INFLUENCE OF SPATIALLY RANDOM SOIL STIFFNESS ON FOUNDATION SETTLEMENTS

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ABSTRACT: The effect of stochastic soil stiffness on the total settlement under the corner of a uniformly loaded flexible strip footing has been studied. The generated Young's modulus random fields with constant Poisson's ratio have been combined with finite element analyses to perform a Monte Carlo statistical analysis of the settlement response. The results of parametric studies to investigate the effect of the standard deviation, correlation structure and geometry on the statistics of the settlement are presented for different values of Poisson's ratio.

INTRODUCTION

Of all engineering materials, soil is one of the most spatially variable. The great majority of geotechnical analyses however are deterministic, in that they ignore this variability, and proceed on the basis of 'average' properties. In reality, properties vary from point to point and can only be established precisely through numerous field tests. Since this is both expensive and impractical, random field models may be used as a rational method of incorporating the soil variability into geotechnical analyses (Vanmarcke 1984). The parameters of these models can be estimated from limited numbers of test results.

The settlement and stress response of a shallow foundation due to a spatially

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variable soil mass has been investigated through the use of stochastic finite element methods by Baecher and Ingra (1981), Righetti and Harrop-Williams (1988), and Zeitoun and Baker (1992).

The analyses performed herein use a technique called Local Average Subdivision (LAS) to generate realizations of the random Young's modulus with a given mean, standard deviation and correlation structure. This technique has been described fully by Fenton (1990) and Fenton and Vanmarcke (1990). The discrete nature of the finite element method allows the mapping of a discrete random field realization to individual elements. The Monte Carlo method, which involves multiple realizations of the problem, is then used to estimate the statistics of the settlement response.

The effect of a variable Young's modulus on the simple 1-dimensional problems of a single column (elements loaded in series) and a single row of elements (elements loaded in parallel) defines the two possible limits of vertical response to the application of a vertical stress at the top. The response leads to an effective Young's modulus for the whole sample of soil with an upper limit of the arithmetic mean for a row of elements and a lower limit of the harmonic mean for a column.

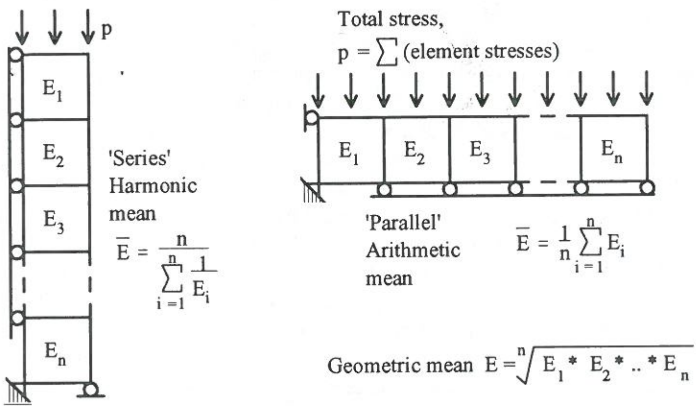


FIG. 1. 1-Dimensional Limits of Effective Young's Modulus

GENERATION OF YOUNG'S MODULUS VALUES

For the purpose of the present work, the soil was treated as a linear elastic body with spatially random Young's modulus and a constant Poisson's ratio. The latter assumption is consistent with previous work since little is currently known about the variability of Poisson's ratio (Cambou 1975; Baecher and Ingra 1981; Zeitoun and Baker 1992). The coefficient of variation (σ_E/μ_E) of the Young's modulus has been estimated through laboratory and in situ tests as having a range from 2% (0.02) to 42% (0.42) with a recommended value of 30% (Lee et al. 1983).

The following assumptions are made for the generation of the Young's modulus fields:

(1) $E(\tilde{t})$ is a homogeneous isotropic random field where the mean is independent of position $\tilde{t}$, and the correlation function is only dependent on the distance $\tau = |\tilde{\tau}|$, where $\tilde{\tau} = \tilde{t} - \tilde{t}'$ is the physical lag.

