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Observations on FORM in a simple geomechanics example

Jinsong Huang *, D.V. Griffiths [1]

Division of Engineering, Colorado School of Mines, Golden, CO 80401, USA

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ABSTRACT

This technical note studies the First Order Reliability Method (FORM) applied to a plane strain Mohr–Coulomb drained (c', tan ϕ') element test. The influences of distribution types and linear correlation between random variables are studied. The approximation of assuming a "first order" limit state function is assessed by comparison with direct integration of the probability distribution function in the failure region. The results indicate that FORM overestimates p_f when random variables are lognormally distributed and underestimate p_f when random variables are normally distributed.

1. Introduction

Many workers in the reliability field are increasingly using the Hasofer–Lind reliability index [4], referred to as FORM (e.g., [6,8]). The Hasofer–Lind reliability index is defined as the shortest distance from the origin to the failure surface in the space of reduced variables. An advantage of the Hasofer–Lind FORM is that it avoids the non-uniqueness observed in FOSM (First Order Second Moment method) associated with different formulations of the factor of safety (or limit state function). These issues have been discussed in detail elsewhere (e.g. [1,7]).

In this paper we describe a simple model of a drained triaxial (CD) test on a c', ϕ' soil, where the sample is subjected to radial (confining) stress followed by an increase in axial stress to failure. We assume two random variables c' and tan ϕ' and investigate the influences of distribution types and the linear correlation between the properties on the probability of failure. The approximation of assuming a "first order" limit state function is also studied by comparing results to those obtained using direct integration of the joint Probability Density Function (PDF) in the region of failure. Results obtained by Second Order Reliability Method (SORM) [2] are also provided for comparison. The results indicate that FORM overestimates p_f when random variables are lognormal distributed and underestimate p_f when random variables are normal distributed.

2. Normal and lognormal distributions

The best known PDF is the Normal or Gaussian distribution. If X is normally distributed with mean and standard deviation, μ_X and σ_X, the PDF is given by:

$$f_X(x) = \frac{1}{\sigma_X\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2\right\} \quad (1)$$

If X is lognormally distributed with mean and standard deviation, μ_X and σ_X, and coefficient of variation

$$v_X = \frac{\sigma_X}{\mu_X} \quad (2)$$

the PDF is given by:

$$f_X(x) = \frac{1}{x\sigma_{\ln X}\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{\ln X-\mu_{\ln X}}{\sigma_{\ln X}}\right)^2\right\} \quad (3)$$

where the mean and standard deviation of ln X are given by:

$$\sigma_{\ln X} = \sqrt{\ln\{1+v_X^2\}} \quad (4)$$

$$\mu_{\ln X} = \ln\mu_X - \frac{1}{2}\sigma_{\ln X}^2 \quad (5)$$

3. Reliability computation

The conventional FORM based on the Hasofer–Lind reliability index [4], β_{HL}, assumes that the mean values of the random variables lie on the safe side of the limit state function. The method

* Corresponding author. Tel.: +1 303 273 3173; fax: +1 303 273 3602.
E-mail addresses: jinsong.huang@newcastle.edu.au (J. Huang), d.v.griffiths@mines.edu (D.V. Griffiths).
[1] Tel.: +1 303 273 3669; fax: +1 303 273 3602.

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then obtains the reliability index, which is related to the minimum distance between the origin and the failure surface in the space of reduced variables. Calculation of the reliability index involves an iterative optimization process as shown in Eq. (7) below. Commonly used software packages (e.g. Excel and Matlab) are easily adapted to perform the optimization (see e.g. http://www.mines.edu/~vgriffit/FORM for examples of Excel spreadsheets). Once the reliability index (the distance between the means and the closest failure point) has been determined, the method assumes a "first order" limit state function tangent to the β_{HL} contour, and the probability of failure, p_f follows from

$$p_f = 1 - \Phi(\beta_{HL}) \tag{6}$$

where $\Phi(\beta_{HL})$ is the cumulative standard normal distribution function evaluated at β_{HL}.

