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BEARING CAPACITY OF HETEROGENEOUS SOILS BY FINITE ELEMENTS

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ABSTRACT: By merging elasto-plastic finite element analysis with random field theory, an investigation has been performed into the bearing capacity of undrained clays with statistically varying shear strength. The main issue is to determine the extent to which variance and correlation length of the soil shear strength impacts on the statistics of the computed bearing capacity. For low coefficients of variation in the soil shear strength, the expected value of the bearing capacity tends to the Prandtl solution of $N_c = 5.14$. For high values of COV however, the expected value of the bearing capacity can fall quite steeply. The results of Monte-Carlo simulations on this nonlinear problem are presented in the form of histograms which enable the interpretation to be expressed in a probabilistic context.

INTRODUCTION

The paper presents a sample of results obtained using code recently developed by the authors called rbear2d. The program computes the bearing capacity of a smooth rigid strip footing (plane strain) at the surface of a soil with statistically defined shear strength parameters. The particular study described in this paper concentrates on undrained clay soil with a variable undrained shear strength c_u ($\phi_u = 0$).

The distribution of the undrained shear strength is assumed to be lognormal with the following three parameters:

Mean	μ_{c_u}
Standard Deviation	σ_{c_u}
Scale of fluctuation	θ_{c_u}

The Scale of Fluctuation refers to the distance over which the spatially random values will tend to be strongly correlated in the underlying normal field. Thus, a large value of θ_{c_u} will imply a smoothly varying field, while a small value will imply a ragged field.

In the parametric studies that follow, the mean has generally been held constant while the standard deviation and scale of fluctuation are varied. The value of the standard deviation is conveniently expressed in terms of the dimensionless Coefficient of Variation defined:

$$C.O.V._{c_u} = \frac{\sigma_{c_u}}{\mu_{c_u}} \tag{1}$$

It has been suggested by Lee *et al* 1983 and Kulhawy *et al* 1991 that $C.O.V._{c_u}$ for actual soils lies in the range 0.2-0.5.

For each set of statistics given by $C.O.V._{c_u}$ and θ_{c_u}, Monte-Carlo simulations have been performed, typically involving 1000 realizations of the random field. Each realization computes a value of bearing capacity, which in turn can be analyzed statistically.

Of particular interest in the present study, is the *probability* that the bearing capacity, q_f, will be *less* than the deterministic value, q_{det}, that would be obtained assuming a homogeneous soil with undrained shear strength everywhere equal to the mean value, μ_{c_u}.

Theoretically, a smooth strip footing on a homogeneous undrained clay of shear strength μ_{c_u} will have a bearing capacity given by the Prandtl solution, where:

$$q_{det} = N_c \, \mu_{c_u} \tag{2}$$

$$N_c = 5.14 \tag{3}$$

BRIEF DESCRIPTION OF FE METHOD USED

The bearing capacity analysis uses an elastic-perfectly plastic stress-strain law with a Tresca failure criterion. Plastic stress redistribution is accomplished using a viscoplastic algorithm. The program uses 8-node quadrilateral elements and reduced integration in both the stiffness and stress redistribution parts of the algorithm. The theoretical basis of the method is described more fully in Chapter 6 of the text by Smith and Griffiths (1998). To download the source code for the (deterministic) bearing capacity program `p60.f90`, see web site:

`http://magma.Mines.EDU/fs_home/vgriffit/3rd_ed/chap6/`

The model incorporates three parameters; Young's modulus (E), Poisson's ratio (ν) and the undrained shear strength (c_u). The program rbear2d allows for statistical distributions of all three parameters, however in the present study, E and ν are held constant while only c_u is randomized. A typical mesh is shown in Figure 1.

