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TECHNICAL NOTE

One-dimensional consolidation theories for layered soil and coupled and uncoupled solutions by the finite-element method

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One-dimensional consolidation theories for layered soil have been re-examined. Coupled (settlement and excess pore pressure), uncoupled (excess pore pressure only) and the classical Terzaghi equation are solved by the finite-element method. By accounting only for changes in the coefficient of consolidation (c_v), the classical Terzaghi approach is unable to satisfy the flow continuity conditions at the interface between layers.

KEYWORDS: consolidation; compressibility; finite-element method; permeability

On a réexaminé des théories de consolidations unidimensionnelles (1-D) pour sols stratifiés. On résout les équations couplées (tassement et excès de pression interstitielle), non couplées (excès de pression interstitielle seulement) et l'équation classique de Terzaghi avec la méthode aux éléments finis (FE). En ne tenant compte que des changements dans le coefficient de consolidation (c_v), l'équation de Terzaghi ne permet pas de satisfaire les conditions de continuité de débit à l'interface entre les couches.

INTRODUCTION

Gray (1945) first discussed the nature of the consolidation of two contiguous layers of unlike compressible soils. The general analytical solution for the one-dimensional (1D) consolidation of a layered system has been developed by Schiffman & Stein (1970). Desai & Saxena (1977) analysed the consolidation behaviour of layered anisotropic foundations. Abid & Pyrah (1988) presented some guidelines for using the finite-element (FE) method to predict 1D consolidation behaviour using both diffusion and coupled approaches. Lee *et al.* (1992) developed a more efficient analytical solution technique, which showed that the effects of the permeability and coefficient of volume compressibility of soil on the consolidation of layered systems are different and cannot be embodied into a single coefficient of consolidation. The compressibility of the soil layer m_v also plays an important role in the rate of consolidation. Xie & Pan (1995) further developed an analytical solution for a layered system under time-dependent loading. Pyrah (1996) showed that the 1D consolidation behaviour of layered soils consisting of two layers with the same value of the c_v, but different k and m_v were quite different. Zhu & Yin (1999) gave more analytical solutions for different loading cases.

In the present work coupled, uncoupled and the Terzaghi 1D consolidation theories have been re-examined using the FE method and it is shown that the Terzaghi FE solutions do not satisfy the flow continuity conditions at the interfaces between soil layers. Numerical results show that applying the Terzaghi FE solution to a layered system can lead to incorrect results. It is also shown that the average degree of consolidation, as defined by settlement and excess pore pressure, are different for layered systems.

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UNCOUPLED AND COUPLED 1D CONSOLIDATION THEORIES

Consider a thin strip of soil within a layered saturated soil undergoing consolidation. At any time t by equilibrium

$$p = u + \sigma' \tag{1}$$

where σ' and u are, respectively, the effective stress and the excess pore pressure at any given depth z and p is the total load on top of the soil layers.

For the sake of simplicity, p is assumed to be constant. By taking a derivative of equation (1) with respect to time and depth

$$\frac{\partial u}{\partial t} + \frac{\partial \sigma'}{\partial t} = 0 \tag{2}$$

$$\frac{\partial u}{\partial z} + \frac{\partial \sigma'}{\partial z} = 0 \tag{3}$$

Assuming 'small strains'

$$\sigma' = \frac{1}{m_v}\frac{\partial s}{\partial z} \tag{4}$$

where s is the settlement at any given depth z.

From equations (3) and (4)

$$\frac{\partial u}{\partial z} + \frac{\partial}{\partial z}\left(\frac{1}{m_v}\frac{\partial s}{\partial z}\right) = 0 \tag{5}$$

The net flow rate from Darcy law is

$$Q = \frac{\partial}{\partial z}\left(\frac{k}{\gamma_w}\frac{\partial u}{\partial z}\right) \tag{6}$$

where γ_w is the unit weight of water.

The rate of volume change of soil is

$$\frac{\partial V}{\partial t} = \frac{\partial}{\partial t}\frac{\partial s}{\partial z} \tag{7}$$

by continuity

$$\frac{\partial}{\partial z}\left(\frac{k}{\gamma_w}\frac{\partial u}{\partial z}\right) + \frac{\partial}{\partial t}\frac{\partial s}{\partial z} = 0 \tag{8}$$

Equations (5) and (8) are the coupled governing equations of 1D consolidation (Griffiths, 1994).