The determination of β_{HL} is an iterative process defined by

$$\beta_{HL} = \min_{g=0} \sqrt{\left\{\frac{X_i' - \mu_i^N}{\sigma_i^N}\right\}^T [R]^{-1} \left\{\frac{X_i' - \mu_i^N}{\sigma_i^N}\right\}} \quad i = 1, 2, \ldots, n \tag{7}$$

where X_i' is the i^{th} random variable, μ_i^N is the equivalent normal mean of the i^{th} random variable, σ_i^N is the equivalent normal standard deviation of the i^{th} random variable, $\{(X_i' - \mu_i^N)/\sigma_i^N\}$ is a vector of n random variables reduced to standard normal space, $[R]$ is a matrix of correlations coefficients between the standard normal variables and g is the limit state function ($g < 0$ means failure).

FORM approximates the limit state function by a linear function at the design point. Accuracy problems can arise however, when the limit state function is strongly nonlinear, in which case SORM can improve the accuracy by using a second order approximation of the limit state surface at the design point (e.g., [7]). In order to estimate the probability of failure in the SORM, Breitung [2] provided an approximate formula based on asymptotic analysis as follows

$$p_f = \Phi(-\beta_{HL}) \prod_{i=1}^{n-1} (1 + \beta_{HL}\kappa_i)^{-1/2} \tag{8}$$

where κ_i are the principal curvatures of the limit state function.

The software used in this paper for the FORM and SORM implementations was developed by the authors using Matlab.

4. Limit state for a single element drained test

Assuming any system of consistent units, Fig. 1 shows a single element of soil subjected to lateral and axial stresses of $\sigma_3' = 1.0$ and σ_1' respectively. The drained strength parameters are to be treated as random variables with means of $\mu_{c'} = 1.0$ and $\mu_{\tan\phi'} = 0.3$. Although the coefficients of variation of cohesion and friction may typically lie in the range 0.05–0.5 (e.g., [5]), in the present study for the sake of illustration, the coefficients of variation of both strength parameters have been assumed to be equal and varied in the range $0 < v_{c'} = v_{\tan\phi'} \leqslant 1.5$, with a linear correlation set equal to zero ($\rho = 0$).

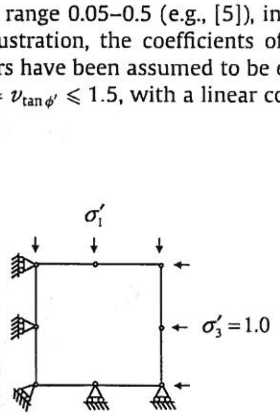

Fig. 1. Single element test.

Given the Mohr–Coulomb failure criterion

$$\sigma_{1f}' = \sigma_3' \tan^2(45 + \phi'/2) + 2c' \tan(45 + \phi'/2) \tag{9}$$

and defining the factor of safety as

$$FS = \frac{\sigma_{1f}'}{\sigma_1'} \tag{10}$$

Table 1 shows the variation in the factor of safety FS for different values of axial compressive stress σ_1' with the shear strength parameters fixed at their mean values.

In order to estimate the reliability using FORM/SORM with random variables c' and $\tan\phi'$, the Mohr–Coulomb failure criterion can be written as a limit state function of the form [3]

$$g(c', \tan\phi') = \sigma_3' - \sigma_1' + (\sigma_1' + \sigma_3')\frac{\tan\phi'}{\sqrt{1 + \tan^2\phi'}} + \frac{2c'}{\sqrt{1 + \tan^2\phi'}} = 0 \tag{11}$$

The influences of distribution type and FS on p_f are shown in Fig. 2, where it can be noted that for a given value of FS, there is a critical value of v below which the lognormal distribution gives lower values of p_f than the normal distribution. Fig. 2 also shows that the critical v increases as FS increases.

