Woodward, P. K. & Griffiths, D. V. (2000). *Géotechnique* **50**, No. 3, 301–303

DISCUSSION

Observations on the computation of the bearing capacity factor N_γ by finite elements

P. K. WOODWARD and D. V. GRIFFITHS (1998). *Géotechnique* **48**, No. 1, 137–141.

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In finite-element analysis the governing stiffness equation gives the relationship between nodal displacements and nodal forces. Applied boundary conditions must be converted to equivalent nodal forces or nodal displacements. The solution of the stillness equation gives the unknown displacements at unrestrained nodes and the unknown reaction forces at restrained nodes. The bearing capacity problem may be solved by applying the footing load as equivalent nodal forces or by applying nodal displacements and determining the nodal reactions. The former simulates a load controlled test of an infinitely flexible footing and the latter simulates a displacement controlled test of an infinitely rigid footing. The latter is computationally easier for the determination of failure loads and is used by the authors. In this case, the total footing load applied to the ground surface is equal to the sum of the nodal reaction forces.

The authors have calculated the footing load from the vertical component of stress at the first row of Gauss points. However, the vertical components of stress in the ground immediately below the footing is not necessarily in equilibrium with the vertical nodal reactions. This is because the reactions in equilibrium with shear stress on vertical planes as well as the direct vertical stress. In the writers' experience the use of the vertical component of stress in the ground is not reliable for the calculation of bearing capacity in displacement controlled analysis.

The writers agree with the authors' findings that bearing capacity calculations are mesh size dependent. However, the writers have found it is the size of the element at the corner of the footing that is instrumental in determining the magnitude of the failure load when displacement control is used.

To illustrate these points Tables 6 and 7 give the results of displacement controlled finite-element analyses performed by the writers. The material properties and initial stress are the same as adopted by the authors. Results for N_γ are presented for $\phi = 25°$. Bearing capacity factor N_c has also been calculated for a cohesionless material to illustrate the points made above. The bearing capacity factors have been calculated, first from the nodal reactions, and secondly from the vertical stress (interpolated from the vertical stress at the Gauss points) at various depths beneath the footing. The stress due to the weight of the soil above the point (γd) has been subtracted to give that portion of the stress resulting from the footing load. Different mesh configurations (Fig. 5) show the dependence of the finite-element calculation on the corner element size (the whole mesh is 10 m deep and 20 m wide. A 1·2 m × 2·4 m detail near the footing is shown in each diagram).

The following observations are made from Table 6:

- The vertical component of stress beneath the footing is significantly less than the value implied by the vertical nodal reactions.
- The value of N_c approaches the theoretical value as the size of the corner element is reduced.
- The vertical stress beneath the footing is less sensitive to the mesh size than the nodal reactions. The value at the Gauss point depth is remarkably close to the theoretical solution.

The following observations are made from Table 7:

- The nodal reactions and the vertical stress beneath the footing give much the same results (contrary to N_c).
- N_γ calculated from the nodal reaction forces approaches the stress field solution (Hansen & Christiensen, 1969, Bolton & Lau, 1993) when the size of the corner element is reduced.
- The effect of the footing width B on the value of N_γ is insignificant.

Table 6. Finite element values of N_c (theoretical solution, $N_c = 5{\cdot}14$)

Mesh number (Fig. 5) (corner element size, a: m)	1 (0·20)	2 (0·10)	2 (0·05)	3 (0·40)	4 (0·20)	4 (0·10)	4 (0·05)
From nodal reactions	5·33	5·22	5·19	5·50	5·29	5·22	5·19
From σ_y at surface	5·13	5·13	5·14	5·00	5·12	5·13	5·10
From σ_y at Gauss point depth	5·12	5·11	5·12	4·98	5·08	5·11	5·09
From σ_y at 0·2 m depth	5·02	4·97	4·95	4·94	5·01	5·01	4·99

Table 7. Finite element values of N_γ (stress field solution, $N_\gamma = 3{\cdot}5$)