From equations (2), (4) and (8)

$$\frac{\partial}{\partial z}\left(\frac{k}{\gamma_w}\frac{\partial u}{\partial z}\right) = m_v \frac{\partial u}{\partial t} \tag{9}$$

Equation (9) is the uncoupled governing equation of 1D consolidation with excess pore pressure u as the only dependent variable (e.g. Schiffman & Arya, 1977; Verruijt, 1995).

If m_v and k/γ_w are constant throughout the soil layer, the settlement variable s can be eliminated from equations (5) and (8) to give the classical Terzaghi consolidation equation

$$c_v \frac{\partial^2 u}{\partial z^2} = \frac{\partial u}{\partial t} \tag{10}$$

where c_v, the coefficient of consolidation, is defined as

$$c_v = \frac{k}{m_v \gamma_w} \tag{11}$$

and m_v and k are the coefficient of volume compressibility and the soil permeability and γ_w is the unit weight of water.

LAYERED 1D CONSOLIDATION THEORY

Let u_i be the excess pore pressure at any given depth z at any given time t. The governing equations of the layered system can be expressed as

$$c_{vi} \frac{\partial^2 u_i}{\partial z^2} = \frac{\partial u_i}{\partial t}, \qquad i = 1, 2, \ldots, n \tag{12}$$

where c_{vi} is the coefficient of consolidation of the ith layer.

The boundary conditions are

$$\begin{aligned} z = 0: \quad & \frac{\partial u_1}{\partial z} = 0 \text{ (impermeable)} \quad \text{or } u_1 = 0 \text{ (drained)} \\ z = H: \quad & \frac{\partial u_n}{\partial z} = 0 \text{ (impermeable)} \quad \text{or } u_n = 0 \text{ (drained)} \end{aligned} \tag{13}$$

The interface flow continuity conditions are

$$k_i \frac{\partial u_i}{\partial z} = k_{i+1} \frac{\partial u_{i+1}}{\partial z} \qquad i = 1, 2, \ldots, n-1 \tag{14}$$

The initial conditions at $t = 0$, assumed in this case to be uniform with depth, are given by

$$u_i = u_0 \tag{15}$$

The analytical solutions of equation (12), with the above conditions, have been developed by Schiffman & Stein (1970), Lee *et al.* (1992), Xie & Pan (1995) and Zhu & Yin (1999) among others.

COUPLED AND UNCOUPLED FE SOLUTIONS

Solution of the uncoupled equation

If the FE method is used, equation (9) can be written in an uncoupled form as (Schiffman & Arya, 1977)

$$\frac{1}{\gamma_w}\frac{\partial}{\partial z} k \frac{\partial u}{\partial z} = m_v \frac{\partial u}{\partial t} \tag{16}$$

It is noted that equation (16) does not use the coefficient of consolidation c_v.

After solution by the Galerkin weighted residual process, equation (16) leads to the element matrix form

$$[\mathbf{k_c}]\{\mathbf{u}\} + [\mathbf{m_m}]\left\{\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}\right\} = \{\mathbf{0}\} \tag{17}$$

where $[\mathbf{k_c}]$ and $[\mathbf{m_m}]$ are the fluid conductivity and 'mass' matrices respectively.

There are many ways of integrating this set of ordinary differential equations (e.g. Smith & Griffiths, 2004). If linear interpolations and fixed time steps are used, equation (17) can be written at two consecutive time steps '0' and '1' as follows

$$[\mathbf{k_c}]\{\mathbf{u}\}_0 + [\mathbf{m_m}]\left\{\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}\right\}_0 = \{\mathbf{0}\} \tag{18}$$

$$[\mathbf{k_c}]\{\mathbf{u}\}_1 + [\mathbf{m_m}]\left\{\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}\right\}_1 = \{\mathbf{0}\} \tag{19}$$

A third equation advances the solution from '0' to '1' using a weighted average of the gradients at the beginning and the end of the time interval, thus

$$\{\mathbf{u}\}_1 = \{\mathbf{u}\}_0 + \Delta t\left((1-\theta)\left\{\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}\right\}_0 + \theta\left\{\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}\right\}_1\right) \tag{20}$$

where $0 \leqslant \theta \leqslant 1$.

