Semi-Analytical Integration of the Elastic Stiffness Matrix of an Axisymmetric Eight-Noded Finite Element

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The finite element method (FEM) is a numerical method for approximate solution of partial differential equations with appropriate boundary conditions. This work describes a methodology for generating the elastic stiffness matrix of an axisymmetric eight-noded finite element with the help of Computer Algebra Systems. The approach is described as "semi analytical" because the formulation mimics the steps taken using Gaussian numerical integration techniques. The semianalytical subroutines developed herein run 50% faster than the conventional Gaussian integration approach. The routines, which are made publically available for download,[1] should help FEM researchers and engineers by providing significant reductions of CPU times when dealing with large finite element models. Numer Methods Partial Differential Eq 26: 1624–1635, 2010

Keywords: axisymmetric quadrilateral finite elements; semi-analytical integration; stiffness matrix; symbolic manipulation

I. INTRODUCTION

Usually stiffness matrices integration of quadrilateral finite elements are carried out by using numerical integration. The explicit integration is only available for very simple elements (rectangles). Okabe [1] was the first researcher who presented explicit formulae to integrate rational integrals over a convex isoparametric quadrilateral having four nodes. These formulas require the

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[1] Source code of the program described in this paper can be downloaded from the second author's web site at www.mines.edu/~vgriffit/analytical

evaluation of terms of the form $\phi(\alpha\beta)\ln[f(\alpha,\beta)]/(\alpha^m\beta^n)$, where ϕ and f are rational functions and $m \geq 0, n \geq 0$. For quadrilateral elements having straight sides, usually encountered in practical applications, the parameters $|\alpha|$ and $|\beta|$ are small, thus leading to difficulties when evaluating the logarithmic terms. Babu and Pinter [2], based on the Okabe's work, presented semianalytical integration formulas to evaluate integrals over straight-sided quadrilateral elements, thus improving the accuracy of the results as compared with those obtained with Gaussian integration. Mizukami [3] worked with parallelograms and presented semianalytical integration formulas to generate the stiffness matrix. In this type of element, the Jacobian of the coordinate transformation is a constant function, which simplified the formulas presented by the author. No comparison on CPU times was carried out. As well, Rathod [4] generalized the results obtained by Mizukami [3] and Babu and Pinter [2], to present analytical integration formulas for the four-noded isoparametric finite element. To obtain those formulas, Rathod used basic methods of integration (integration by parts), thus transforming all the integrals involved to unidimensional integrals, which were further expressed as a linear combination of four basic integrals. Using the REDUCE [5] computer algebra system, Kikuchi [6] obtained explicit formulas for the integration of a four-noded finite element. Yagawa et al. [7] combined both analytical and numerical methods to integrate the stiffness matrix of a four node finite element in plane elasticity. These authors expanded and grouped the integrand, obtaining a 15% reduction in CPU time when compared against Gaussian numerical integration. Griffiths [8] presented a semianalytical formula to calculate the stiffness matrix of a four-noded plane elasticity finite element. This author classified the stiffness matrix terms into six groups, each one related to specified conditions of their degrees of freedom. The semianalytical expression was obtained by symbolic manipulation of the Gaussian integration technique for four points. He used the software MAPLE [9]. The technique lead to a significant reduction in CPU times. After, Videla et al. [10] generated analytical formulas to integrate the stiffness matrix of a four-noded plane elasticity finite element. These authors, by using symbolic manipulation of the partial derivatives of the shape functions, obtained general expressions for each one of them. The results were codified in Fortran and a reduction of 50% in CPU time was reported when compared with numerical Gaussian integration.

Lozada et al. [11] generalized Griffiths results and presented semianalytical formulas for the evaluation of the stiffness matrix of eight-noded finite elements in 2D elasticity. CPU times were improved about 37% when compared with numerical Gaussian integration. Videla et al. [12] presented explicit and analytical formulas for the stiffness matrix of eight-noded finite elements in plane elasticity. These authors reported 50% savings in CPU times, again compared with numerical integration.

This work extend the formulation obtained by Lozada et al. [11] to the integration of the eight-noded finite elements for axial symmetry.

