Possibilities and limitations of finite elements for limit analysis

R. DE BORST and P. A. VERMEER (1984). *Géotechnique* **34**, No. 2, 199–210

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de Borst and Vermeer have presented solutions to a variety of collapse–load problems of interest in soil mechanics. Of particular note was the stable solution to a footing on a c–ϕ soil with an associated flow rule as a previous attempt (Griffiths, 1982) to use a high friction angle of 40° resulted in very slow convergence.

Some of the solutions quoted in the paper and elsewhere (de Borst 1982, de Borst & Vermeer, 1982), however, are worthy of comment, as they experienced an overshoot of the 'correct' failure load before levelling out at larger displacements. It is accepted that the solutions were obtained from a displacement controlled approach, but is it not likely that this type of numerical softening would cause difficulties with load control? In such cases, load control would predict a failure load corresponding to the peak of the load displacement response.

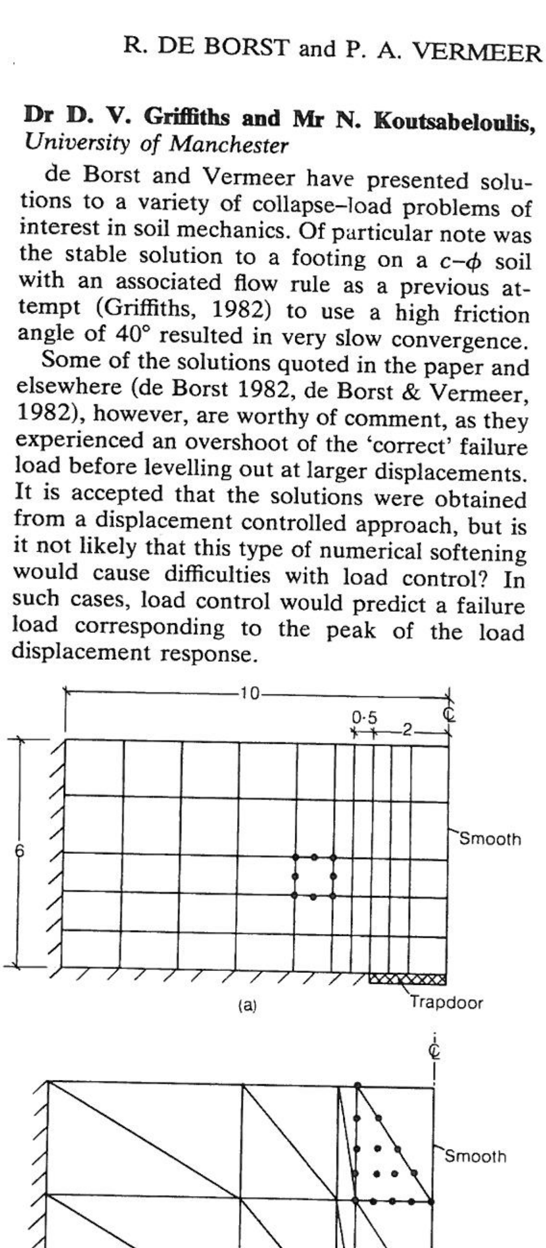

Fig. 1. Meshes for the trapdoor problem: (a) eight-node rectangles; (b) fifteen-node triangles

Although generally stable results were obtained using associated flow in the problems considered, this was not the case when the degree of non-association became large. Indeed, de Borst and Vermeer assert that in other publications the use of non-associated flow rules has been restricted to friction angles of 30° 'at the most'. This is not true in general as stable results have been published by Zienkiewicz, Humpheson & Lewis (1975) for triaxial tests ($\phi = 45°$, $\psi = 0$) and Griffiths (1980, 1982) for earth pressures ($\phi = 40°$, $\psi = 0$) and bearing capacity ($\phi = 35°$, $\psi = 0$). In the latter cases, it was observed that convergence was slightly improved by the use of a non-associated flow rule.

The trapdoor problem was particularly interesting in that the post-peak softening was thought to be mainly due to a change in the mechanism to a localized vertical shear band. Physically, such softening has been explained by Vardoulakis, Graf & Gudehus (1981) to be due to a changing mechanism in conjunction with a reduction in the mobilized friction angle from its peak value to its critical or residual value. It is difficult then to envisage how a finite element solution could model such a phenomenon without the friction angle being reduced during the computations according to some softening stress–strain law.