Mesh number (Fig. 5) (corner element, a: m)	1 (0·20)	2 (0·10)	2 (0·05)	3 (0·40)	4 (0·20)	4 (0·10)	4 (0·05)	5 (0.05)	6 (0.05)
From nodal reactions	3·74	3·56	3·57	4·03	3·70	3·60	3·57	3·55	3·60
From σ_y at surface	3·72	3·56	3·57	3·96	3·71	3·64	3·66	3·59	3·65
From σ_y at Gauss point depth	3·72	3·55	3·56	3·94	3·68	3·59	3·58	–	3·63
From σ_y at 0·2 m depth	3·71	3·54	3·54	3·92	3·63	3·54	3·52	3·52	3·61

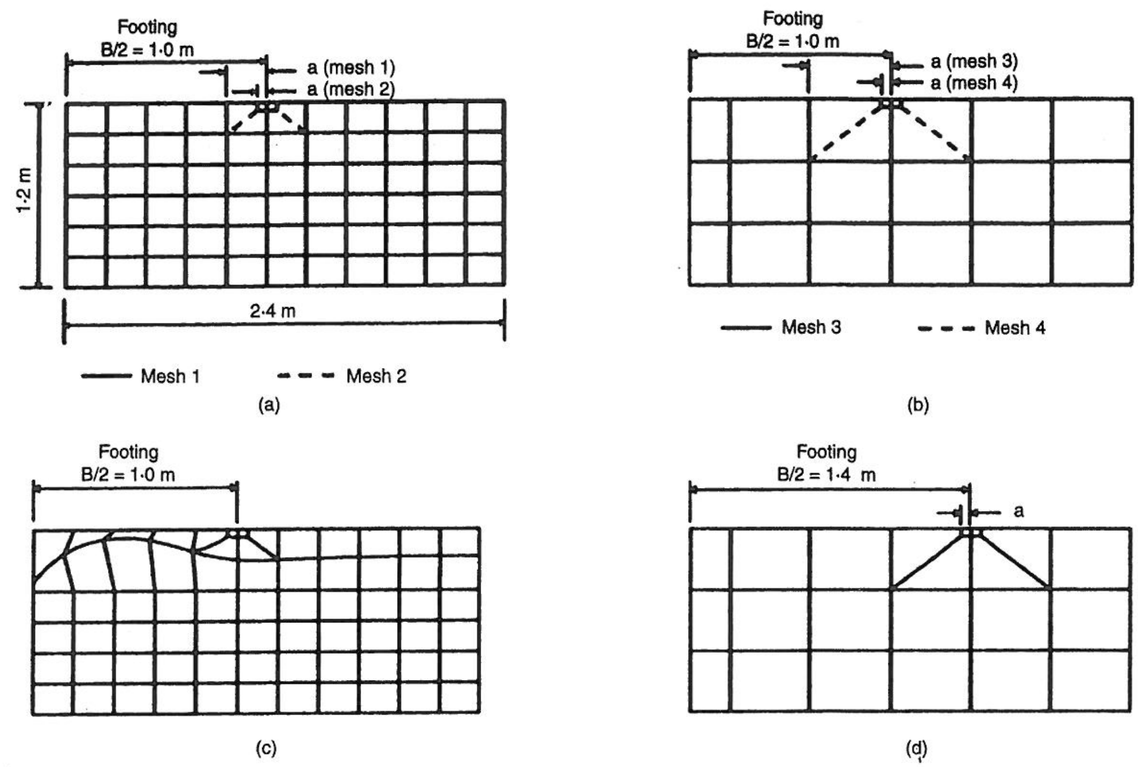

Fig. 5. Mesh configurations used in analyses. (a) Mesh 1 and 2. (b) Mesh 3 and 4. (c) Mesh 5—irregular mesh. (d) Mesh 6—$B = 2{\cdot}8$ m

- The value of N_γ is independent of the depth of the Gauss points.

The results and the discussion above demonstrate that the authors statement 'we are approximating a footing at the Gauss point depth' is incorrect. If we consider the irregular mesh (Fig. 5(c)) the following question must be asked: What is the depth of the footing we are approximating? Consequently, the procedure adopted by the authors of subtracting $\gamma d' N_q$ from the vertical stress cannot be justified. It brings the authors' finite-element results into agreement with stress field solutions. The writers' results agree with the stress field solutions without such a procedure (Table 7).