II. FORMULATION

Three-dimensional solids having axial symmetry and loading axially symmetric can be analized by using 2D models. This work consider the quadrilateral finite element of eight nodes, with two DOF per node, to be used in axisymmetric problems. The element is depicted in Fig. 1.

By using the coordinate transformation (see Eq. 1), the element is transformed into a square element, as shown in Fig. 2.

The coordinate transformation between the plane rz and the plane $\xi\eta$ is given by

$$r = \sum_{i=1}^{8} N_i(\xi,\eta) r_i; \quad z = \sum_{i=1}^{8} N_i(\xi,\eta) z_i \tag{1}$$

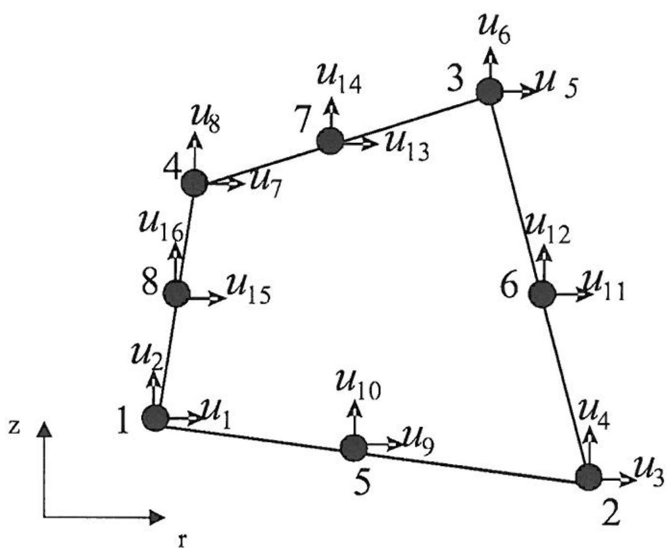

FIG. 1. Quadrilateral finite element.

where N_i is the element shape functions which interpolate both the displacements and element geometry. For eight-noded elements, the shape functions are:

$$
\begin{aligned}
N_1 &= -\frac{1}{4}(1-\xi)(1-\eta)(\xi+\eta+1); & N_2 &= -\frac{1}{4}(1+\xi)(1-\eta)(-\xi+\eta+1)\\
N_3 &= -\frac{1}{4}(1+\xi)(1+\eta)(-\xi-\eta+1); & N_4 &= -\frac{1}{4}(1-\xi)(1+\eta)(\xi-\eta+1)\\
N_5 &= \frac{1}{2}(1-\eta)(1-\xi^2); & N_6 &= \frac{1}{2}(1+\xi)(1-\eta^2)\\
N_7 &= \frac{1}{2}(1+\eta)(1-\xi^2); & N_8 &= \frac{1}{2}(1-\xi)(1-\eta^2)
\end{aligned}
\tag{2}
$$

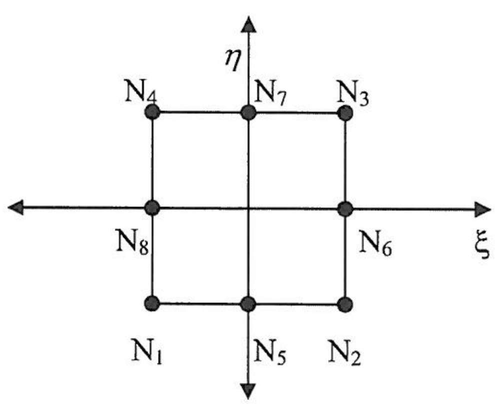

FIG. 2. Finite element in the Gauss-plane.