5. Assessment of the first order approximation

The first order approximation assumed in FORM could lead to an underestimation of the probability of failure if the actual limit state function curves towards the safe region. A more accurate, yet more time consuming, method of determining the probability of failure is to numerically integrate the probability distribution function in the region of failure. The results of direct integrations are compared to FORM in Figs. 3 and 4 and it can be seen that FORM overestimates p_f when the random variables are lognormally distributed and underestimates p_f when the random variables are normal distributed. Also plotted in Figs. 3 and 4 (dashed lines)

Table 1
FS vs. σ_1' ($\mu_{c'} = 1.0$, $\mu_{\tan\phi'} = 0.3$, $\sigma_3' = 1.0$).

FS	σ_1'
1.00	4.50
1.25	3.60
1.50	3.00
1.75	2.57

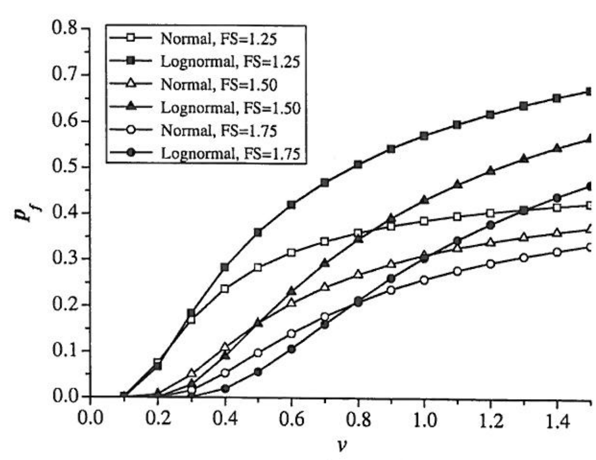

Fig. 2. Influence of normal and lognormal distributions on p_f.

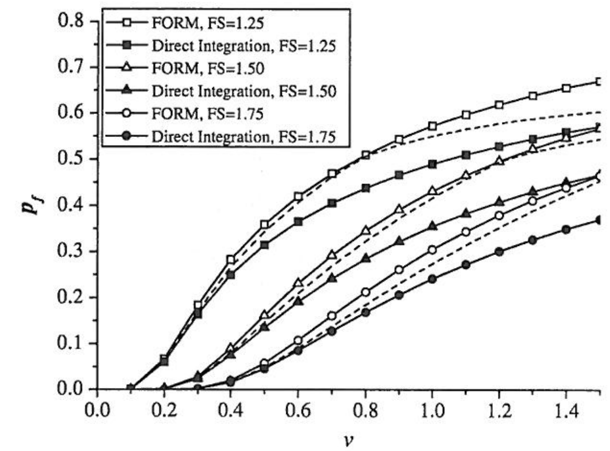

Fig. 3. FORM vs. direct integration (lognormal distribution, dashed lines obtained by SORM).

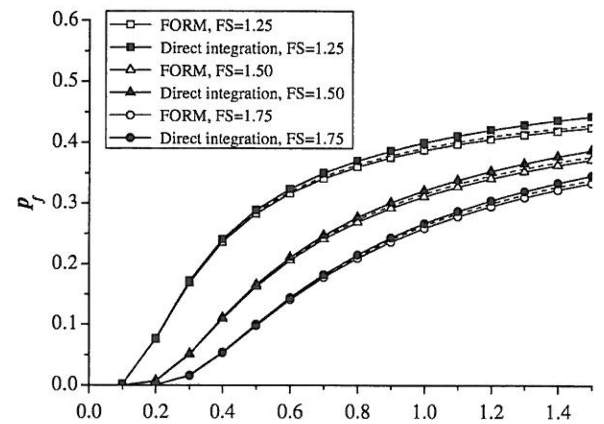

Fig. 4. FORM vs. direct integration (normal distribution, dashed lines obtained by SORM).