(2) $Y = \ln E$ is a Gaussian process that is fully characterized by its mean $\mu_{\ln E}$, variance $\sigma^2_{\ln E}$, and correlation function $\rho(\tilde{\tau})$.

(3) a point process - the statistics for the Young's modulus come from field and laboratory measurements which are obtained through samples much smaller than the site and smaller than the elements used to discretize the model.

The lognormal distribution is convenient as a model of the distribution of the Young's modulus since there is no need to censor the random field in any way to avoid values of less than zero. The Young's modulus field is obtained through the transformation

$$E_i = \exp\{\mu_{\ln E} + \sigma_{\ln E} g_i\} \tag{1}$$

where E_i is the Young's modulus assigned to the i[th] element, g_i, is the local average of a standard Gaussian random field, g, over the domain of the i[th] element, and $\mu_{\ln E}$ and $\sigma_{\ln E}$ are the mean and standard deviation of the logarithm of E_i (obtained from the 'target' mean and standard deviation μ_E and σ_E).

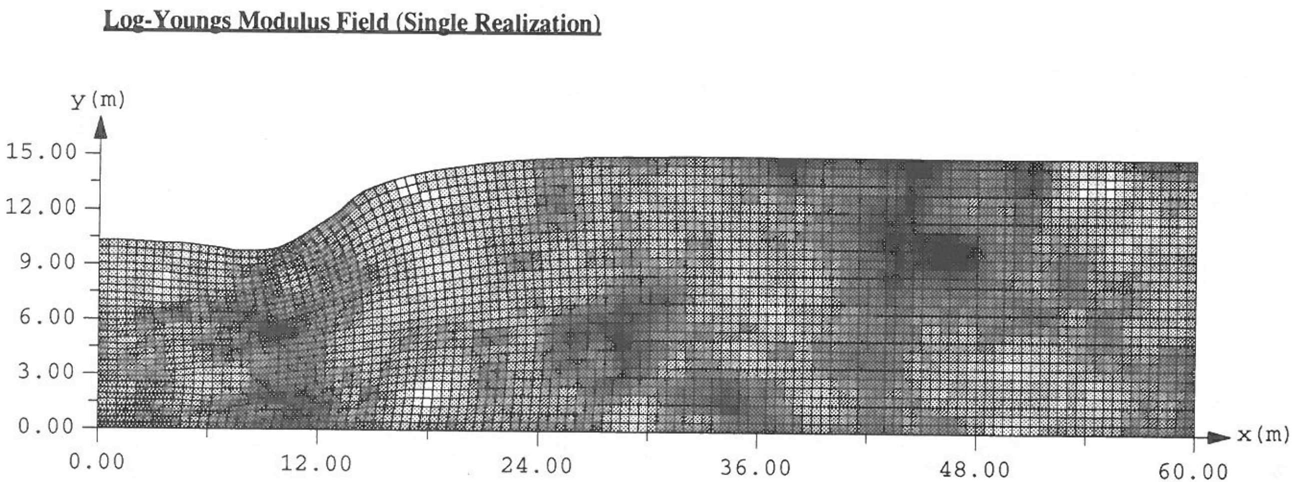
The LAS technique renders local averages, g_i, derived from the random field, g, having zero mean, unit variance, and a spatial correlation controlled by the scale of fluctuation (θ_E) of the Young's modulus. As the value of the scale of fluctuation increases the field becomes more uniform due to the greater distance over which the field is significantly correlated. A very small scale of fluctuation yields element values that are largely independent of one another.

For this study, the soil was represented by a 2-D random field and thus the out-of-plane variation of the soil Young's modulus has not been taken into account. This limitation on the analysis implies that the scale of fluctuation in this direction is infinite - the soil's Young's modulus is constant in this direction. In addition, due to the assumption of symmetry about the centerline, the settlement response is identical on each side of the central boundary. This symmetry could have been avoided by doubling the size of the finite element mesh, however the purpose of the current work was to observe the influence of a variable Young's modulus and the current model was considered acceptable in order to show the trends without loss of validity.