The trapdoor problem has been repeated with the two quite different meshes in Fig. 1. The

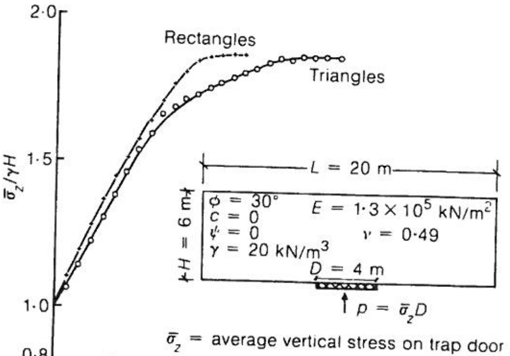

Fig. 2. Stress vs. displacement for the trapdoor problem

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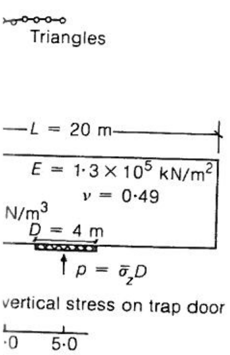

ent for the trapdoor

algorithm used was of the initial stress type with correction factors applied as described by Nayak & Zienkiewicz (1972). Although a global convergence criterion was applied to the whole mesh, close scrutiny was continually made of the individual Gauss points to eliminate overshoot almost completely. Displacements were applied to the trapdoor and the average stress was computed from the nodal reactions. As shown in Fig. 2 both meshes generated a reasonably smooth build-up of stress giving failure loads in agreement with the range of solutions given by Vardoulakis *et al.* (1981).

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Authors' reply

In most of our solutions we observed an overshoot of the correct failure load, followed by a softening until a limiting value had been reached after which we could impose displacement increments at constant load. Such a procedure is possible for displacement control. We agree that for load control the iterative procedure would diverge at the peak of the load–displacement curve and this load level would incorrectly be identified with the ultimate failure load. This is because we are not able to trace post-peak responses with load control. To overcome this basic deficiency of load control, the load level is replaced as the parameter that controls the loading process by the arc length of the load–displacement curve in the generalized load–displacement space (Fig. 3). Such an approach was followed originally by Riks (1979) and was modified for implementation in existing finite element codes amongst others by Crisfield (1981). The technique was applied to geotechnical problems by Casciaro & Cascini (1982) and by de Borst (1984). An example of an originally load-controlled problem is the slope stability analysis of Fig. 4. When the weight of the soil is incremented to simulate a centrifuge test the numerical procedure will fail near peak. With the arc length method we are able to obtain a converged solution also at peak strength and to continue the solution beyond the peak.

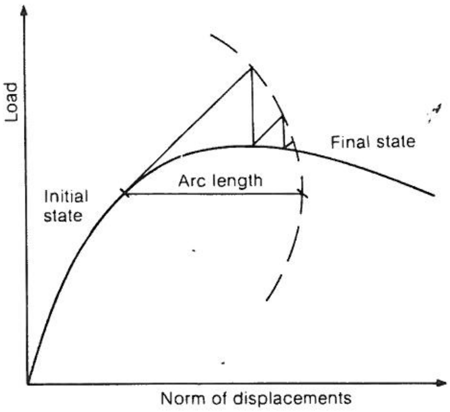


Fig. 3. Arc length as control parameter

The slope stability problem is also of interest in that it shows that the overshoot of the correct collapse load diminishes if we employ a tighter convergence tolerance. This is another point raised by Griffiths and Koutsabeloulis, i.e. whether the post-peak softening that we experience is physical or merely numerical. For most problems that we considered, the softening was thought to be due to the convergence tolerance. A clear example of numerical softening is shown in Fig. 4. We made an exception for the trapdoor problem where the softening was thought to be due to the use of a non-associated flow law at least partially. Griffiths and Koutsabeloulis have recalculated this problem with a tight convergence tolerance and obtained a failure load which was below the load presented in our Paper. Here it is unfortunate that an error has crept into the scale of the vertical axis. The number 3 should be replaced by 2, and the number 2 should be read as 1·5 for the value of the normalized uplift force. With this correction we obtain a peak strength of 1·95 which is slightly in excess of the 1·87 calculated by Griffiths and Koutsabeloulis. This is probably