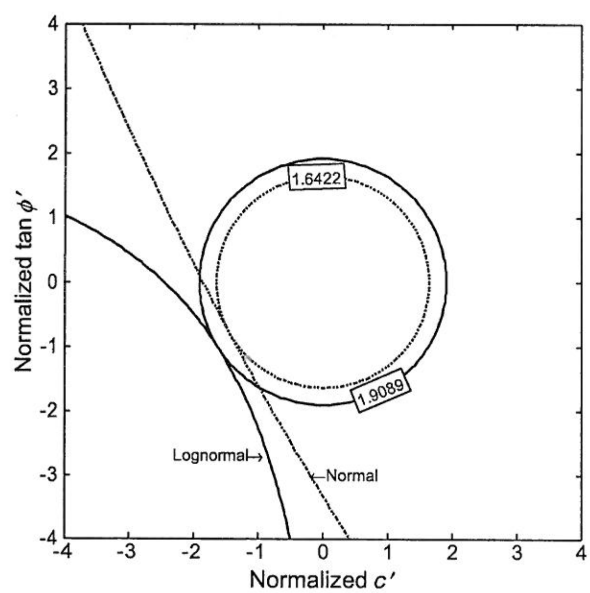

Fig. 5. Comparison of normal and lognormal distributions in normalized space (v = 0.3, FS = 1.5). Lognormal distribution gives lower p_f. SORM gives reliability indices of 1.6353 and 1.9605 for normal and lognormal distributions respectively.

are the results obtained by SORM [2]. It can be seen that SORM can improve the accuracy of FORM by taking into account the curvatures of the limit state function at the design point.

This observation is explained for normal and lognormal random variables, for v = 0.3 and 0.7 in Figs. 5 and 6 respectively. In these figures, the limit state functions, given as the loci of $g = 0$, have been plotted with respect to the normalized random variables [7], together with optimized contours of the reliability index β_{HL}. Using FORM, the design points correspond to the smallest contour of β_{HL} that is tangent to the limit state functions.

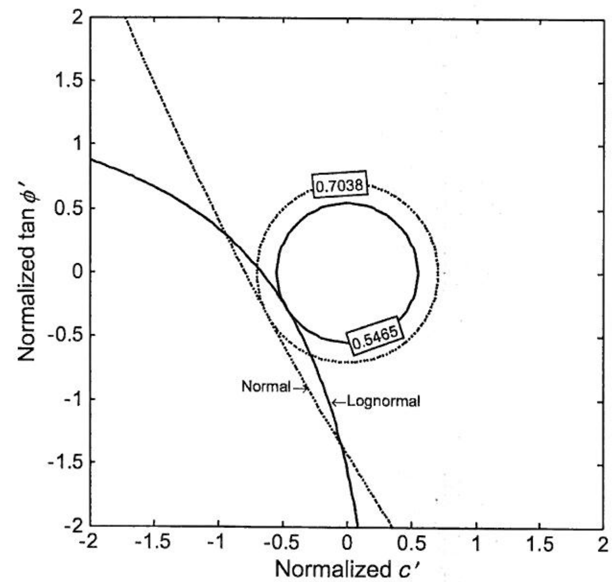

Fig. 6. Comparison of normal and lognormal distributions in normalized space (v = 0.7, FS = 1.5). Lognormal distribution gives higher p_f. SORM gives reliability indices of 0.6927 and 0.6136 for normal and lognormal distributions respectively.

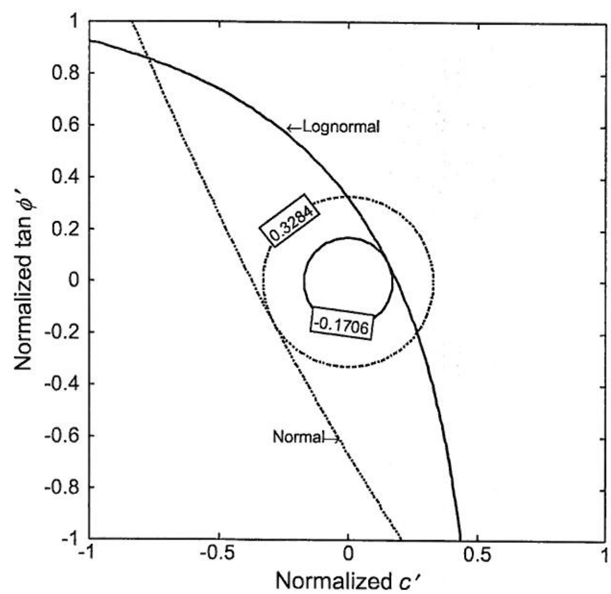

Fig. 7. Comparison of normal and lognormal distributions in normalized space (v = 1.5, FS = 1.5). Lognormal distribution gives significantly higher $p_f > 0.5$. SORM gives reliability indices of 0.3143 and −0.1130 for normal and lognormal distributions respectively.