Finite Element Analyses

A typical finite element mesh used in this study is shown in Fig. 2. It contains 3600 elements and represents a 2-d model for the plane strain settlement under a flexible strip footing. The boundary conditions assume a rigid interface between the soil and the underlying stratum and only vertical freedom at the left and right boundaries.

The finite element code for the solution of the problem is similar to that published by Smith and Griffiths (1988). For the discretization of the problem, the



n_x = 120 elements, n_y = 30 elements, Δx_e = 0.50 m, Δy_e = 0.50 m, 3600 elements in total

FIG. 2. Typical Finite Element Mesh with Young's Modulus Gray-Scale Superimposed on the Displaced Mesh; $\mu_E = 1.00\times10^6$ kN/m^2, $\sigma_E = 4.20\times10^5$ kN/m^2, θ_E = 8.0 m, B/h = 1.0, ν = 0.2; Displacement Magnification = 3.00×10^5

4-noded quadrilateral element has been used with algebraic closed-form stiffness, (Griffiths 1994) based on a Selective Reduced Integration (SRI) formulation.

The SRI 'K-G' method was implemented where the element stiffness is divided into volumetric and deviatoric parts, the volumetric part being integrated with one order less than the deviatoric. The effect of this was to allow problems with high Poisson's ratios to be studied without numerical locking.

For each boundary value problem considered, multiple solutions were obtained using successive realizations of the Young's modulus field. The random Young's modulus field is characterized by the following parameters,

mean	μ_E
standard deviation	σ_E
scale of fluctuation	θ_E
correlation structure	$\rho(\tilde{\tau})$

For the generation of the Young's modulus fields an exponential correlation function has been used.

To obtain a sufficiently large number of results for the calculation of the output statistics, it was decided that each 'run' would consist of the analysis of 2000 realizations. In Fig. 2 a typical displaced finite element mesh is shown with the corresponding gray-scale of the Young's modulus field realization superimposed onto it. The darker grays signify regions of lower Young's modulus and the lighter grays signify regions of higher Young's modulus. A region of low Young's modulus can clearly be seen at x = 10 m and y = 5 m due to the dark gray and the large deformations of the finite element mesh.

PARAMETRIC STUDIES

In all of the analyses that follow, the soil is assumed to be isotropic in the plane with a mean Young's Modulus of $\mu_E = 10^6$ kN/m^2. Parametric studies of a flexible strip footing carrying a uniformly distributed load were performed to determine the effect of σ_E and θ_E on the total vertical settlement under the left edge. For each case in the parametric studies, statistics were obtained relating to the vertical settlement under the edge.

The range of σ_E/μ_E considered was:

$$0.015 \leq \frac{\sigma_E}{\mu_E} \leq 8.0 \qquad (2)$$

however it should be noted that Lee et al. (1983) suggested that for typical soils:

$$0.02 \leq \frac{\sigma_E}{\mu_E} \leq 0.42 \qquad (3)$$

The use of coefficients of variation above the recommended upper value stated by Lee et al. (1983) has been used to demonstrate the rapid increase in the expected vertical settlement that could occur under more variable conditions.

The values of θ_E used were:

$$\theta_E \quad = \quad 0.125,\ 0.5,\ 2.0 \text{ and } 8.0 \text{ m} \tag{4}$$

The statistics of the settlement are presented in the form of an influence coefficient (I) which is obtained in the same manner as Poulos and Davis (1974):

$$\rho \quad = \quad \frac{ph}{\pi E}\, I \tag{5}$$

where ρ = vertical settlement under the edge of the strip footing; E = Young's modulus, taken as μ_E in Eq. (5) and I = influence coefficient associated with the problem geometry (B/h).