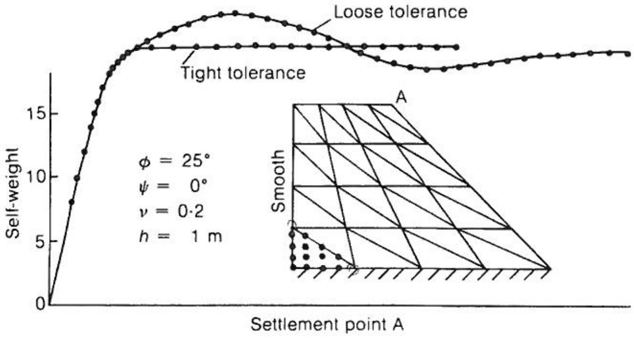


Fig. 4. Load-settlement curves for the top of a slope and finite element mesh

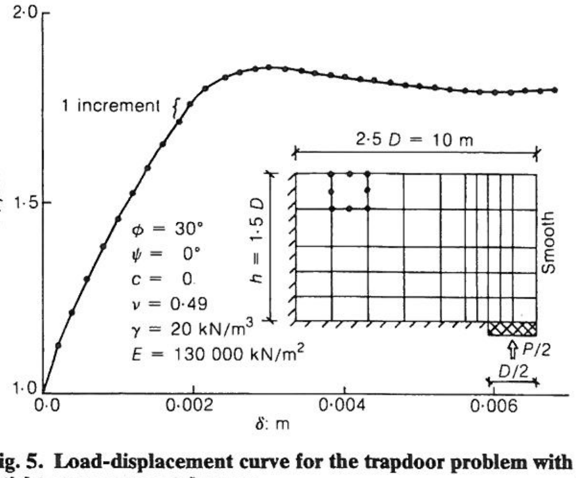


Fig. 5. Load-displacement curve for the trapdoor problem with a tight convergence tolerance

due to the fact that they used a tighter convergence tolerance. To clarify this matter, we also performed a calculation with a very tight convergence tolerance, and we imposed a tight energy norm in addition to monitoring each individual Gauss point. This resulted in the curve shown in Fig. 5. The peak is now at 1·86 which indicates that our tolerance for this calculation is at least as tight as the criterion employed by Griffiths and Koutsabeloulis. Nevertheless we again observe a post-peak softening when we continue the calculation beyond peak strength. For a greater embedment ratio, the softening effect becomes more pronounced. This is represented in Fig. 6, which gives results for the same material properties but for an embedment ratio of $h/D = 4{\cdot}0$.

In our Paper we stated that the global

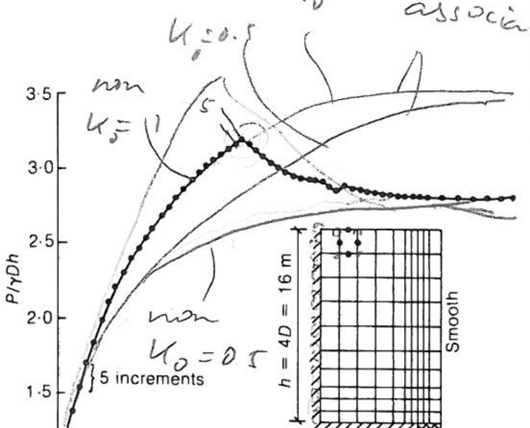


Fig. 6. Load–displacement curve for the trapdoor problem with $h/D = 4$

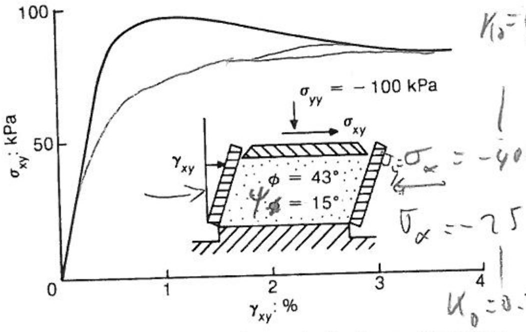


Fig. 7. Shear stress vs. shear strain for a shear box experiment on an overconsolidated soil

post-peak softening response for the trapdoor problem was partly physical without giving an explanation for the phenomenon. If we employ an associated flow rule in conjunction with a perfectly plastic model, softening is impossible. Then, Drucker's postulate holds, so that

$$\dot{\boldsymbol{\sigma}}^{\mathrm{T}}\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}=0 \rightarrow \dot{\boldsymbol{\sigma}}^{\mathrm{T}}\dot{\boldsymbol{\varepsilon}}=\dot{\boldsymbol{\sigma}}^{\mathrm{T}}\dot{\boldsymbol{\varepsilon}}^{\mathrm{e}} \geqslant 0 \quad (1)$$

so that softening is precluded. For non-associated flow rules, we may have $\dot{\boldsymbol{\sigma}}^{\mathrm{T}}\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}<0$, and as a consequence we have the possibility that

$$\dot{\boldsymbol{\sigma}}^{\mathrm{T}}\dot{\boldsymbol{\varepsilon}}=\dot{\boldsymbol{\sigma}}^{\mathrm{T}}(\dot{\boldsymbol{\varepsilon}}^{\mathrm{e}}+\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}})<0 \quad (2)$$