Authors' reply

The authors would like to thank the writers for their discussion and for the opportunity to clarify some of the points raised in the paper.

The authors agree that the bearing capacity of surface footings can be determined by the vertical component of the stress at the first row of Gauss points or by determining the equivalent nodal reactions (this was mentioned in the paper). The writers' comment that 'in the writers' experience the use of the vertical component of the stress in the ground is not reliable for the calculation of the bearing capacity in displacement controlled analysis'. However, if we compare the results presented by the writers themselves for the calculation of both N_c and N_γ (rows 1 and 3 in Tables 6 and 7) we see that, in general, the Gauss point depth method actually gives better results when compared to the analytical solutions (Tables 8 and 9).

The authors have recently found that provided a correction is made for the singularity at the footing edge the values computed by either method give remarkably similar results at failure. This is shown in Fig. 6, where a non-linear elastic strain-hardening/strain-softening constitutive soil model is used to predict N_γ for smooth surface strip foundations on Nevada sand using a Lade and Duncan failure criterion. The model is more suitable to granular foundation analysis as it accurately simulates dilation effects and the reduction in ϕ with confining pressure.

The main aim of the paper was to show that using a linear

Table 8. Percentage errors for each method for the calculation of N_c (positive unconservative result/negative conservative result), from Table 6

Mesh number	1	2	2	3	4	4	4
From nodal reactions	3·7	1·6	1·0	7·0	2·9	1·6	1·6
From σ_y at the Gauss point depth	−0·4	−0·6	−0·4	−3·1	−1·2	−0·6	−1·0

Table 9. % errors for each method for the calculation of N_γ (positive unconservative result/negative conservative result), from Table 7

Mesh number	1	2	2	3	4	4	4	5	6
From nodal reactions	7·0	1·7	2·0	15	5·7	2·9	2·0	1·4	2·9
From σ_y at the Gauss point depth	6·3	1·4	1·7	13	5·1	2·6	2·3	–	3·7

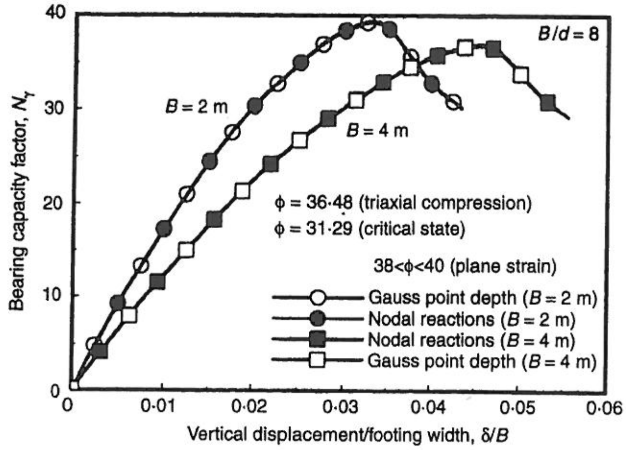

Fig. 6. Comparison of the Gauss point and nodal reaction methods for the computation of N_γ at a constant B/d ratio using an advanced soil model