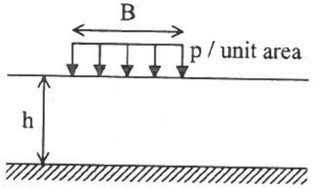


FIG. 3. Flexible Strip Footing

RESULTS

Fig. 4a shows the behavior of the mean of the influence coefficient (I) for a Poisson's ratio equal to 0.2 and B/h = 0.25 associated with four different values of θ_E over a range of $0.015 \leq \sigma_E/\mu_E \leq 1.0$. For all scales of fluctuation, as the coefficient of variation approaches zero the expected value of I approaches the constant 0.940, the deterministic value. The percentage increases of I at a coefficient of variation of 0.42 (42%) for different Poisson's ratios and scales of fluctuation are shown in Table 1. For θ_E varying from 0.125 m to 8.0 m, the percentage increase of I (and thus the vertical settlement) changes from approximately 8.5% to 14.5%. This change is significant and can be explained by larger areas of more extreme values of Young's modulus governing the behavior of the soil mass with stiffer areas acting like rigid bodies and softer areas deforming to a greater extent. The results show that the percentage increase of μ_I is largely independent of ν. Fig. 4a also shows that as the value of σ_E/μ_E rises, the value of μ_I also rises with a rate of increase of μ_I that is also rising. At a value of σ_E/μ_E equal to 0.5 (marginally above the recommended upper value stated by Lee et al. (1983)), the percentage increases in μ_I for scales of fluctuation equal to 0.125 m and 8.0 m are 12.1% and 20.4% respectively. These percentage increases in comparison with those for σ_E/μ_E=0.42 show the possible dangers in the underestimation of the soil variability of a site.

Fig. 4b shows the behavior of the standard deviation of I shown in Fig. 4a. Again as σ_E/μ_E approaches zero the standard deviation approaches zero and the

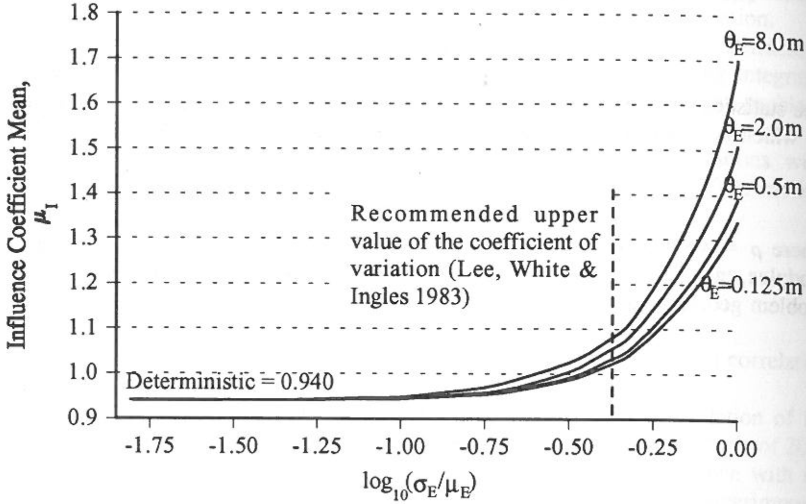


FIG. 4a. Variation of μ_I with $\log_{10}(\sigma_E/\mu_E)$ for B/h = 0.25, Poisson's ratio = 0.2