Now, the finite element formulation for axially symmetric problems, is summarized below

$$
\begin{aligned}
K_{ij} &= \iiint_e B_i^T D B_j dV \\
&= \iint_A \int_0^{2\pi} B_i^T D B_j r d\theta dA \\
&= 2\pi \iint_A B_i^T D B_j r dA \\
&= 2\pi \int_{-1}^{1}\int_{-1}^{1} r\ B_i^T(\xi,\eta)\ D\ B_j(\xi,\eta)\ \det J\ d\xi\ d\eta
\end{aligned} \tag{3}
$$

where

$$
B_i^T = \begin{bmatrix} \dfrac{\partial N_i}{\partial r} & 0 & \dfrac{N_i}{r} & \dfrac{\partial N_i}{\partial z} \\ 0 & \dfrac{\partial N_i}{\partial z} & 0 & \dfrac{\partial N_i}{\partial r} \end{bmatrix}; \quad D = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \dfrac{1-2\nu}{2} \end{bmatrix}
$$

$$
\underset{\sim}{u} = \{u \quad w\}^T; \quad \underset{\sim}{\varepsilon} = \{\varepsilon_r \quad \varepsilon_z \quad \varepsilon_\theta \quad 2\varepsilon_{rz}\}^T \tag{4}
$$

Here, J, E, and ν are the Jacobian matrix, Young modulus, and Poisson ratio, respectively. From (3) and (4) we obtain

$$
K_{ij} = 2\pi \int_{-1}^{1}\int_{-1}^{1} \begin{bmatrix} c_{11}^{ij} & c_{12}^{ij} \\ c_{21}^{ij} & c_{22}^{ij} \end{bmatrix} \frac{1}{det J} d\eta d\xi \tag{5}
$$

where

$$
c_{11}^{ij} = \left(\frac{(\det J)^2\ N_i N_j}{\sum_{i=1}^{n} r_i N_i} + \left(\sum_{i=1}^{n} r_i N_i\right) \tilde{t}_i\ \tilde{t}_j \right) E_1 + \det J\ (\tilde{t}_i N_j + \tilde{t}_j N_i)\ E_2 + \left(\sum_{i=1}^{n} r_i N_i\right) \tilde{s}_i\ \tilde{s}_j\ E_3 \tag{6}
$$

$$
c_{12}^{ij} = \left(\left(\sum_{i=1}^{n} r_i N_i\right) \tilde{s}_j\ \tilde{t}_i + \tilde{s}_j N_i \det J \right) E_2 + \left(\sum_{i=1}^{n} r_i N_i\right) \tilde{s}_i \tilde{t}_j\ E_3 \tag{7}
$$

$$
c_{21}^{ij} = \left(\left(\sum_{i=1}^{n} r_i N_i\right) \tilde{s}_i \tilde{t}_j + \tilde{s}_i N_j \det J \right) E_2 + \left(\sum_{i=1}^{n} r_i N_i\right) \tilde{t}_i\ \tilde{s}_j E_3 \tag{8}
$$

$$
c_{22}^{ij} = \left(\sum_{i=1}^{n} r_i N_i\right) (\tilde{s}_i \tilde{s}_j E_1 + \tilde{t}_i \tilde{t}_j E_3) \tag{9}
$$

and

$$\lambda = \frac{E}{(1+\nu)(1-2\nu)}; \quad E_1 = \lambda\,(1-\nu); \quad E_2 = \lambda\nu; \quad E_3 = \frac{\lambda\,(1-2\nu)}{2} \tag{10}$$

$$\tilde{t}_i = \left(\frac{\partial z}{\partial \eta}\frac{\partial N_i}{\partial \xi} - \frac{\partial z}{\partial \xi}\frac{\partial N_i}{\partial \eta}\right) = \det J\left(\frac{\partial N_i}{\partial r}\right) \tag{11}$$

$$\tilde{s}_i = \left(-\frac{\partial r}{\partial \eta}\frac{\partial N_i}{\partial \xi} + \frac{\partial r}{\partial \xi}\frac{\partial N_i}{\partial \eta}\right) = \det J\left(\frac{\partial N_i}{\partial z}\right) \tag{12}$$

