Discretization Errors of Random Fields in Finite Element Analysis

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Abstract. The mechanical properties of natural materials such as rocks and soils vary spatially. This randomness is usually modelled by random field theory so that the material properties can be specified at each point in space. When these point-wise material properties are mapped onto a finite element mesh, discretization errors are inevitable. In this study, the discretization errors are studied and suggestions for element sizes in relation with spatial correlation lengths are given.

Introduction

The handling of uncertainties is a research area of great importance and interest within civil engineering and material engineering. Uncertainty is related to the inherent randomness involved in material and geometric properties of engineering structures and systems. The present work concentrates on issues regarding the modeling of uncertainties in material properties and investigates the influence of such uncertainties on overall structural performance. Randomness in material properties arises from random microstructure and phases distribution at micro scale (typically nano-meters to micro-meters). Ideally, we can directly model micro scale randomness to predict structure performance (macro scale). But this is obviously too computational demanding to be practical. On the other hand, we can use experimental tests on real structures to decide the performance/capacity of the structures. This is not practical either because the structures are usually expensive. In most cases, we have to rely on a meso scale (i.e., the size of element in finite element method (FEM)), in which the micro scale randomness is homogenized, to predict macro scale performance. This is where the concept of Representative Volume Element (RVE) comes from. In order for the equivalent continuum to be a meaningful representation of a heterogeneous body, the three scales have to satisfy the Micro-Meso-Macro (MMM) principle as shown in Fig. 1 (Hashin 1983).

Micro scale (nano to micro meters) (i.e., material phases)	<<	Meso scale (centimeters) (i.e., finite elements)	<<	Macro scale (meters to kilometres) (i.e., structures)

Fig. 1 Micro, Meso and Macro scales

At micro scale, material properties are random due to random microstructures and phase distributions. This randomness may or may not propagate into meso scale. When meso scale is large enough to include all micro randomness, the overall material properties at meso scale are spatially constant. This is usually the cases in material engineering. The second condition of MMM principle (i.e., meso scale is much less than macro scale) can be satisfied. In civil engineering especially in geotechnical engineering, however, most of the time the material properties at meso scale show significant spatial variability. When the scale of spatial fluctuation is close to the macro scale, the second condition of MMM principle cannot be satisfied. This is because, on one hand, the meso scale needs to be much larger than the scale of spatial fluctuation to include enough randomness (the first condition of MMM principle). On the other hand, according to the second condition of MMM principle, the meso scale should be much smaller than macro scale. To resolve this dilemma, instead of chasing a meso scale, random field theory is usually used to capture micro scale randomness. The starting point for a discussion of random field is the "point" statistics that are assumed for the model. These are the hypothetical statistical properties of the soil/rock that might be measured if very many tests could be performed at a site on very small samples of the soil. Numerous studies have been undertaken in recent years to develop probabilistic methods that deal with spatial variability in a systematic way (e.g., Griffiths et al. 2009; Huang et al. 2010). Of particular importance has been the development of the random finite element (RFEM) to model the spatial variability of soil properties (e.g., Fenton and Griffiths 2008). Despite these investigations, the discretization error still remains unexplored; therefore, the validity of the corresponding simulation results is questionable. The major aim of the present work is to stimulate a systematic effort towards establishing a more formal and complete link between the maximum element size and the scale of spatial fluctuation.

The Karhunen-Loeve expansion method

There are several random field generation methods available (see, for example, Fenton and Griffiths 2008). In Random Finite Element Method (RFEM) (Fenton and Griffiths 2008), each element is given a constant property (i.e. no property variation is assumed across an individual element), hence a proper local averaging strategy has been included to take account of this. The Karhunen-Loeve expansion method was chosen in this study because it doesn't require discretization at the stage of random field generation. In Karhunen-Loeve expansion method, random fields are generated as functions of point coordinates. This allows us to study the discretization errors, which will be shown in the next section.