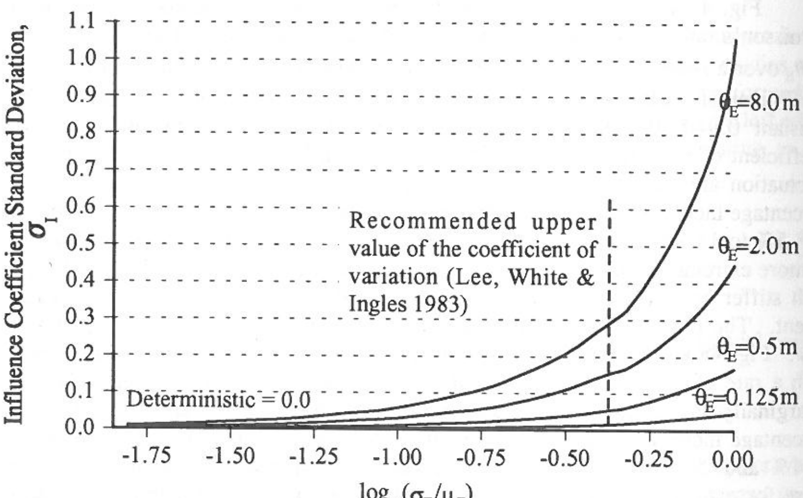


FIG. 4b. Variation of σ_I with $\log_{10}(\sigma_E/\mu_E)$ for B/h = 0.25, Poisson's ratio = 0.2

analysis becomes deterministic. Of note is the rapid increase in the variation of I with increasing σ_E/μ_E for higher values of θ_E which reflects the degree of uncertainty of the result and the possibility of areas of extreme values.

TABLE 1. Percentage Increase of the Mean Influence Coefficient (μ_I) with θ_E for Different Poisson's Ratios; B/h = 0.25, σ_E/μ_E = 0.42

θ_E (m)	Percentage increase in μ_I at σ_E/μ_E=0.42			
	Poisson's ratio = 0.0	Poisson's ratio = 0.2	Poisson's ratio = 0.4	Poisson's ratio ≈ 0.5
0.125	8.5	8.5	8.5	8.3
0.5	9.5	9.5	9.5	8.2
2.0	12.0	12.0	12.0	9.3
8.0	14.5	14.5	14.0	11.1

Figs. 5a and 5b show the behavior of the normalized Young's modulus means of the stochastic analysis over a larger range of σ_E/μ_E for a scale of fluctuation equal to 0.125 m and 8.0 m. It can be seen that the effective mean Young's modulus, which is back figured from the observed centerline settlements, shows close agreement with the geometric mean of the Young's modulus random field values. Of note is the movement of the other measures of the mean, the arithmetic and harmonic means. As θ_E increases, the arithmetic mean increases and will eventually become a normalized value equal to 1.0, i.e. the input mean value will be modeled by the random process. The harmonic mean decreases as θ_E increases providing the limits of the possible response of the soil mass to the spatially variable Young's modulus.

Figs. 6a and 6b show the variation of μ_I with B/h and σ_E/μ_E for two values of θ_E in the form of Poulos and Davis (1974). The two limits corresponding to σ_E/μ_E = 0.02 and σ_E/μ_E = 0.42 indicate the range of possible values of μ_I for coefficients of variation below the recommended upper value that has been stated by Lee et al. (1983). It can be seen from the figures and Table 2 that the behavior of the stochastic system at a Poisson's ratio of 0.5 appears to be largely independent of θ_E for $\theta_E \leq 2.0$ m. For values of θ_E at a Poisson's ratio less than 0.5, the percentage increase in μ_I from the deterministic value appears to be largely independent of both the Poisson's ratio and the geometry of the problem (B/h) when the constant values of Poisson's ratio are used as input into the soil properties.

CONCLUDING REMARKS

Random field theory has been combined with finite element analysis to model the settlement of a flexible footing on soil with spatially random stiffness. For low stiffness variance, classical or deterministic soil settlement values were retrieved, however as the stiffness variance increased so did the expected settlement. Within