The analytical integration of Eq. 5 is very complex and, thus, computer codes use Gaussian numerical integration to evaluate it. Now, by using Gaussian integration of order 2×2, the following semianalytical expression is obtained:

$$k_{ij} = 3\left\{\frac{A_3(E_1S_1 + E_2S_2 + E_3S_3) + f_1(E_1S_4 + E_2S_5 + E_3S_6)}{3A_3^2 - f_1^2} + \frac{A_3(E_1T_1 + E_2T_2 + E_3T_3) + f_2(E_1T_4 + E_2T_5 + E_3T_6)}{3A_3^2 - f_2^2}\right\} \tag{13}$$

where

$$f_1 = (r_1 + r_3)(z_4 - z_2) - (z_1 + z_3)(r_4 - r_2) - 2(r_2z_4 - r_4z_2) \tag{14}$$

$$f_2 = (z_2 + z_4)(r_3 - r_1) - (r_2 + r_4)(z_3 - z_1) - 2(r_3z_1 - r_1z_3) \tag{15}$$

$$A_3 = \frac{1}{8}[(z_2 - z_4)(r_1 - r_3) + (z_3 - z_1)(r_2 - r_4)] \tag{16}$$

The functions S_i and T_i $(i = 1\ldots6)$ depend on the Cartesian coordinates. These functions were calculated by using symbolic manipulation and they are used to generate the parent terms (see Eqs. 17–26). For instance, the functions used to generate the term $k_{2,2}$ (Eq. 17) in the eight-noded element are as follows:

$$S_1 = \frac{1}{96}(r_4 - r_2)^2(7r_1 + 2(r_2 + r_4) + r_3)$$

$$S_2 = 0$$

$$S_3 = \frac{1}{96}(z_4 - z_2)^2(7r_1 + 2(r_2 + r_4) + r_3)$$

$$S_4 = \frac{1}{96}(r_4 - r_2)^2(4r_1 + r_2 + r_4)$$

$$S_5 = 0$$

$$S_6 = \frac{1}{96}(z_4 - z_2)^2(4r_1 + r_2 + r_4)$$

$$T_1 = \frac{1}{2592}\big(2r_1^3 + 7r_2^3 + 8r_3^3 + 12r_3^2r_4 - 18r_3r_4^2 + 7r_4^3 + 6r_1^2(r_2 + r_4 - r_3) - 3r_2^2(6r_3 + r_4) + 3r_2\left(4r_3^2 + 2r_3r_4 - r_4^2\right) + 6r_1\left(2r_2^2 - r_2(3r_3 + r_4) + r_4(-3r_3 + 2r_4)\right)\big)$$

$$T_2 = 0$$

$$T_3 = \frac{1}{2592}\left(2\left(z_1^2 + 4z_3(z_3 - z_1) - z_2z_4\right)(r_1 + 2(r_2 + r_4) + r_3) + 2((r_1 + r_3) + 5r_4 - r_2)(z_1 - 2z_3)z_4 + 2\,(-r_1 - 5r_2 + r_4 - r_3)(2z_3 - z_1)z_2 + (2(r_1 + r_3) + r_4 + 7r_2)\,z_2^2 + (2(r_1 + r_3) + 7r_4 + r_2)z_4^2\right)$$

$$T_4 = \frac{-1}{2592}(r_4 - r_2)\left(4\left(r_1^2 + r_2^2 + r_3^2 + r_4^2\right) + 7r_1(r_2 + r_4) + 2r_2r_4 - 10r_1r_3 - 11r_3(r_2 + r_4)\right)$$

$$T_5 = 0$$

$$T_6 = \frac{1}{2592}\left(2(3(r_2 + r_4) + r_1 + r_3)(2z_3z_4 - z_1z_4) + 2(6r_4 - r_2)z_2z_4 + 2z_2(z_1 - 2z_3)(r_1 + r_4 + r_3 + 3r_2) + 8(r_4 - r_2)z_1z_3 + (r_1 + r_3 + 4r_2)z_2^2 - 2(r_4 - r_2)z_1^2 - 8z_3^2(r_4 - r_2) - z_4^2(r_1 + r_3 + 4r_4)\right)$$

III. STIFFNESS MATRIX TERMS GENERATION

We consider now the classification of terms presented by Lozada et al. [11] for the eight-noded element, as shown in Table I:

TABLE I. Stiffness matrix terms classification in eight-noded element.

