k_{1,15} \xrightarrow{T_1} k_{3,9} \xrightarrow{T_1} k_{5,11} \xrightarrow{T_4} k_{8,16} \xrightarrow{T_1} k_{6,14} \xrightarrow{T_1} k_{4,12} \xrightarrow{T_1} k_{2,10},$$

with

$$s_1 := \{\tfrac{1}{216}\}(-7y_1^2 + (4y_3 - 4y_4 + 14y_2)y_1 - 10y_2^2 + (7y_4 - y_3)y_2 - y_3y_4 - y_4^2 - y_3^2)$$

$$s_2 := \{\tfrac{1}{216}\}(-7x_1^2 + (14x_2 + 4x_3 - 4x_4)x_1 - 10x_2^2 + (7x_4 - x_3)x_2 - x_4^2 - x_3^2 - x_3x_4)$$

$$s_3 := \{\tfrac{1}{72}\}(-4y_1^2 + (9y_2 + 2y_3 - 3y_4)y_1 - 2y_2^2 + (-y_4 - 4y_3)y_2 + 2y_3y_4 + y_4^2)$$

$$s_4 := \{\tfrac{1}{72}\}(-4x_1^2 + (-3x_4 + 2x_3 + 9x_2)x_1 - 2x_2^2 + (-x_4 - 4x_3)x_2 + x_4^2 + 2x_3x_4) \tag{43}$$

$$t_1 := \{\tfrac{1}{24}\}((-4y_4 + y_2 + 3y_3)y_1 - 3y_3^2 - y_2y_3 + 4y_3y_4)$$

$$t_2 := \{\tfrac{1}{24}\}((x_2 - 4x_4 + 3x_3)x_1 - 3x_3^2 - x_2x_3 + 4x_3x_4)$$

$$t_3 := \{\tfrac{1}{24}\}(y_1^2 + (2y_2 - 7y_4 + 3y_3)y_1 + 7y_3y_4 - 4y_3^2 - 2y_2y_3)$$

$$t_4 := \{\tfrac{1}{24}\}(x_1^2 + (-7x_4 + 3x_3 + 2x_2)x_1 + 7x_3x_4 - 4x_3^2 - 2x_2x_3). \tag{44}$$

2.1.8. Group H

Now, using "Parent" term $K_{7,14}$, all the terms of this group can be generated:

$$k_{7,14} \xrightarrow{T_1} k_{1,16} \xrightarrow{T_1} k_{3,10} \xrightarrow{T_1} k_{5,12} \xrightarrow{T_3} k_{2,9} \xrightarrow{T_1} k_{4,11} \xrightarrow{T_1} k_{6,13} \xrightarrow{T_1} k_{8,15}$$

and

$$k_{7,14} \xrightarrow{T_2} k_{7,16} \xrightarrow{T_1} k_{1,10} \xrightarrow{T_1} k_{3,12} \xrightarrow{T_1} k_{5,14} \xrightarrow{T_4} k_{8,13} \xrightarrow{T_1} k_{6,11} \xrightarrow{T_1} k_{4,9} \xrightarrow{T_1} k_{2,15},$$

with

$$s_1 := \{\tfrac{1}{216}\}((y_1 - y_4 + 2y_2 - 2y_3)x_1 + (-y_1 - 5y_4 + 20y_2 - 4y_3)x_2 + (7y_3 - 2y_1 + 5y_4 - 10y_2)x_3 + (-2y_2 + 2y_1 - y_3 + y_4)x_4)$$

$$s_2 := \{\tfrac{1}{216}\}(-(y_2 - y_1 + 2y_3 - 2y_4)x_1 + (10y_2 - 10y_3 + 2y_1 - 2y_4)x_2 + (7y_3 - 4y_2 - 2y_1 - y_4)x_3 + (-y_1 + y_4 - 5y_2 + 5y_3)x_4)$$