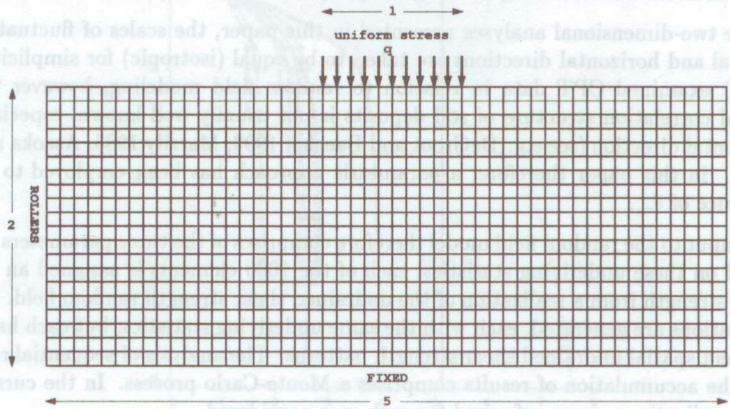

Fig 1. Mesh used in probabilistic bearing capacity analyses

The mesh consisted of 1000 elements, with 50 columns and 20 rows. Each element was a square of side length 0.1 units. The strip footing occupied 10 elements, giving a footing width of 1 unit.

At each realization of the Monte-Carlo process, the footing is incrementally displaced vertically (δ_v) into the soil and the sum of the nodal reactions (q) back-figured from the converged stress state. When the sum of the nodal reactions levels out to within quite strict tolerances, "failure" is said to have occurred and the sum of the nodal reactions is considered to be the "bearing capacity" (q_f) of that particular realization.

BRIEF DESCRIPTION OF THE OF THE RANDOM FIELD MODEL

The undrained shear strength is obtained through the transformation

$$c_{u_i} = \exp\{\mu_{\ln c_u} + \sigma_{\ln c_u}\, g_i\} \tag{4}$$

in which c_{u_i} is the undrained shear strength assigned to the ith element, g_i is the local average of a standard Gaussian random field, g, over the domain of the ith element, and $\mu_{\ln c_u}$ and $\sigma_{\ln c_u}$ are the mean and standard deviation of the logarithm of c_u (obtained from the 'point' mean and standard deviation μ_{c_u} and σ_{c_u}).

The LAS technique (Fenton 1990, Fenton and Vanmarcke 1990) generates realizations of the local averages g_i which are derived from the random field g having zero mean, unit variance, and a spatial correlation controlled by the scale of fluctuation, θ_{c_u}. As the scale of fluctuation goes to infinity, g_i becomes equal to g_j for all elements i and j – that is the field of permeabilities tends to become uniform on each realization. At the other extreme, as the scale of fluctuation goes to zero, g_i and g_j become independent for all $i \neq j$ – the soil's undrained shear strength changes rapidly from point to point.

In the two-dimensional analyses presented in this paper, the scales of fluctuation in the vertical and horizontal directions are taken to be equal (isotropic) for simplicity. Fenton (1999) examined CPT data in relation to random field modeling, however the actual spatial correlation structure of soil deposits is not usually well known, especially in the horizontal direction (see e.g. DeGroot and Baecher 1993, Marsily 1985, Asaoka and Grivas 1982). In this paper therefore, a parametric approach has been employed to study the influence of θ_{c_u}.

The input to the random field model therefore comprises of the three parameters $(\mu_{c_u}, \sigma_{c_u}, \theta_{c_u})$. Based on these underlying statistics, each of the 1000 elements is assigned an undrained shear strength from a realization of the undrained shear strength random field. A series of realizations are generated, each with the same underlying statistics, but each having quite different spatial undrained shear strength patterns. The analysis of sequential realizations and the accumulation of results comprises a Monte-Carlo process. In the current study, 1000 realizations were performed for each case considered.

Following Monte-Carlo simulation of each parametric combination, 1000 values of the bearing capacity q_f were obtained, which were then analyzed statistically to give the mean, standard deviation and probability of low values that might lead to an unconservative design based on the mean value.

The plane strain model used herein implies that the out-of-plane scale of fluctuation is infinite – soil properties are constant in this direction. This is clearly a deficiency, however previous studies by the authors (Griffiths and Fenton 1997) involving seepage through two- and three-dimensional random fields has indicated that the difference may not be very great. The role of the third dimension is an area of ongoing research by the authors.