Group	Terms	Description	Degree of adjacency
A	$k_{1,1}, k_{2,2}, k_{3,3}, k_{4,4}, k_{5,5}, k_{6,6}, k_{7,7}, k_{8,8}, k_{9,9}, k_{10,10}, k_{11,11}, k_{12,12}, k_{13,13}, k_{14,14}, k_{15,15}, k_{16,16}$	Parallel DOF at the same node	0
B	$k_{1,2}, k_{3,4}, k_{5,6}, k_{7,8}, k_{9,10}, k_{11,12}, k_{13,14}, k_{15,16}$	Orthogonal DOF at the same node	0
C	$k_{1,3}, k_{3,5}, k_{5,7}, k_{1,7}, k_{9,11}, k_{11,13}, k_{13,15}, k_{9,15}, k_{2,4}, k_{4,6}, k_{6,8}, k_{2,8}, k_{10,12}, k_{12,14}, k_{14,16}, k_{10,16}$	Parallel DOF at nodes separated by one node.	2
D	$k_{2,3}, k_{3,6}, k_{6,7}, k_{2,7}, k_{10,11}, k_{11,14}, k_{14,15}, k_{10,15}, k_{4,5}, k_{5,8}, k_{1,8}, k_{1,4}, k_{12,13}, k_{13,16}, k_{9,16}, k_{9,12}$	Orthogonal DOF at nodes separated by one node	2
E	$k_{1,5}, k_{9,13}, k_{3,7}, k_{11,15}, k_{2,6}, k_{4,8}, k_{10,14}, k_{12,16}$	Parallel DOF at opposite nodes	4
F	$k_{1,6}, k_{9,14}, k_{3,8}, k_{11,16}, k_{2,5}, k_{10,13}, k_{4,7}, k_{12,15}$	Orthogonal DOF at opposite nodes	4
G	$k_{1,9}, k_{3,9}, k_{3,11}, k_{5,11}, k_{5,13}, k_{7,13}, k_{7,15}, k_{1,15}, k_{2,10}, k_{4,10}, k_{4,12}, k_{6,12}, k_{6,14}, k_{8,14}, k_{8,16}, k_{2,16}$	Parallel DOF at adjacent nodes	1
H	$k_{1,10}, k_{3,10}, k_{3,12}, k_{5,12}, k_{5,14}, k_{7,14}, k_{7,16}, k_{1,16}, k_{2,9}, k_{4,9}, k_{4,11}, k_{6,11}, k_{6,13}, k_{8,13}, k_{8,15}, k_{2,15}$	Orthogonal DOF at adjacent nodes	1
I	$k_{1,11}, k_{7,11}, k_{7,9}, k_{5,9}, k_{5,15}, k_{3,15}, k_{3,13}, k_{1,13}, k_{2,12}, k_{8,12}, k_{8,10}, k_{6,10}, k_{6,16}, k_{4,16}, k_{4,14}, k_{2,14}$	Parallel DOF at nodes separated by two nodes	3
J	$k_{2,11}, k_{8,11}, k_{8,9}, k_{6,9}, k_{6,15}, k_{4,15}, k_{4,13}, k_{2,13}, k_{1,12}, k_{7,12}, k_{7,10}, k_{5,10}, k_{5,16}, k_{3,16}, k_{3,14}, k_{1,14}$	Orthogonal DOF at nodes separated by two nodes	3

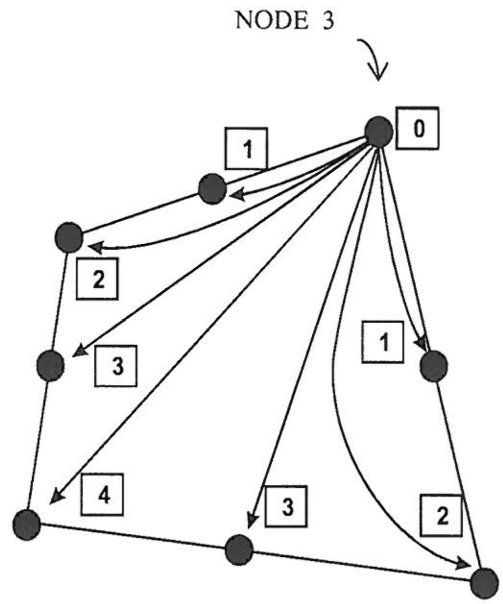

FIG. 3. Degrees of adjacency for node 3 and DOF in each node.