In both figures it can be seen that when the random variables are normally distributed, the limit state function curves towards the safe region (albeit very slightly in Fig. 5) and the first order approximation underestimates p_f. Conversely, if the random variables are lognormally distributed, the limit state function curves away from the unsafe region and the first order approximation overestimates p_f.

For significantly higher values of v, the means of the lognormal distribution may even lie in the unsafe region as shown in Fig. 7, leading to a negative β_{HL} contour.

It is interesting to redraw the results in real variable space as shown in Figs. 8–10 respectively. In normalized space, Figs. 5–7 indicated fixed locations for the contours of β_{HL}, while the limit state function was shifted for different values of v. This is in contrast to Figs. 8–10 in real space, where the limit state function is fixed and the contours of β_{HL} are shifted for different values of v.

6. Influence of linear correlation

All the previous results assumed no correlation ($\rho = 0$) between c' and $\tan\phi'$. In this section we demonstrate the influence of ρ as shown in Fig. 11. It can be seen that the greater the value of ρ, the lower the reliability index β_{HL} and hence the higher the probability of failure p_f. This is because when $\rho > 0$ the major axis of the elliptical contours of β_{HL} point towards the limit state function, thus the point of tangency occurs at a lower value of β_{HL}.

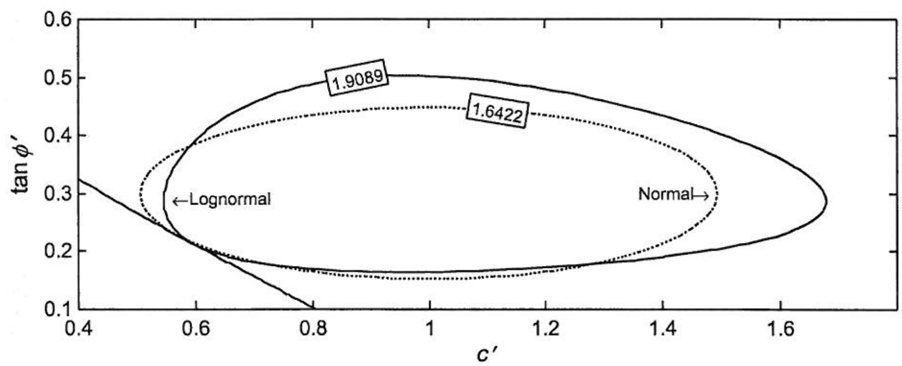

Fig. 8. Comparison of normal and lognormal distributions in real space ($v = 0.3$, $FS = 1.5$). Lognormal distribution gives lower p_f.

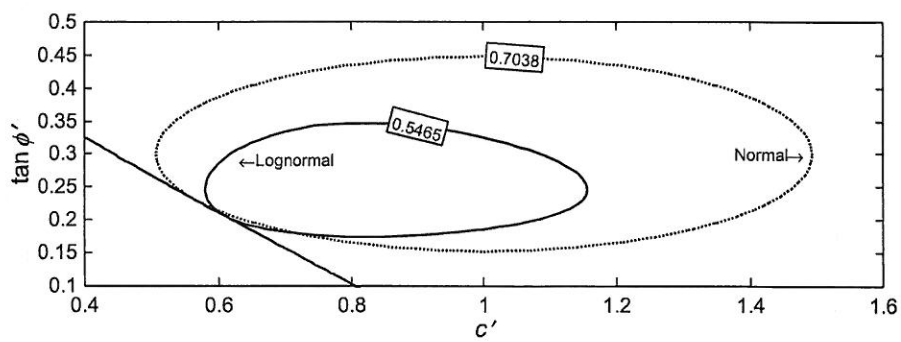

Fig. 9. Comparison of normal and lognormal distributions in real space ($v = 0.7$, $FS = 1.5$). Lognormal distribution gives higher p_f.