PARAMETRIC STUDIES

Analyses were performed using the mesh of Figure 1 with the parameters taking the following values:

$$\begin{aligned} \theta_{c_u} &= 0.5,\ 1,\ 2,\ 4,\ 8,\ \infty \\ C.O.V._{c_u} &= 0.125,\ 0.25,\ 0.5,\ 1,\ 2,\ 4,\ 8 \end{aligned} \qquad (5)$$

Load/deformation results for 10 realizations are shown in Figure 2 for the case when $\theta_{c_u} = 1$ and $C.O.V._{c_u} = 1$. The load has been non-dimensionalized by dividing the footing stress (q) by the mean undrained shear strength μ_{c_u}. The bold vertical line corresponds to the Prandtl solution of $N_c = 5.14$, which gives the theoretical bearing capacity factor of a homogeneous soil. It is clear that the majority of the computed bearing capacity values fall below the Prandtl value. This trend will be confirmed in all the results shown in this paper.

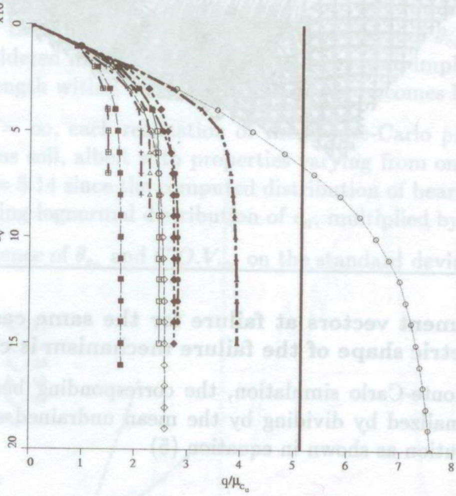

Fig 2. Typical load/deformation curves corresponding to different realizations in the bearing capacity analysis of an undrained clay with $\theta_{c_u} = 1$ and $C.O.V._{c_u} = 1$.

Figure 3 shows a typical deformed mesh at failure with a superimposed greyscale in which lighter regions indicate stronger soil and darker regions indicate weaker soil. It is clear in this case that the weak (dark) region near the ground surface to the right of the footing has triggered a quite non-symmetric failure mechanism.

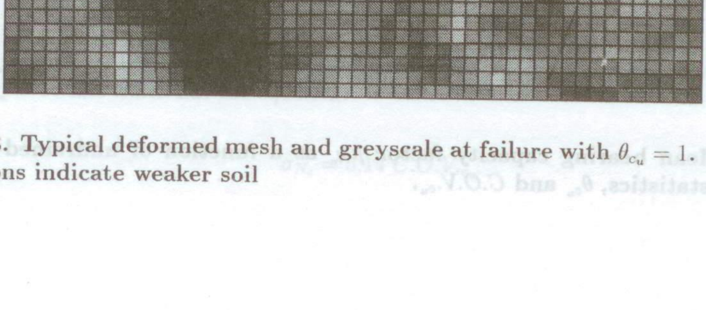

Fig 3. Typical deformed mesh and greyscale at failure with $\theta_{c_u} = 1$. The darker regions indicate weaker soil

The shape of the mechanism is emphasised further by the plot of displacement vectors for the same realization shown in Figure 4.

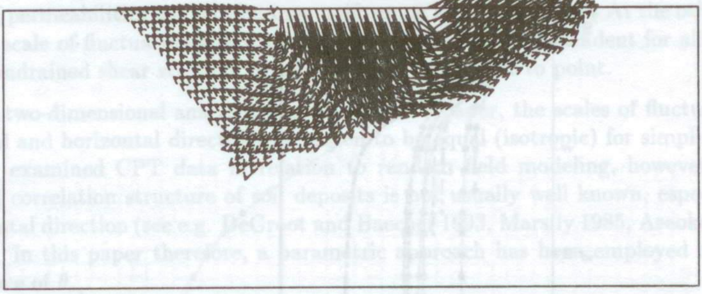

Fig 4. Displacement vectors at failure for the same case shown in Figure 3. The non-symmetric shape of the failure mechanism is clearly visible.