$$s_3 := \{\tfrac{1}{72}\}((-2y_2 + y_4 + y_3)x_1 + (-2y_2 - 2y_1 + y_4 + 3y_3)x_2 + (-3y_4 + 6y_2 - 4y_3 + y_1)x_3 + (-y_2 + y_1)x_4)$$

$$s_4 := \{\tfrac{1}{72}\}((-2y_2 + y_4 + y_3)x_1 + (-2y_2 - 2y_4 - 2y_1 + 6y_3)x_2 + (-4y_3 + 3y_2 + y_1)x_3 + (y_2 + y_4 + y_1 - 3y_3)x_4) \tag{45}$$

$$t_1 := \{\tfrac{1}{24}\}(y_3 - y_1)(-x_1 - x_2 + 4x_4)$$

$$t_2 := \{\tfrac{1}{24}\}(-4y_4 + y_2 + 3y_1)(x_1 - x_3)$$

$$t_3 := \{\tfrac{1}{24}\}(2(-2x_1 - x_2)(y_3 - y_1) - (y_3 - y_1)(x_3 - 7x_4))$$

$$t_4 := \{\tfrac{1}{24}\}((y_3 + 2y_2 - 7y_4 + 4y_1)x_1 + (-4y_1 + 7y_4 - y_3 - 2y_2)x_3). \quad (46)$$

2.1.9. Group I

The terms of this group are generated from the "Parent" term $K_{7,9}$:

$$k_{7,9} \xrightarrow{T_1} k_{1,11} \xrightarrow{T_1} k_{3,13} \xrightarrow{T_1} k_{5,15} \xrightarrow{T_3} k_{4,16} \xrightarrow{T_1} k_{6,10} \xrightarrow{T_1} k_{8,12} \xrightarrow{T_1} k_{2,14}$$

and

$$k_{7,9} \xrightarrow{T_2} k_{7,11} \xrightarrow{T_1} k_{1,13} \xrightarrow{T_1} k_{3,15} \xrightarrow{T_1} k_{5,9} \xrightarrow{T_4} k_{8,10} \xrightarrow{T_1} k_{6,16} \xrightarrow{T_1} k_{4,14} \xrightarrow{T_1} k_{2,12},$$

with

$$s_1 := \{\tfrac{1}{216}\}\{(7y_1 - 11y_2 - 4y_3 + y_4)y_1 + y_2(-2y_2 + 4y_3 + 11y_4) + y_3(-2y_4 + y_3) - 5y_4^2\}$$

$$s_2 := \{\tfrac{1}{216}\}\{x_1(7x_1 - 4x_3 + x_4 - 11x_2) + x_2(-2x_2 + 4x_3 + 11x_4) + x_3^2 - x_4(2x_3 + 5x_4)\}$$

$$s_3 := \{\tfrac{1}{72}\}((4y_1 - 6y_2 - 2y_3)y_1 + (2y_2 + y_4 + y_3)y_2 + y_4(y_3 - y_4))$$

$$s_4 := \{\tfrac{1}{72}\}(x_1(4x_1 - 2x_3 - 6x_2) + x_2(2x_2 + x_4 + x_3) + x_4(x_3 - x_4)) \quad (47)$$

$$t_1 := \{\tfrac{1}{24}\}(y_1(-y_1 - y_2 + 3y_3) + y_3(-2y_3 + y_2))$$

$$t_2 := \{\tfrac{1}{24}\}(x_1(-x_1 + 3x_3 - x_2) + x_3(-2x_3 + x_2))$$

$$t_3 := \{\tfrac{1}{24}\}(y_1(-2y_1 + 3y_3 + 2y_2 - y_4) + y_3(-2y_2 - y_3 + y_4))$$

$$t_4 := \{\tfrac{1}{24}\}(x_1(-2x_1 + 3x_3 + 2x_2 - x_4) + x_3(-2x_2 - x_3 + x_4)). \quad (48)$$