In the classification above, the symmetry of the stiffness matrix was considered. The eight-noded element was classified into 10 groups, considering the adjacency between DOF in the element nodes and the type of DOF (parallel or orthogonal). To illustrate this, Fig. 3 shows the different degrees of adjacency of node 3 with other nodes.

To generate the stiffness matrix terms, the following simple coordinate transformations are used, which modify only the node position and not the element geometry (Table II). The notation used here is that the symbol $\Leftarrow$ means "is overwritten by"

Thus, given any term in a group (parent term), the other terms of this group can be obtained by using the functions S_i and T_i (generating functions) and coordinate transformations. By comparing the number of algebraic operations carried out by the generating functions of each term in a group, it is more efficient to generate each group using several parent terms, as the number of algebraic operations needed by functions S_i and T_i are different for each group.

In what follows, the way the stiffness matrix terms in each group are obtained, using Eq. 13 and the transformation R is shown:

TABLE II. Simple coordinate transformation.

Transformation (Rotation R)	
Transformation	Terms affected
$(r_1, z_1) \Leftarrow (r_4, z_4)$	f_1, f_2
$(r_2, z_2) \Leftarrow (r_1, z_1)$	$S_1, S_2, S_3, S_4, S_5, S_6$
$(r_3, z_3) \Leftarrow (r_2, z_2)$	$T_1, T_2, T_3, T_4, T_5, T_6$
$(r_4, z_4) \Leftarrow (r_3, z_3)$	

Group A (parent terms: $k_{1,1}$, $k_{2,2}$, $k_{9,9}$, and $k_{10,10}$)

$$k_{1,1} \rightarrow k_{7,7} \rightarrow k_{5,5} \rightarrow k_{3,3}$$
$$k_{2,2} \rightarrow k_{8,8} \rightarrow k_{6,6} \rightarrow k_{4,4}$$
$$k_{9,9} \rightarrow k_{15,15} \rightarrow k_{13,13} \rightarrow k_{11,11}$$
$$k_{10,10} \rightarrow k_{16,16} \rightarrow k_{14,14} \rightarrow k_{12,12} \quad (17)$$

Group B (parent terms: $k_{1,2}$ and $k_{9,10}$)

$$k_{1,2} \rightarrow k_{7,8} \rightarrow k_{6,5} \rightarrow k_{4,3}$$
$$k_{9,10} \rightarrow k_{15,16} \rightarrow k_{13,14} \rightarrow k_{11,12} \quad (18)$$

Group C (parent terms: $k_{1,3}, k_{6,8}, k_{9,11}$ and $k_{10,12}$)

$$k_{1,3} \rightarrow k_{1,7} \rightarrow k_{5,7} \rightarrow k_{3,5}$$
$$k_{6,8} \rightarrow k_{6,4} \rightarrow k_{2,4} \rightarrow k_{2,8}$$
$$k_{9,11} \rightarrow k_{9,15} \rightarrow k_{13,15} \rightarrow k_{11,13}$$
$$k_{10,12} \rightarrow k_{10,16} \rightarrow k_{14,16} \rightarrow k_{12,14} \quad (19)$$

Group D (parent terms: $k_{1,4}, k_{1,8}, k_{9,12}$, and $k_{14,15}$)

$$k_{1,4} \rightarrow k_{2,7} \rightarrow k_{5,8} \rightarrow k_{3,6}$$
$$k_{1,8} \rightarrow k_{6,7} \rightarrow k_{4,5} \rightarrow k_{2,3}$$
$$k_{9,12} \rightarrow k_{10,15} \rightarrow k_{13,16} \rightarrow k_{11,14}$$
$$k_{14,15} \rightarrow k_{12,13} \rightarrow k_{10,11} \rightarrow k_{9,16} \quad (20)$$

Group E (parent terms: $k_{1,5}$, $k_{2,6}$, $k_{9,13}$, and $k_{12,16}$)

$$k_{1,5} \rightarrow k_{3,7}$$
$$k_{2,6} \rightarrow k_{4,8}$$
$$k_{9,13} \rightarrow k_{11,15}$$
$$k_{12,16} \rightarrow k_{10,14} \quad (21)$$