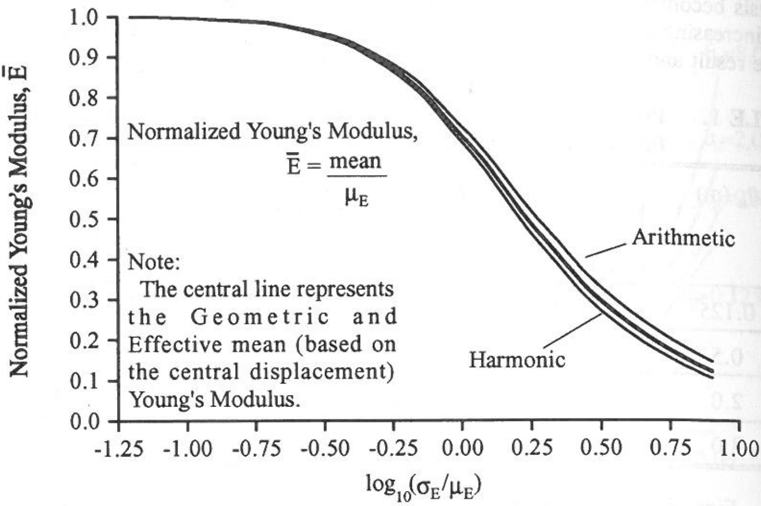


FIG. 5a. Variation of Normalized Young's Modulus Means with $\log_{10}(\sigma_E/\mu_E)$ for B/h = 0.25, Poisson's ratio = 0.2, θ_E = 0.125 m

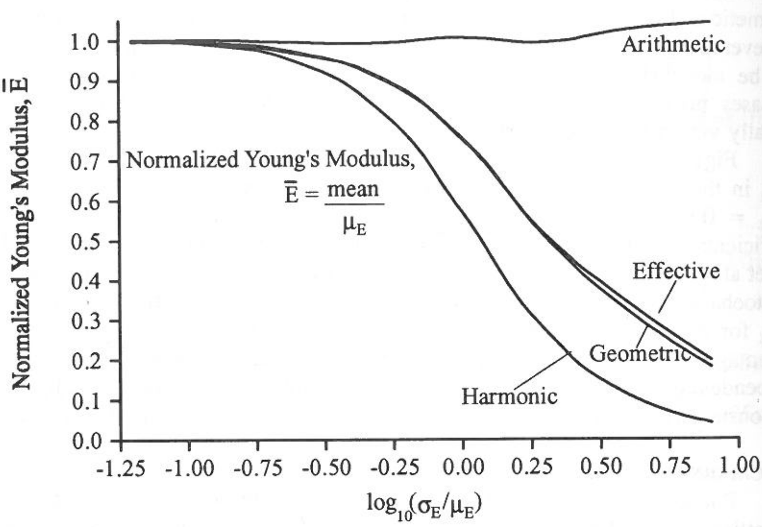


FIG. 5b. Variation of Normalized Young's Modulus Means with $\log_{10}(\sigma_E/\mu_E)$ for B/h = 0.25, Poisson's ratio = 0.2, θ_E = 8.0 m

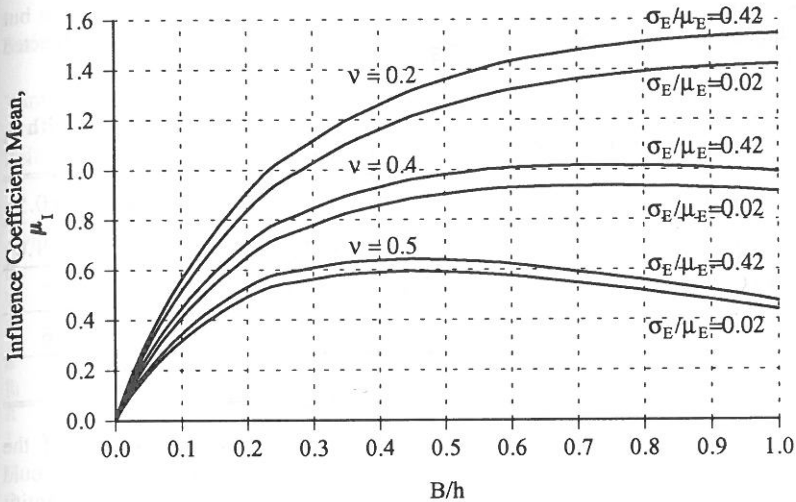


FIG. 6a. Variation of μ_I with B/h for various values of Poisson's ratio and σ_E/μ_E, θ_E = 0.125 m