Elimination of $\{\mathrm{d}\mathbf{u}/\mathrm{d}t\}_0$ and $\{\mathrm{d}\mathbf{u}/\mathrm{d}t\}_1$ from equations (18) to (20) leads to the following recurrence equation after assembly between time steps '0' and '1'

$$([\mathbf{M_m}] + \theta\Delta t[\mathbf{K_c}])\{\mathbf{u}\}_1 = ([\mathbf{M_m}] - (1-\theta)\Delta t[\mathbf{K_c}])\{\mathbf{u}\}_0 \tag{21}$$

The above solution gives the distribution of excess pore pressure. The corresponding settlement distribution can be obtained at every time step from

$$s = \int_z^{D-z} m_v(p-u)\mathrm{d}z \tag{22}$$

Applying the above FE solution to Terzaghi equation (10) for layered systems will give the wrong excess pore pressure distribution. The solution is unable explicitly to model changes in the permeability k, and is therefore unable to enforce the interface flow continuity conditions given by equation (14). A numerical example is given in the Appendix to show that combining the permeability and coefficient of volume compressibility into a single coefficient of consolidation will give wrong excess pore pressure distributions.

Solution of the coupled equations

After solution by the Galerkin weighted residual method, equations (5) and (8) lead to the element matrix form

$$\begin{aligned} &[\mathbf{k_m}]\{\mathbf{s}\} + [\mathbf{c}]\{\mathbf{u}\} = \{\mathbf{f}\} \\ &[\mathbf{c}]^{\mathrm{T}}\left\{\frac{\mathrm{d}\mathbf{s}}{\mathrm{d}t}\right\} - [\mathbf{k_c}]\{\mathbf{u}\} = \{\mathbf{0}\} \end{aligned} \tag{23}$$

where $[\mathbf{k_m}]$ and $[\mathbf{k_c}]$ are the solid stiffness and fluid conductivity matrices. The matrix $[\mathbf{c}]$ is the connectivity matrix. $\{\mathbf{f}\}$ is the total force applied.

If $\{\Delta\mathbf{f}\}$ is the change in load between successive times, the incremental form of the first part of equation (23) is

$$[\mathbf{k_m}]\{\Delta\mathbf{s}\} + [\mathbf{c}]\{\Delta\mathbf{u}\} = \{\Delta\mathbf{f}\} \tag{24}$$

where $\{\Delta\mathbf{s}\}$ and $\{\Delta\mathbf{u}\}$ are the resulting changes in displacement and excess pore pressure respectively. Linear interpolation in time using the 'θ-method' yields

$$\{\Delta s\} = \Delta t\left((1-\theta)\left\{\frac{ds}{dt}\right\}_0 + \theta\left\{\frac{ds}{dt}\right\}_1\right) \quad (25)$$

and the second part of equation (23) can be written at the two time levels to give expressions for the derivatives which can then be eliminated to give the following incremental recurrence equations (e.g. Sandhu & Wilson, 1969; Griffiths, 1994)

$$\begin{bmatrix} [k_m] & [c] \\ [c]^T & -\theta\Delta t[k_c] \end{bmatrix}\begin{Bmatrix} \{\Delta s\} \\ \{\Delta u\} \end{Bmatrix} = \begin{Bmatrix} \{\Delta f\} \\ \Delta t[k_c]\{u\}_0 \end{Bmatrix} \quad (26)$$

At each time step, all that remains is to update the dependent variables using

$$\begin{aligned} \{s\}_1 &= \{s\}_0 + \{\Delta s\} \\ \{u\}_1 &= \{u\}_0 + \{\Delta u\} \end{aligned} \quad (27)$$

If applying equation (23) to a layered system, the FE solution not only enforces the excess pore pressure continuity, but also the interface flow continuity conditions (14), because the second line of equation (23) is derived from the flow continuity condition.

AVERAGE DEGREE OF CONSOLIDATION OF A LAYERED SYSTEM

The average degree of consolidation can be expressed in terms of either excess pore pressure or settlement.

