# Influence of soil strength spatial variability on the stability of an undrained clay slope by finite elements

**D. V. Griffiths [1] and Gordon A. Fenton [2]**

*Abstract*

An investigation has been performed into the stability of an undrained clay slope having spatially randomly varying shear strength. The results of the study lead to a direct comparison between the probability of slope failure and the traditional Factor of Safety for a range of statistically defined input shear strength properties. The results highlight the influence of the spatial correlation length, a variable which is routinely omitted from conventional probabilistic studies in geotechnics.

*Introduction*

The paper presents results obtained using a program developed by the authors which merges nonlinear elasto-plastic finite element analysis (e.g. Smith and Griffiths 1998) with random field theory (e.g. Fenton 1990, Vanmarcke 1984). Some initial work using this approach has been reported by Paice and Griffiths (1997), however the problem to be considered in this paper is an undrained clay slope ($\phi_u = 0$) of height $H$ with a gradient of 2:1 resting on a foundation layer, also of depth $H$. A typical finite element mesh is shown in Figure 1.

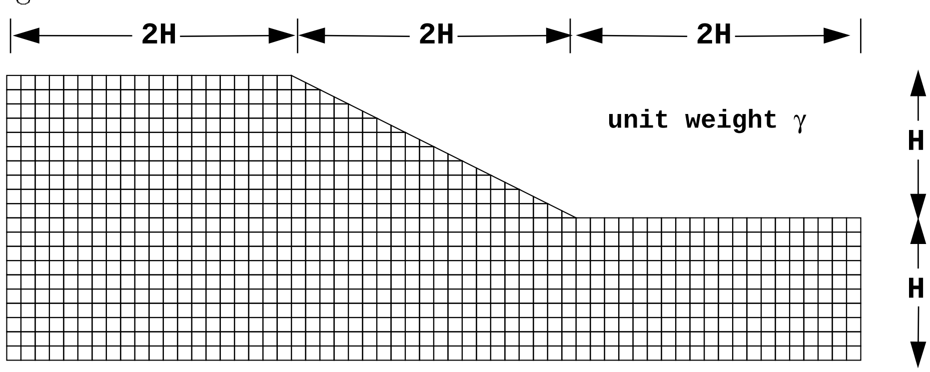


**Figure 1:** Mesh used for slope stability analyses.

[1]Professor, Geomechanics Research Center, Colorado School of Mines, Golden, CO 80401
[2]Professor, Department of Engineering Mathematics, Dalhousie University, Canada

In this study, the variability of the undrained shear strength ($c_u$) is assumed to be characterized by a lognormal distribution with the following three parameters:

| | | Units |
|---|---|---|
| Mean | $\mu_{c_u}$ | Stress |
| Standard Deviation | $\sigma_{c_u}$ | Stress |
| Spatial Correlation Length | $\theta_{\ln c_u}$ | Length |

The mean and standard deviation can conveniently be expressed in terms of the dimensionless coefficient of variation defined as

$$C.O.V._{c_u} = \frac{\sigma_{c_u}}{\mu_{c_u}} \tag{1}$$

Since the actual undrained shear field is assumed lognormally distributed, taking its logarithm yields an "underlying" normally distributed (or Gaussian) field. The spatial correlation length is measured with respect to this underlying field, that is, with respect to $\ln c_u$. In particular, the spatial correlation length ($\theta_{\ln c_u}$) describes the distance over which the spatially random values will tend to be significantly correlated in the underlying Gaussian field. Thus, a large value of $\theta_{\ln c_u}$ will imply a smoothly varying field, while a small value will imply a ragged field. The spatial correlation length can be estimated from a set of shear strength data taken over some spatial region simply by performing the statistical analyses on the log-data. In practice, however, $\theta_{\ln c_u}$ is not much different in magnitude from the correlation length in real space and, for most purposes, $\theta_{c_u}$ and $\theta_{\ln c_u}$ are interchangeable given their inherent uncertainty in the first place. In the current study, the spatial correlation length has been non-dimensionalized by dividing it by the height of the embankment $H$.

It should be emphasised that the spatial correlation length is rarely taken into account in routine probabilistic studies relating to geotechnical engineering. In the majority of these cases, a Single Random Variable approach (e.g. Harr 1987, Duncan 2000) is used, which is equivalent to setting $\theta_{\ln c_u} = \infty$.