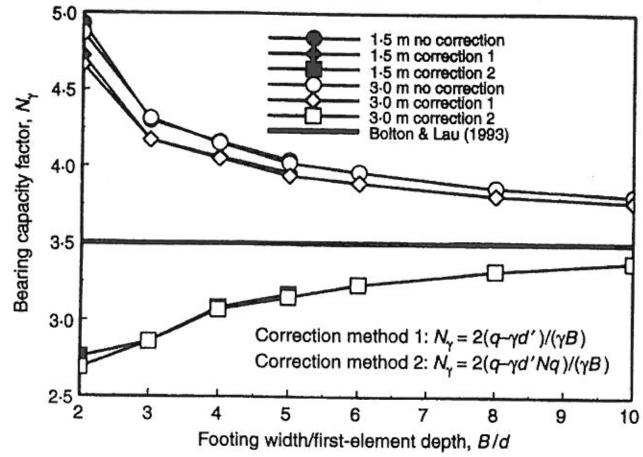

Fig. 7. Plots of N_γ for various B/d ratios from Tables 3 and 4

elastic perfectly plastic Mohr–Coulomb constitutive soil model the bearing capacity factor N_γ is not a function of the foundation width. This was achieved by using the non-dimensional ratio B/d (footing width/first element depth), as it was observed that the computation of N_γ was mesh-dependent, explaining why previous authors had reported a variation with footing width. The paper showed that if the B/d ratio was kept constant then, for a 'uniform' mesh, the same value of N_γ could be computed regardless of the foundation width. The paper went on to show that values of N_γ reduce towards those proposed by Bolton & Lau (1993) as B/d increases. The writers agree with the authors that N_γ is mesh-dependent and does not vary with the footing width. However, they comment that 'the value of N_γ is independent of the depth of the Gauss points'. This suggests that the same value of N_γ can be computed regardless of the depth of the first row of elements. This is clearly incorrect. For example, one would not expect to compute the same value of N_γ if the depth of the first row of elements increased from $d = 0{\cdot}1$ m to $d = 10$ m.

As commented earlier, the authors have found that the singularity at the footing edge can significantly effect the results due to the development of out-of-balanced forces Frydman & Burd (1997) described a stress correction procedure to account for this. They commented that;

> the contact stress at the edge of the footing should be zero. It appears then, that the upward displacement of the curves may correspond to errors introduced due to the singularity at the footing edge, and that a correction to the bearing pressure may be obtained by moving the curves downwards so as to give a zero edge stress. This is equivalent to subtracting the edge stress from the calculated bearing capacity.

The curves referred to are the contact normal stresses at failure under the footing. This type of stress correction was not applied in the authors' paper, explaining the higher values of N_γ. The authors are unsure as to whether this type of correction was used by the writers. Furthermore, while the writers quote results to two decimal places, they give no information on their plasticity algorithm. The writers suggest that a small element should be included next to the footing edge to 'refine' the mesh. However, as the writers did not put this approach into a non-dimensional form for other foundation widths, it is unwise to apply it for general foundation analysis. If refinement of individual elements is to be considered, a better approach would be to use adaptive mesh refinement. The approach taken by the authors' is therefore justified in its simplicity and practicality, requiring no special treatment of the mesh.

The authors' comment, 'we are approximating a footing at the Gauss point depth' simply refers to the fact that a correction must be applied to account for the initial surcharge at the Gauss point depth where N_γ is actually computed, this is especially important when the friction angle is small. Fig. 7 shows plots of N_γ from Tables 3 and Table 4. Two approaches were considered: (a) subtract $\gamma d'$ or (b) subtract $\gamma d' N_q$. In the paper the former correction still resulted in higher values of N_γ, and so was not included. However, the latter correction is justified as it resulted in conservative estimates of N_γ when compared to Bolton & Lau (1993), as stated in the paper. It should be noted that the bearing capacity factor N_q would be calculated in exactly the same way as N_γ if all Gauss points were set to this initial surcharge value. The latter correction can therefore be used to ensure conservative estimates of N_γ whenever the exact solution is unknown, especially when considering dilatancy.

In the search for accuracy, more advanced constitutive soil models become attractive in their ability to reproduce shear–dilatancy coupling and the dependence of the friction angle on the mean effective stress. However, with a correction for the singularity and an appropriate B/d ratio, good estimates of N_γ are still possible for uniform meshes with simple linear elastic perfectly plastic models when compared with analytical solutions.

Finally, it should be remembered that the rigid-plastic theories (e.g. Prandtl), take no account whatsoever of the material stiffness. It is hardly surprising therefore, that elastoplastic calculations with back-figured nodal reactions always come out higher due to the additional contribution of shear stresses in the material just outside the footing edge. Stress averaging therefore, where only those Gauss points beneath the footing are considered, is a practical way of avoiding these excess loads.

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