2.1.10. Group J

This last group is generated by using the "Parent" term $K_{7,10}$:

$$k_{7,10} \xrightarrow{T_1} k_{1,12} \xrightarrow{T_1} k_{3,14} \xrightarrow{T_1} k_{5,16} \xrightarrow{T_3} k_{4,15} \xrightarrow{T_1} k_{6,9} \xrightarrow{T_1} k_{8,11} \xrightarrow{T_1} k_{2,13}$$

$$k_{8,9} \xrightarrow{T_1} k_{2,11} \xrightarrow{T_1} k_{4,13} \xrightarrow{T_1} k_{6,15} \xrightarrow{T_3} k_{3,16} \xrightarrow{T_1} k_{5,10} \xrightarrow{T_1} k_{7,12} \xrightarrow{T_1} k_{1,14},$$

with

$$s_1 := \{\tfrac{1}{216}\}(y_3 - y_1)(-2x_1 + 2x_2 + x_3 - x_4)$$

$$s_2 := \{\tfrac{1}{216}\}((10y_2 - 5y_4 + 2y_3 - 7y_1)x_1 + (y_1 - y_4 + 2y_2 - 2y_3)x_2 - (y_3 - y_4 + 2y_2 - 2y_1)x_3 + (5y_4 - 10y_2 + y_3 + 4y_1)x_4)$$

$$s_3 := \{\tfrac{1}{72}\}((3y_4 - 4y_1 + y_3)x_1 + 2x_2(-y_2 - y_4 + 3y_1 - y_3) + (y_2 + y_1 - 2y_4)x_3 + (y_2 + y_3 + y_4 - 3y_1)x_4)$$

$$s_4 := \{\tfrac{1}{72}\}((6y_2 - 3y_4 + y_3 - 4y_1)x_1 + (-2y_2 + y_4 + y_3)x_2 + (y_4 + y_1 - 2y_2)x_3 + (-2y_2 - 2y_3 + 3y_1 + y_4)x_4) \quad (49)$$

$$t_1 := \{\tfrac{1}{24}\}(y_2 + y_1 - 2y_3)(x_1 - x_3)$$

$$t_2 := \{\tfrac{1}{24}\}(y_3 - y_1)(-x_1 - x_2 + 2x_3)$$

$$t_3 := \{\tfrac{1}{24}\}((-2y_2 + 2y_1 + y_3 - y_4)x_1 + (y_3 - y_4 + 2y_2 - 2y_1)x_3)$$

$$t_4 := \{\tfrac{1}{24}\}(y_3 - y_1)(-2x_1 + 2x_2 + x_3 - x_4). \quad (50)$$

3. COMPARISON ON CPU INTEGRATION TIMES

To verify the advantages of semi-analytical integration over standard numerical integration, a comparison between both techniques is included herein. Table III shows the CPU times required for numerical and symbolic integration. The last column of Table III displays the CPU savings using semi-analytical integration. It can be noted that the saving in CPU time is close to one third, which is a relevant achievement in finite element analysis. In all cases the results obtained

TABLE III. CPU time comparisons.

Elements	Numerical (sec)	Symbolic (sec)	Saving (%)
1000	5.5102E-02	4.0047E-02	27
10000	4.7565E-01	3.2290-01	32
100000	4.0425	2.5405	37
1000000	39.6342	24.790	37

using semi-analytical integration were exactly the same as those obtained using 2×2 numerical quadrature, as expected.

4. CONCLUDING REMARKS

Semi-analytical integration of a plane strain eight-node finite element stiffness matrix was performed using symbolic-manipulation *Maple* code. The methodology used was the one developed by Griffiths [22], based on *adjacency* rules relating to element nodes. The power of the method lies in its generality, which can be applied to higher order plane elements and 3D elements.

Comparison between numerical and semi-analytical computation times showed that the semi-analytical approach resulted in a reduction in CPU time of approximately one third over the numerical approach.

This kind of improvement can be very significant when dealing with large problems (i.e., FEM meshes having thousands of elements), and other analyses in which highly repetitive stiffness formulations is needed, such as dynamic nonlinear analyses and Monte-Carlo processes.

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