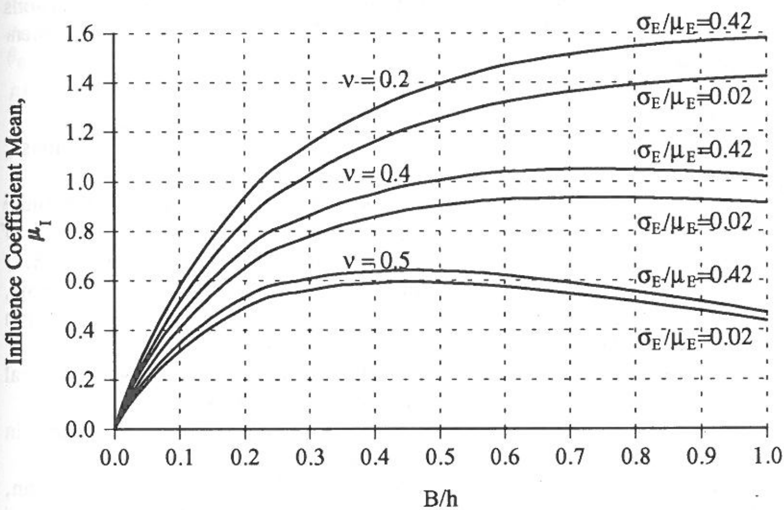


FIG. 6b. Variation of μ_I with B/h for various values of Poisson's ratio and σ_E/μ_E, θ_E = 2.0 m

a 'recommended' range, the increase in expected settlements was quite modest but for higher values of stiffness variance, a quite significant increase in expected settlements was observed.

TABLE 2. Percentage Increase of the Mean Influence Coefficient (μ_I) with θ_E for B/h = 0.5 and B/h = 1.0. σ_E/μ_E = 0.42

θ_E (m)	Poisson's ratio=0.2		Poisson's ratio=0.4		Poisson's ratio≈0.5	
	B/h=0.5	B/h=1.0	B/h=0.5	B/h=1.0	B/h=0.5	B/h=1.0
0.125	8.5	8.6	8.6	8.6	8.3	8.2
0.5	9.4	9.3	9.4	9.6	7.9	7.6
2.0	11.3	11.0	11.6	11.7	8.0	7.8

One interpretation of the results was to express each realization of the random process in terms of an effective stiffness, namely the value of E that would have given the same centerline deflection on a deterministic basis. This quantity was found to be reproduced to a good degree of accuracy by the geometric mean of the stiffness field over a wide range of scales of fluctuation.

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Appendix I. References

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Appendix II. Notation

B	width of flexible strip footing;
E	Young's modulus (Elastic modulus);
E_i	Young's modulus assigned to the i^{th} element;
g	standard Gaussian random field;
g_i	local average of g over the i^{th} element;
h	depth of soil layer to the rigid stratum;
I	edge settlement influence coefficient;
n_x, n_y	number of elements in the x- and y-directions, respectively;
p	load/unit area applied to flexible footing;
Δx_e, Δy_e	dimensions of elements in the x- and y-directions, respectively;
ρ	settlement under flexible strip footing edge;
ν	Poisson's ratio;
μ_E, σ_E	Young's modulus mean and standard deviation;
$\mu_{\ln E}$, $\sigma_{\ln E}$	mean and standard deviation of ln E_i;
σ_E/μ_E	coefficient of variation of Young's modulus;
μ_I, σ_I	influence coefficient mean and standard deviation;
θ_E	scale of fluctuation of Young's modulus;
$\rho(\tilde{\tau})$	correlation function; and
$\tilde{\tau}$	physical lag vector.