If the initial (uniform) excess pore pressure is given by u_0 and the maximum drainage path by D, the average degree of consolidation defined by excess pore pressure is

$$U_{avp} = 1 - \frac{1}{D}\int_0^D \frac{u}{u_0}dz \quad (28)$$

and defined by settlement

$$U_{avs} = \frac{s_t}{s_u} \quad (29)$$

where s_u is the long-term (ultimate) settlement and s_t is the settlement at time t.

For layered systems, equations (28) and (29) become

$$U_{avp} = 1 - \frac{1}{D}\sum_{i=1}^{n}\int_{z_{i-1}}^{z_i}\frac{u_i}{u_0}dz \quad (30)$$

$$U_{avs} = \frac{\sum_{i=1}^{n} m_{vi}\int_{z_{i-1}}^{z_i}(u_0 - u_i)dz}{\sum_{i=1}^{n} h_i m_{vi} u_0} = 1 - \frac{\sum_{i=1}^{n} m_{vi}\int_{z_{i-1}}^{z_i} u_i dz}{\sum_{i=1}^{n} h_i m_{vi} u_0} \quad (31)$$

where h_i is the thickness of ith layer.

If m_{vi} is constant throughout the layers

$$\sum_{i=1}^{n} h_i m_{vi} u_0 = D m_v u_0 \quad (32)$$

$$\sum_{i=1}^{n} m_{vi}\int_{z_{i-1}}^{z_i} u_i dz = m_v \sum_{i=1}^{n}\int_{z_{i-1}}^{z_i} u_i dz \quad (33)$$

U_{avp} and U_{avs} will be exactly the same (e.g. Xie & Pan, 1995).

NUMERICAL EXAMPLES

Two 1D consolidation FE programs using two-node 'rod' elements were developed in the same style as program 9·3 in the text written by Smith & Griffiths (2004) (see Appendix for link to downloadable version of this program). The coupled program was based on equations (5) and (8) and was called 8·1_c. The uncoupled program was based on equation (16) and was called 8·1_uc. Program 8·1 from the same source was used for solving the Terzaghi equation (10). The results obtained by the uncoupled program were omitted since they are the same as the coupled ones. In all the following analyses, the time interpolation parameter $\theta = 0{\cdot}5$.

The example used by Schiffman & Stein (1970) was reanalysed. The soil profile shown in Fig. 1 consists of four compressible layers with double-drainage and properties shown in Table 1. An instantaneous applied load (u_0) of unit magnitude was applied and maintained constant with time. The thickness of the ith layer is given by h_i. A time step of $\Delta t = 1$ day was used.

The calculated results for a set of excess pore pressure at different times after loading are shown in Fig. 2. It can be seen that the coupled results are essentially the same as those presented by Schiffman & Stein (1970). Also plotted in Fig. 2 are the results which were obtained by Terzaghi's 1D equation (10) and the c_{vi} values listed in Table 1. The results are quite different from the coupled ones except at the earliest time of $t = 740$ days.

Figure 3 plots the average degree of consolidation as a function of time. It can be seen that the coupled approach predicts faster consolidation with respect both to settlement and excess pore pressure than the Terzaghi approach.

CONCLUDING REMARKS

The paper has examined the differences between coupled, uncoupled and Terzaghi 1D consolidation modelling by the FE method in layered systems. Results show that applying the Terzaghi FE equation to a layered system can lead to incorrect results because interface flow continuity conditions are violated. Coupled analyses show that the average degree of consolidation is different depending on whether it is defined by settlement or excess pore pressure for a layered system.

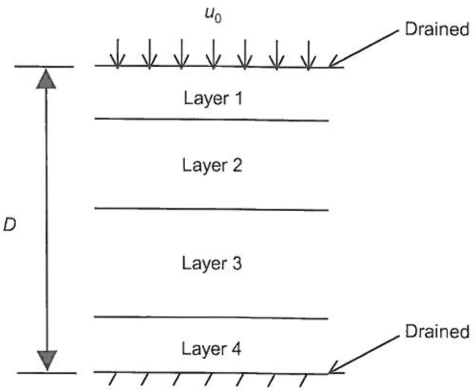

Fig. 1. Four-layer system

Table 1. Geotechnical data of the four-layer system (Schiffman & Stein, 1970)