Group F (parent terms: $k_{1,6}$ and $k_{9,14}$)

$$k_{1,6} \rightarrow k_{4,7} \rightarrow k_{2,5} \rightarrow k_{3,8}$$
$$k_{9,14} \rightarrow k_{12,15} \rightarrow k_{10,13} \rightarrow k_{11,16} \quad (22)$$

Group G (parent terms: $k_{7,15}$, $k_{7,13}$, $k_{2,10}$, and $k_{6,12}$)

$$\begin{aligned}
&k_{7,15} \rightarrow k_{5,13} \rightarrow k_{3,11} \rightarrow k_{1,9} \\
&k_{7,13} \rightarrow k_{5,11} \rightarrow k_{3,9} \rightarrow k_{1,15} \\
&k_{2,10} \rightarrow k_{8,16} \rightarrow k_{6,14} \rightarrow k_{4,12} \\
&k_{6,12} \rightarrow k_{4,10} \rightarrow k_{2,16} \rightarrow k_{8,14}
\end{aligned} \tag{23}$$

Group H (parent terms: $k_{2,9}$, $k_{6,11}$, $k_{1,10}$, and $k_{5,12}$)

$$\begin{aligned}
&k_{2,9} \rightarrow k_{8,15} \rightarrow k_{6,13} \rightarrow k_{4,11} \\
&k_{6,11} \rightarrow k_{4,9} \rightarrow k_{2,15} \rightarrow k_{8,13} \\
&k_{1,10} \rightarrow k_{7,16} \rightarrow k_{5,14} \rightarrow k_{3,12} \\
&k_{5,12} \rightarrow k_{3,10} \rightarrow k_{1,16} \rightarrow k_{7,14}
\end{aligned} \tag{24}$$

Group I (parent terms: $k_{1,11}$, $k_{5,9}$, $k_{2,12}$, and $k_{6,10}$)

$$\begin{aligned}
&k_{1,11} \rightarrow k_{7,9} \rightarrow k_{5,15} \rightarrow k_{3,13} \\
&k_{5,9} \rightarrow k_{3,15} \rightarrow k_{1,13} \rightarrow k_{7,11} \\
&k_{2,12} \rightarrow k_{8,10} \rightarrow k_{6,16} \rightarrow k_{4,14} \\
&k_{6,10} \rightarrow k_{4,16} \rightarrow k_{2,14} \rightarrow k_{8,12}
\end{aligned} \tag{25}$$

Group J (parent terms: $k_{2,11}$, $k_{2,13}$, $k_{1,12}$, and $k_{1,14}$)

$$\begin{aligned}
&k_{2,11} \rightarrow k_{8,9} \rightarrow k_{6,15} \rightarrow k_{4,13} \\
&k_{2,13} \rightarrow k_{8,11} \rightarrow k_{6,9} \rightarrow k_{4,15} \\
&k_{1,12} \rightarrow k_{7,10} \rightarrow k_{5,16} \rightarrow k_{3,14} \\
&k_{1,14} \rightarrow k_{7,12} \rightarrow k_{5,10} \rightarrow k_{3,16}
\end{aligned} \tag{26}$$

where the groups of terms of stiffness matrix are arranged as follows:

$$k = \begin{bmatrix}
A & B & C & D & E & F & G & H & I & J & G & H & E & F & C & D \\
 & A & D & C & F & E & H & G & J & I & H & G & F & E & D & C \\
 & & A & B & C & D & E & F & G & H & I & J & G & H & E & F \\
 & & & A & D & C & F & E & H & G & J & I & H & G & F & E \\
 & & & & A & B & C & D & E & F & G & H & I & J & G & H \\
 & & & & & A & D & C & F & E & H & G & J & I & H & G \\
 & & & & & & A & B & C & D & E & F & G & H & I & J \\
 & & & & & & & A & D & C & F & E & H & G & J & I \\
 & & & & & & & & A & B & C & D & E & F & G & H \\
 & & & & & & & & & A & D & C & F & E & H & G \\
 & & & & & \text{symmetrical} & & & & & A & B & C & D & E & F \\
 & & & & & & & & & & & A & D & C & F & E \\
 & & & & & & & & & & & & A & B & C & D \\
 & & & & & & & & & & & & & A & D & C \\
 & & & & & & & & & & & & & & A & B \\
 & & & & & & & & & & & & & & & A
\end{bmatrix} \tag{27}$$