Following each Monte-Carlo simulation, the corresponding bearing capacity value, q_f, was non-dimensionalized by dividing by the mean undrained shear strength μ_{c_u} of the underlying distribution as shown in equation (5)

$$N_c = \frac{q_f}{\mu_{c_u}} \quad (6)$$

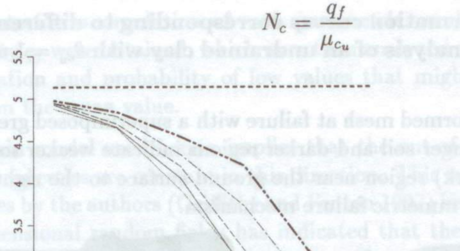

Fig 5. Mean bearing capacity factor μ_{N_c} as a function of undrained shear strength statistics, θ_{c_u} and $C.O.V._{c_u}$.

Figure 5 shows how the mean bearing capacity factor μ_{N_c} varies with θ_{c_u} and $C.O.V._{c_u}$. The plot confirms that for low values of $C.O.V._{c_u}$, μ_{N_c} tends to the deterministic Prandtl value of 5.14. For higher values of $C.O.V._{c_u}$, however, the mean bearing capacity factor falls steeply, e.g. when $\theta_{c_u} = 0.5$ and $C.O.V._{c_u} = 4$, μ_{N_c} value is less than unity–more than five times smaller than the Prandtl value! The influence of θ_{c_u} is more subtle, at least for the range considered in this paper. As θ_{c_u} is increased, implying greater spatial correlation of shear strength within the soil, the fall in μ_{N_c} becomes less pronounced.

In the limit, when $\theta_{c_u} = \infty$, each realization of the Monte-Carlo process is essentially analyzing a homogeneous soil, albeit with properties varying from one realization to the next. In this case, $\mu_{N_c} = 5.14$ since the computed distribution of bearing capacity will be identical to the underlying lognormal distribution of c_u, multiplied by 5.14.

Figure 6 shows the influence of θ_{c_u} and $C.O.V._{c_u}$ on the standard deviation of the bearing capacity factor, σ_{N_c}.

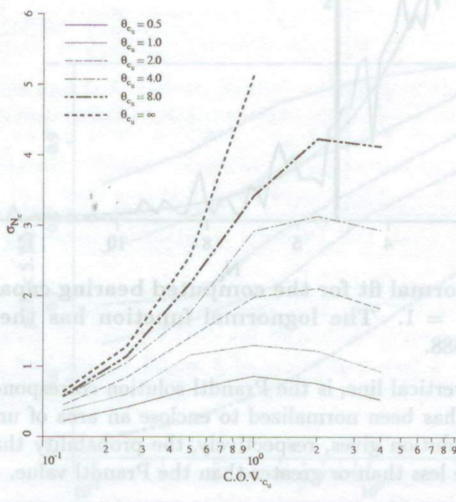

Fig 6. Standard deviation of the bearing capacity factor σ_{N_c} as a function of undrained shear strength statistics, θ_{c_u} and $C.O.V._{c_u}$.

The plots indicate that as $C.O.V._{c_u}$ increases in the range $0 \rightarrow 2$, σ_{N_c} also increases, however for $C.O.V._{c_u} > 2$ a reduction of σ_{N_c} is observed. This interesting result is currently under further investigation.

As θ_{c_u} is increased, σ_{N_c} also increases, although there is a limiting value of σ_{N_c} given by the $\theta_{c_u} = \infty$ line which has the equation:

$$\sigma_{N_c} = 5.14\ C.O.V._{c_u} \tag{7}$$

PROBABILISTIC INTERPRETATION

Following Monte-Carlo simulations for each parametric combination of input parameters (θ_{c_u} and $C.O.V._{c_u}$), the suite of computed bearing capacity factor realizations was plotted in the form of a histogram, and a "best-fit" lognormal distribution superimposed as shown in Figure 7.