i layer	h_i: m	k_i: m/s	m_{vi}: kPa^{-1}	c_{vi}: m^2/s
1	3·05	$2{\cdot}78 \times 10^{-7}$	$6{\cdot}41 \times 10^{-5}$	$4{\cdot}42 \times 10^{-8}$
2	6·10	$8{\cdot}25 \times 10^{-7}$	$4{\cdot}08 \times 10^{-5}$	$2{\cdot}06 \times 10^{-7}$
3	9·14	$1{\cdot}17 \times 10^{-7}$	$2{\cdot}04 \times 10^{-5}$	$5{\cdot}85 \times 10^{-8}$
4	6·10	$2{\cdot}94 \times 10^{-7}$	$4{\cdot}08 \times 10^{-5}$	$7{\cdot}35 \times 10^{-8}$

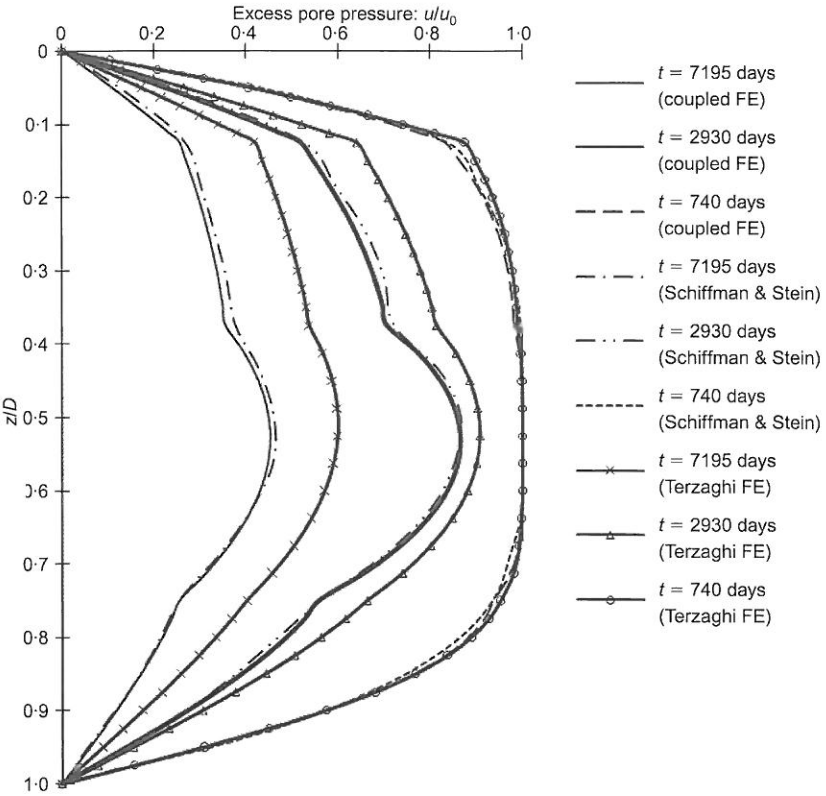

Fig. 2. Excess pore pressure isochrones

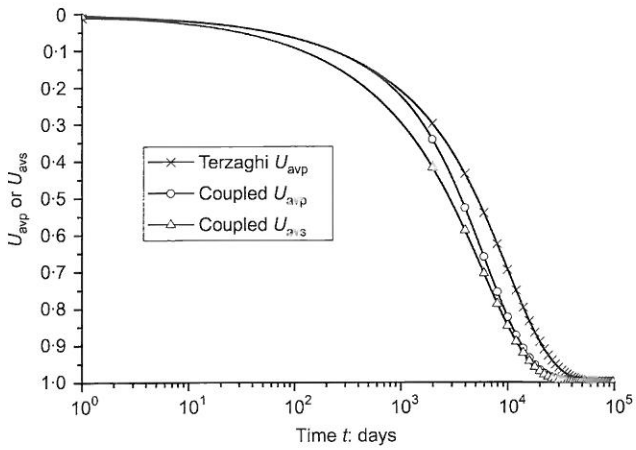

Fig. 3. Coupled and Terzaghi results of the four-layer system

When the uncoupled (excess pore pressure only) approach is used, equation (22) must be used to obtain the average degree of consolidation as defined by settlement, the only exception is when m_v is constant throughout the soil layers.