Let $X(\mathbf{x},\omega)$ be a random field, where $\mathbf{x} \in D$ (physical space) and $\omega \in \Omega$ (a probability space). The covariance function, denoted as $C_X(\mathbf{s},\mathbf{t})$, where $\mathbf{s},\mathbf{t} \in D$, is bounded, symmetric and positively defined. Using Mercer's Theorem, it can be decomposed according to

$$C_X(\mathbf{s},\mathbf{t}) = \sum_{i=1}^{\infty} \lambda_i f_i(\mathbf{s}) f_i(\mathbf{t}) \tag{1}$$

where λ_i and $f_i(\mathbf{x})$ are the eigenvalues and eigenfunctions of $C_X(\mathbf{s},\mathbf{t})$, respectively.

The eigenfunctions $C_X(\mathbf{s},\mathbf{t})$ form a complete orthogonal set satisfying

$$\int_D f_i(\mathbf{s}) f_i(\mathbf{t}) \mathrm{d}x = \delta_{ij} \tag{2}$$

Based on Eq. (2), the eigenvalues and eigenfunctions of $C_X(\mathbf{s},\mathbf{t})$ are the solutions of the following Fredholm equation:

$$\int_D C_X(\mathbf{s},\mathbf{t}) f(\mathbf{s}) = \lambda f(\mathbf{t}) \tag{3}$$

Numerical methods are usually required to solve Eq. (3), although exact solutions are available for some classes of covariance function (see, for example, Ma and Zabaras 2008, Zhang and Lu 2004) presented an analytical solution for an exponential covariance function. The computation involves only the solution of a one-dimensional super characteristic equation (i.e., (7)).

The exponential covariance function in one dimension is

$$C_X(x_1,x_2) = \sigma_X^2 \exp\left(\frac{-|x_1-x_2|}{\theta_x}\right) \quad (4)$$

where σ_X is the standard deviation and θ_x is the spatial correlation length. θ_x is also called the scale of fluctuation, which is equal to twice the so-called autocorrelation distance.
The eigenvalues and their corresponding eigenfunctions can be expressed as

$$\lambda_i = \frac{2\theta_x\sigma_X^2}{\theta_x{}^2 w_i^2 + 1} \quad (5)$$

$$f_i(x) = \frac{1}{\sqrt{\frac{1}{2}\left(\theta_x{}^2 w_i^2 + 1\right) + \theta_x}}\left[\theta_x w_i \cos(w_i x) + \sin(w_i x)\right] \quad (6)$$

where w_n are positive roots of the characteristic equation:

$$\theta_x\left(\theta_x{}^2 w^2 - 1\right)\sin(wL) = 2\theta_x w \cos(wL) \quad (7)$$

and L is the length of the random field.

For two-dimensional problems, Eq. (4) can be written as

$$C_X\left((x_1,y_1),(x_2,y_2)\right) = \sigma_X^2 \exp\left(\frac{-|x_1-x_2|}{\theta_x}\right)\exp\left(\frac{-|y_1-y_2|}{\theta_y}\right) \quad (8)$$

and Eq. (3) can be solved independently for each dimension yielding the eigenvalues and eigenfunctions as

$$\lambda_{i,j} = \lambda_i^{(x)}\lambda_j^{(y)} \quad (9)$$

$$f_{i,j}(x,y) = f_i(x) f_j(y) \quad (10)$$

Example

A standard biaxial test is considered in this section. The sample is assumed to fail under drained conditions. The Mohr-Coulomb failure criterion is used in the FEM calculations. The effective friction angle is assumed to be lognormally distributed and modelled as a random field. The mean and standard deviation of effective friction angle are 30° and 15°. All other parameters are assumed to be deterministic. Effective cohesion (c') was set to zero. A non-associated flow rule was assumed (zero dilation angle). Young's Modulus was fixed at 10^5 $\mathrm{kN/m^2}$. Poisson's ratio was set to 0.4. The sample (see Fig. 2) is initially confined isotropically to 10 $\mathrm{kN/m^2}$ and then subjected to a uniform vertical displacement with an increment of 10^{-7} m. Eight-node plane strain elements with reduced integration scheme were used. The return mapping algorithm for stress integration of plasticity was used (e.g., Huang and Griffiths 2009).The axial stress at failure by Mohr-Coulomb is 30.0 $\mathrm{kN/m^2}$.