In Eq. 13, the coefficient A_3 represents the element area. Moreover, if we denote as $L_1 = f_1(r_1, \dots, r_4, z_1, \dots, z_4)$ and $L_2 = f_2(r_1, \dots, r_4, z_1, \dots, z_4)$ the values of the functions f_1 and f_2 when evaluated on the original coordinates, then by applying the rotations to the finite element, we obtain:

$$f_1 = \begin{cases} L_2 & \text{with one rotation} \\ -L_1 & \text{with two rotations} \\ -L_2 & \text{with three rotations} \end{cases} \quad \text{and} \quad f_2 = \begin{cases} L_1 & \text{with one rotation} \\ -L_2 & \text{with two rotations} \\ -L_1 & \text{with three rotations} \end{cases}$$

IV. VALIDATION OF SEMIANALYTICAL FORMULATION

In this section, semianalytical results will be compared with those obtained using Gaussian numerical integration of order 2×2. The differences are calculated as follows:

$$\text{Error} = \frac{\sqrt{\sum_{i,j} \left(s_{ij}^* - s_{ij}^{EX}\right)^2}}{\sum_{i,j} \left|s_{ij}^*\right|} \tag{28}$$

where s_{ij}^*, stiffness matrix terms obtained with numerical integration and s_{ij}^{EX}, stiffness matrix terms obtained with semi-analytical integration.

Table III displays the differences obtained when evaluating the finite element shown in Fig. 4. The results are essentially the same with any differences occurring due to machine rounding.

TABLE III. Difference between numerical and semianalytical formulations.

a	b	Difference
10	10	0.158075×10^{-7}
10	100	0.220665×10^{-7}
10	1000	0.232560×10^{-7}

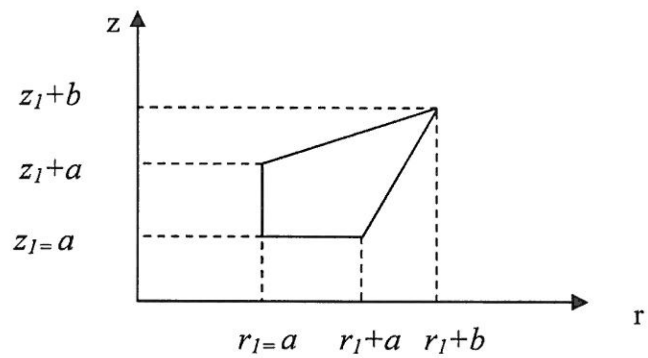

FIG. 4. Test finite element dimensions.

V. CPU TIMES

To evaluate the reduction in CPU times, a processor HP NXG120 Centrino 1.86 GHZ with 1 GB RAM was used. The CPU times by both techniques (Gaussian integration with 2 × 2 points and semianalytical) are given in Table IV.

Results show that the semianalytical formulation for the stiffness matrix of an axisymmetric eight-node element is significantly more efficient than numerical integration, leading to CPU savings about 50%.

VI. CONCLUDING REMARKS

A general methodology using symbolic algebra has been presented for evaluating the stiffness matrix of an elastic axisymmetric eight-noded finite element. The stiffness matrix terms were obtained using a semianalytical approach, that followed the steps of a Gaussian integration algorithm symbolically and took full advantage of simple coordinate transformations and symmetry to populate the matrix. The proposed approach gives exactly the same results as Gaussian integration with 2 × 2 sampling points, but runs 50% faster. This improvements described in this article would be even more impressive if dealing with large nonlinear and dynamic FEM analysis, where regular reformulations of the element stiffness matrices are called for.

VII. SOFTWARE

Source code of the program described in this article, and others generated with the help of Computer Algebra Systems can be downloaded from the second author's web site at www.mines.edu/~vgriffit/analytical.

TABLE IV. CPU times comparison.

Number of elements	Numerical integration (s)	Semianalytical integration (s)	Saving (%)
10,000	1.67	0.84	49.5
100,000	16.04	8.28	48.3
1,000,000	161.44	82.51	48.8

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