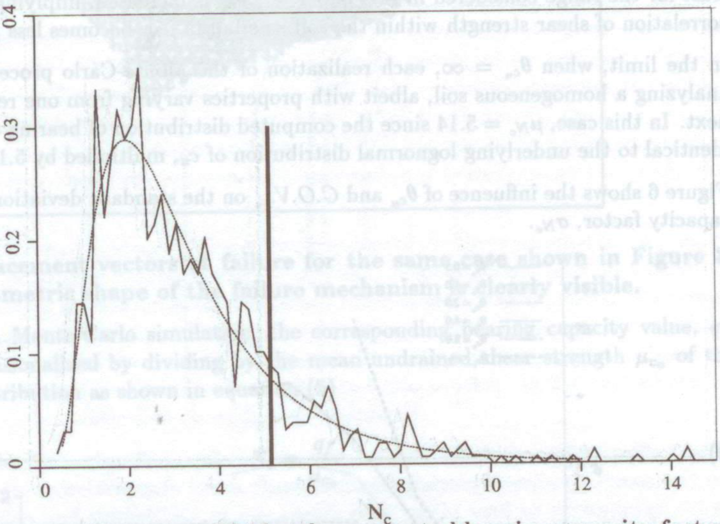

Fig 7. Histogram and lognormal fit for the computed bearing capacity factors when $\theta_{c_u} = 2$ and $C.O.V._{c_u} = 1$. The lognormal function has the properties $\mu_{\ln N_c} = 1.025$ and $\sigma_{\ln N_c} = 0.588$.

Also shown on Figure 7 as a vertical line, is the Prandtl solution corresponding to $N_c = 5.14$. Since the lognormal fit has been normalized to enclose an area of unity, the area to the left and right of this solution gives, respectively, the probability that the actual bearing capacity factor will be less than or greater than the Prandtl value.

From a design viewpoint we may be interested in the probability that the bearing capacity factor will be *less* than the Prandtl solution. Let this quantity be $p(N_c < 5.14)$, and given by the equation:

$$p(N_c < 5.14) = \Phi\left(\frac{\ln 5.14 - \mu_{\ln N_c}}{\sigma_{\ln N_c}}\right) \tag{8}$$

where Φ is the cumulative normal function.

For the particular case shown in Figure 7, equation 8 gives $p(N_c < 5.14) = 0.851$, hence there is an 85% chance the actual bearing capacity factor will be less than the Prandtl value of $N_c = 5.14$.

Figure 8 gives a summary of $p(N_c < 5.14)$ for a range of values of θ_{c_u} and $C.O.V._{c_u}$. The figure indicates that as $C.O.V._{c_u} \to 0$, $p(N_c < 5.14) \to 0.5$. This trend is to be expected, because for very low $C.O.V._{c_u}$ values, the distribution becomes narrow and "centered" on the deterministic result. There is thus an equal chance of the computed bearing capacity factor lying on either side of the Prandtl solution. As $C.O.V._{c_u}$ is increased however, the probability of obtaining an underestimation of the Prandtl solution increases rapidly for low values of θ_{c_u} and less rapidly for higher values of θ_{c_u}. For example, it is virtually certain that the computed bearing capacity will underestimate Prandtl's solution when $\theta_{c_u} = 0.5$ and $C.O.V._{c_u} = 2$.

The result corresponding to the special case of $\theta_{c_u} = \infty$ is also indicated in Figure 8. For a lognormal underlying distribution, the equation of this line is given by:

$$p(N_c < 5.14) = \Phi(\frac{1}{2}(\ln(1 + C.O.V._{c_u}^2))^{\frac{1}{2}}) \tag{9}$$