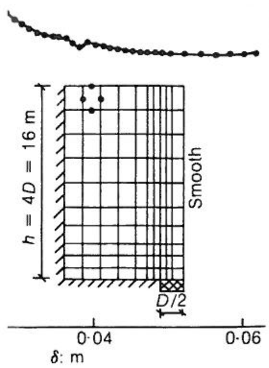
This softening post-peak response may also be demonstrated by considering a shear box experiment on an overconsolidated soil (Fig. 7). Here, the elasto-plastic differential stress–strain law has been integrated by a simple numerical integration formula. The residual shear strength is given by

$$\sigma_{xy}=\sigma_{yy}\frac{\cos\psi\sin\phi}{1-\sin\phi\sin\psi} \quad (3)$$

with σ_{xy} and σ_{yy} the shear and the normal stress respectively, and ϕ and ψ are the friction and dilatancy angles. However, a higher peak strength may be found. Depending on the initial stresses the peak strength can be as high as $\sigma_{xy}=\sigma_{yy}\tan\phi$.

It is recognized that a reduction in the friction angle also leads to global softening behaviour. In experiments, the reduction in the friction angle, geometrical effects and the non-association are probably all responsible for the observed global softening. Yet, global softening can be explained from a non-associated flow rule alone and we have demonstrated this through the examples of a trapdoor and of a shear box. A more detailed treatment of this type of softening is given by Vermeer & de Borst (1984).

A final remark concerns the stability and convergence characteristics of the iterative procedure for higher friction angles ($\phi>30°$). Griffiths and Koutsabeloulis report that some other solutions for high friction angles have been presented. This is true and it is a pity that we were unaware of Griffiths' thesis (1980). An important conclusion from his thesis is that 'convergence was much slower in the analysis of bearing capacity than earth pressure'. Earth pressure problems are relatively simple and so is the compression problem by Zienkiewicz, Humpheson & Lewis (1975). For these problems converged solutions were presented respectively for $\phi=40°$ and for $\phi=45°$. Griffiths (1980), however, also states that 'due to the complex confinements in the bearing capacity problem, convergence could rarely be achieved in a reasonable number of iterations when $\phi>30°$'. This is virtually in line with our experiences, but we also reported the following. When using a friction angle of 40° in conjunction with an associated flow rule both the strip footing and the circular footing problem converged.



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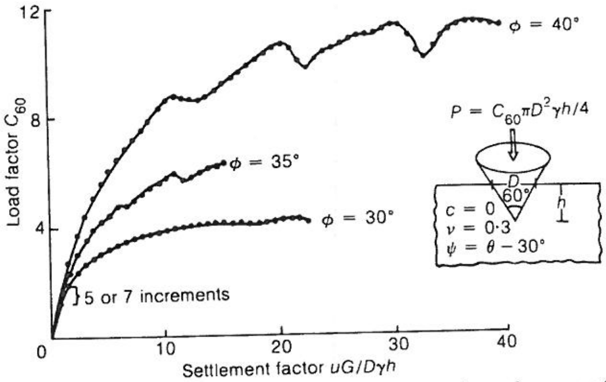

Fig. 8. Finite element results of the indentation of a smooth cone for various friction angles

Moreover we obtained converged solutions for moderate differences between the friction angle and the dilatancy angle. For a large difference between the friction angle and the dilatancy angle ($\phi = 40°$ and $\psi = 0°$) we could not obtain a converged solution. This indicates that the degree of non-normality is an important factor and that a high degree of non-normality deteriorates convergence. The magnitude of the friction angle is another factor which influences convergence. This influence can be demonstrated by the cone indentation problem of Fig. 8 which was studied by Zaadnoordijk (1983). For the same degree of non-normality ($\phi - \psi = 30°$) the solution becomes more unstable and the computational effort increases with increase in friction angle. Zaadnoordijk (1983), however, also found that the solutions became more stable for the same friction angle when he reduced the difference between the friction angle and the dilatancy angle.

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