One single realization of random field ($L = 1.0\mathrm{m}$, $\theta = 0.125\mathrm{m}$) is analysed. Square elements were used with side length l changing in the range $\{L/8,\ L/16, L/32, L/64, L/128\}$ meters. Random fields of friction angle were generated for each Gauss points (4/element) as shown in Fig. 3. Each square with greyscales in Fig. 3 represents a Gauss point (not an element). Fig. 3 also shows the deformations at failure. The failure loads were calculated from stresses in the top layer Gauss points. Fig. 4 shows the failure loads from different mesh densities. It is noted that $l/L = 1/64$ would give stable results. Further investigations were carried out for larger spatial correlation lengths. The results are shown in Fig. 5. It is noted that for $\theta/L \ge 0.25$, $l/L = 1/16$ can give stable results.

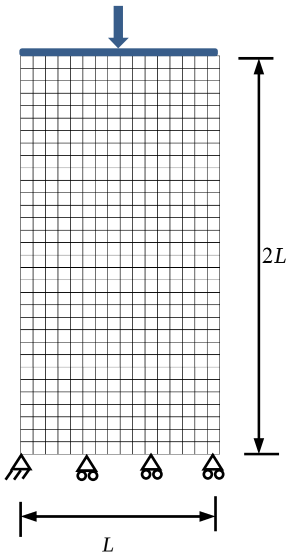
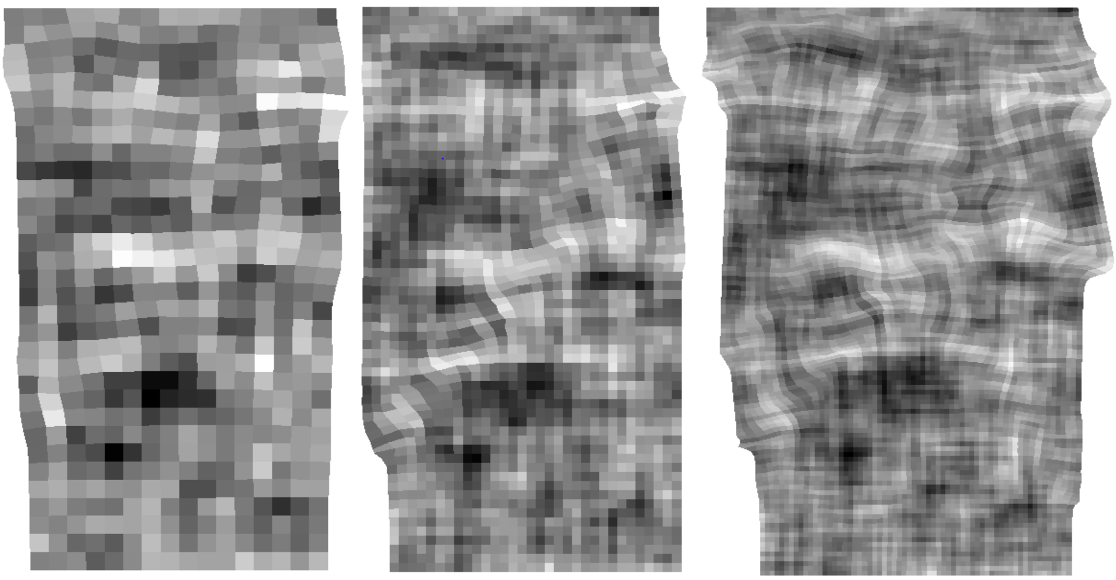
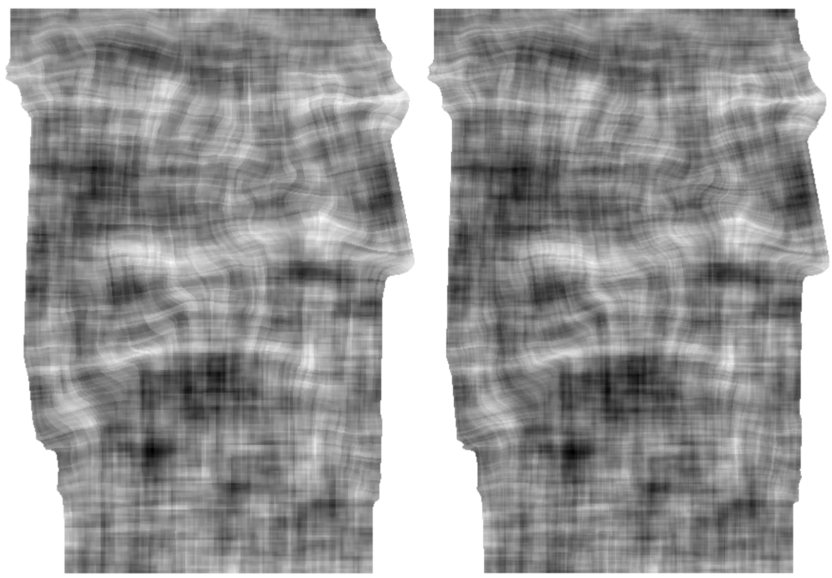