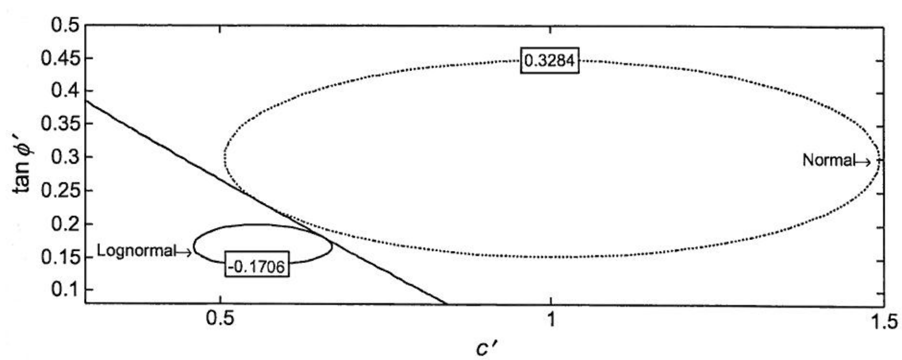

Fig. 10. Comparison of normal and lognormal distributions in real space ($v = 1.5$, $FS = 1.5$). Lognormal distribution gives significantly higher $p_f > 0.5$.

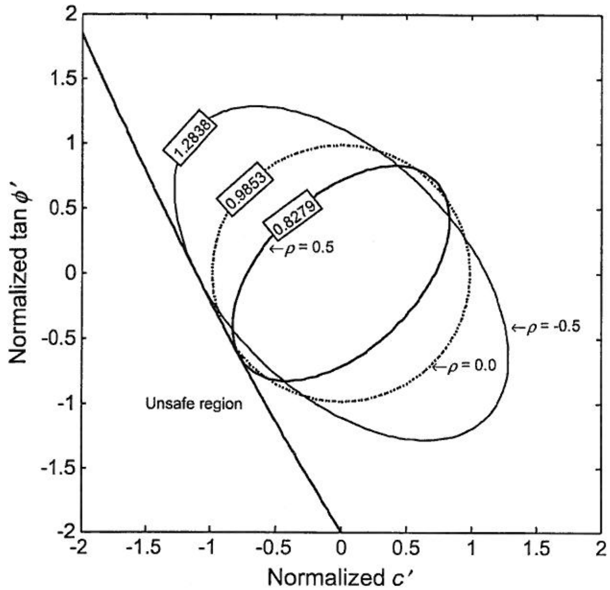

Fig. 11. Influence of linear correlation on β_{HL} ($v = 0.5$, $FS = 1.5$, normal distribution).

7. Concluding remarks

This technical note has made some observations on results given by the First Order Reliability Method (FORM) when applied to a simple geomechanics problem involving drained triaxial compression of a Mohr–Coulomb soil with random c' and $\tan\phi'$. Results indicated that a lognormal distribution of these variables gives lower probabilities of failure than a normal distribution (for the same mean values) when v is relatively low, and higher values when v is relatively high. The accuracy of FORM, which assumes a first order limit state function, was studied in comparison to SORM, which assumes a second order limit state function and with the most accurate solution of all given by direct integration of the probability distribution on the failure side of the limit state function. The results indicated that FORM overestimates p_f when random variables are lognormally distributed and underestimates p_f when random variables are normally distributed. In all cases, SORM gave results that lie between FORM and direct integration. Although it is well documented that positive correlation between c' and $\tan\phi'$ leads to higher probabilities of failure than negative correlation, this trend has been demonstrated in a novel way by plotting the reliability contours for different levels of correlation in normalized space.

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