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APPENDIX

Demonstration of contrasting global FE matrices obtained from equations (10) and (16) for a two-layered system with the same c_v

A two-layer system is modelled by two 'rod' elements as shown in Fig. 4 with parameters given in Table 2. Note that the elements have the same c_v but different k/γ_w and m_v.

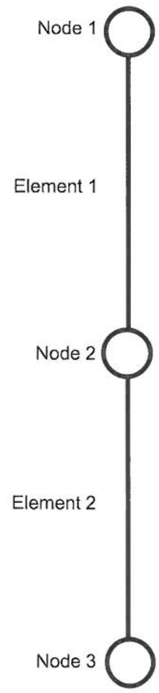

Fig. 4. A two-layer system

Table 2. Parameters of a two-layer system

Element	Length	k/γ_w	m_v	c_v
1	1	10	10	1
2	1	1	1	1

For the classical Terzaghi consolidation equation (equation (10)), the element fluid conductivity matrices are given as

$$[\mathbf{k}_c]_1 = [\mathbf{k}_c]_2 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad (34)$$

and the 'mass' matrices as

$$[\mathbf{m}_m]_1 = [\mathbf{m}_m]_2 = \begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix} \quad (35)$$

After assembly, the global fluid conductivity and 'mass' matrices are

$$[\mathbf{K}_c] = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \quad (36)$$

$$[\mathbf{M}_m] = \begin{bmatrix} 1/3 & 1/6 & 0 \\ 1/6 & 2/3 & 1/6 \\ 0 & 1/6 & 1/3 \end{bmatrix} \quad (37)$$

For the uncoupled equation (16), the element fluid conductivity matrices are given as

$$[\mathbf{k}_c]_1 = \begin{bmatrix} 10 & -10 \\ -10 & 10 \end{bmatrix} \quad (38)$$

$$[\mathbf{k}_c]_2 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad (39)$$

and the element 'mass' matrices as

$$[\mathbf{m}_m]_1 = \begin{bmatrix} 10/3 & 10/6 \\ 10/6 & 10/3 \end{bmatrix} \quad (40)$$

$$[\mathbf{m}_m]_2 = \begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix} \quad (41)$$

After assembly, the global fluid conductivity and 'mass' matrices are

$$[\mathbf{K}_c] = \begin{bmatrix} 10 & -10 & 0 \\ -10 & 11 & -1 \\ 0 & -1 & 1 \end{bmatrix} \quad (42)$$

$$[\mathbf{M}_m] = \begin{bmatrix} 10/3 & 10/6 & 0 \\ 10/6 & 11/3 & 1/6 \\ 0 & 1/6 & 1/3 \end{bmatrix} \quad (43)$$

Clearly equations (10) and (16) lead to different FE formulations and will therefore deliver different excess pore pressure distributions. The formulation given by equation (16) is the correct one for layered soils and the authors' program called p90_u.f95 can be downloaded from www.mines.edu/~vgriffit/4th_ed/source/chap09.

NOTATION

- c_v coefficient of consolidation
- $c_{v0\cdot5}$ coefficient of consolidation determined by 'log time method'
- $c_{v0\cdot9}$ coefficient of consolidation determined by 'root time method'
- $\{\mathbf{f}\}$ total force applied
- D maximum drainage path
- h_i thickness of ith layer
- i layer number
- k permeability
- $[\mathbf{k}_c]$ fluid conductivity matrices
- $[\mathbf{m}_m]$ 'mass' matrices
- m_v coefficient of volume compressibility
- n number of layers
- p total load on top of the soil layers
- s settlement
- s_t settlement at time t
- s_u long-term (ultimate) settlement
- T dimensionless 'time factor'
- t time
- U average degree of consolidation
- U_{avp} average degree of consolidation defined by excess pore pressure
- U_{avs} average degree of consolidation defined by settlement
- u excess pore pressure
- u_0 initial (uniform) excess pore pressure
- z depth
- γ_w unit weight of water
- θ time interpolation parameter, $0 \leq \theta \leq 1$

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