Fig. 2 Biaxial test



a) $l / L = 8$ b) $l / L = 16$ c) $l / L = 32$



d) $l / L = 64$ e) $l / L = 128$

Fig. 3 Deformations at failure ($L = 1.0$m, θ=0.125m)

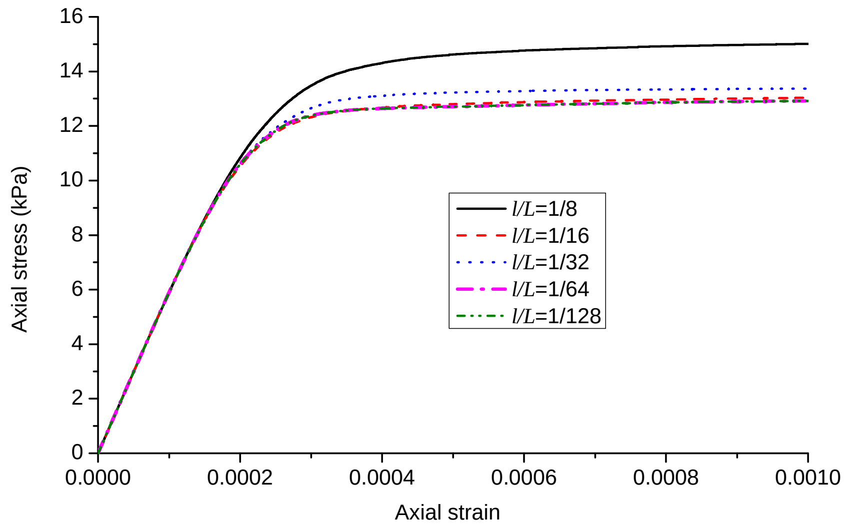


Fig. 4 Axial stress verse axial strain ($L = 1.0$m, θ=0.125m)

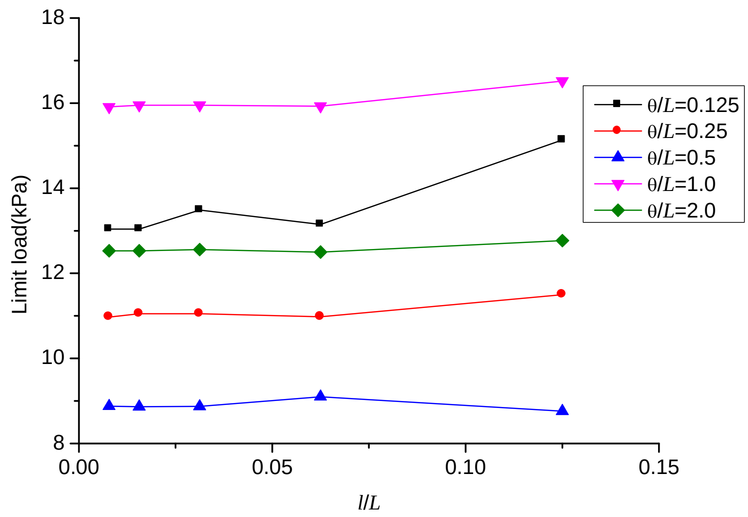


Fig. 5 Influence of spatial correlation length and element size on limit loads.

Concluding remarks

When the scale of spatial fluctuation of material properties is close to the size of structures, the second condition of MMM principle cannot be satisfied. Instead of homogenizing micro randomness, random field theory is usually used to capture micro randomness. This study shows that the element size in random field discretization should be much smaller than the scale of spatial fluctuation. Further statistical investigations on the maximum allowable element size in relation with scale of fluctuation are under way.

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