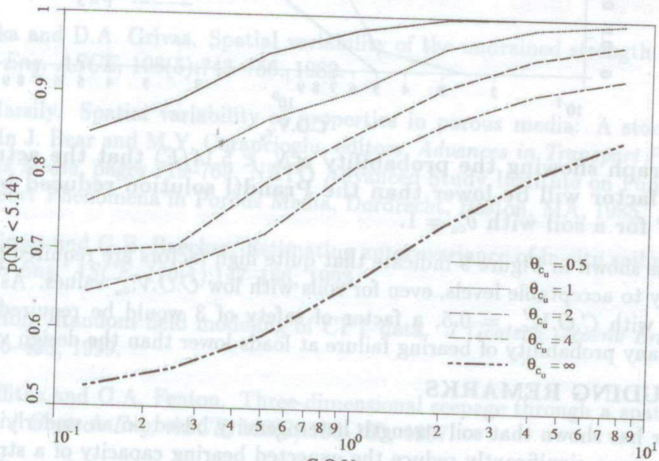

Fig 8. Graph showing the probability $p(N_c < 5.14)$ that the actual bearing capacity factor will be lower than the Prandtl solution for a soil with different values of θ_{c_u} and $C.O.V._{c_u}$

Figure 8 indicates that soil shear strength variability *always* reduces the expected bearing capacity of a strip footing supported by an undrained clay. This figure gives an over-pessimistic impression however, by not incorporating the variance of the computed bearing capacity distribution. Even an essentially deterministic analysis with $C.O.V._{c_u} \to 0$ would indicate that there was still a 50% chance of the bearing capacity being below the Prandtl value. It is more useful therefore, to indicate the probability that a factored Prandtl solution might be underestimated.

Figure 9 takes this into account for the case of $\theta_{c_u} = 1$ by indicating the probability, $p(N_c < 5.14/F)$, that the computed bearing capacity factor will be even less than the Prandtl solution divided by a "safety factor" F, where $F > 1$.

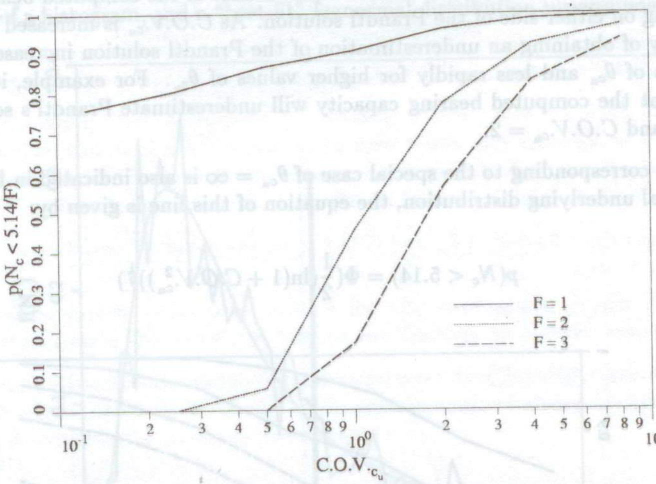

Fig 9. Graph showing the probability $p(N_c < 5.14/F)$ that the actual bearing capacity factor will be lower than the Prandtl solution reduced by a "safety factor" F for a soil with $\theta_{c_u} = 1$.

The results shown in Figure 9 indicate that quite high factors are required to reduce the probability to acceptable levels, even for soils with low $C.O.V._{c_u}$ values. As an example, for a soil with $C.O.V._{c_u} = 0.5$, a factor of safety of 3 would be required to virtually eliminate any probability of bearing failure at loads lower than the design value.

CONCLUDING REMARKS

The paper has shown that soil strength heterogeneity based on an underlying lognormal distribution can significantly reduce the expected bearing capacity of a strip footing on undrained clay.

The following more specific observations have been made:

1. As the soil strength $C.O.V.$ increases, the mean bearing capacity decreases. The rate of decrease of the mean bearing capacity lessens as the scale of fluctuation increases.
2. As the soil strength $C.O.V.$ increases, the standard deviation of the bearing capacity initially increases but later decreases, exhibiting a maximum that requires further investigation. Increasing the scale of fluctuation always has the effect of increasing the standard deviation of the bearing capacity.

3. From a probabilistic viewpoint, there will always be a greater than 50% chance that the actual bearing capacity will be less than the Prandtl solution. The smaller the scale of fluctuation the greater this probability becomes.

4. By investigating the role of a safety factor applied to the Prandtl solution, it was observed that quite high factors are needed to reduce the probability of design "failure" to negligible levels. These results explain in probabilistic terms, why such high factors of safety are needed in the design of foundations against bearing capacity failure.

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