It has been suggested (see e.g. Lee *et al* 1983, Kulhawy *et al* 1991 and Duncan 2000) that typical $C.O.V._{c_u}$ values for the undrained shear strength lie in the range 0.1-0.5, however the spatial correlation length is less well documented, especially in the horizontal direction, and may well exhibit anisotropy. While the analysis tools used in this study have the capability of modeling an anisotropic spatial correlation field, all the results presented in this paper assume that $\theta_{\ln c_u}$ is isotropic. This is not a severe restriction, since the geometry can often be scaled to achieve the desired spatial correlation structure

*Brief description of the finite element method used*

The slope stability analyses use an elastic-perfectly plastic stress-strain law with a Tresca failure criterion. Plastic stress redistribution is accomplished using a viscoplastic algorithm which uses 8-node quadrilateral elements and reduced integration in both the stiffness and stress redistribution parts of the algorithm. The theoretical basis of the method is described more fully in Chapter 6 of the text by Smith and Griffiths (1998), and for a discussion of the method applied to slope stability analysis, the reader is referred to Griffiths and Lane (1999).

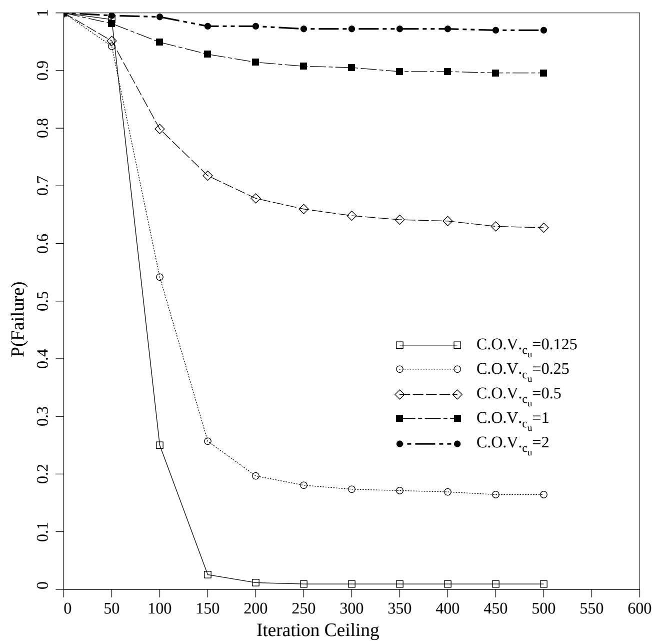


**Figure 2:** Probability of failure vs. Iteration Ceiling.

In brief, the analyses involve the application of gravity loading, and the monitoring of stresses at all the Gauss points. If the Tresca criterion is violated, the program attempts to redistribute those stresses to neighboring elements that still have reserves of strength. This is an iterative process which continues until the Tresca criterion and global equilibrium are satisfied at all points within the mesh under quite strict tolerances.

In this study,"failure" is said to have occurred if, for any given realization, the algorithm is unable to converge within 500 iterations. Following a set of 1000 realizations of the Monte-Carlo process the probability of failure is simply defined as the proportion of these realizations that required 500 or more iterations to converge.

While the choice of 500 as the iteration ceiling is subjective, Figure 2 confirms, for the case of $\theta_{\ln c_u}/H = 1$, that the probability of failure computed using this criterion is quite stable after about 200 iterations.

*Parametric studies*

In the parametric studies described in this section, the mean strength expressed in the form of a Stability Number

$$N_s = \mu_{c_u}/(\gamma H), \tag{2}$$

was given the values 0.15, 0.20, 0.25 and 0.30, and in each case, a range of $C.O.V._{c_u}$ and $\theta_{\ln c_u}/H$ values were investigated as follows:

$$\begin{aligned} \theta_{\ln c_u}/H &= 0,\ 0.5,\ 1,\ 2,\ 4,\ 8,\ \infty \\ C.O.V._{c_u} &= 0.125,\ 0.25,\ 0.5,\ 1,\ 2,\ 4 \end{aligned} \quad (3)$$

To put the probabilistic results in context, Table 1 shows the Factor of Safety $F$ from conventional limit equilibrium analysis for the slope in Figure 1 assuming a homogeneous shear strength defined by the Stability Number $N_s$.

**Table 1:** Factors of Safety Assuming Homogeneous Soil

| $N_s$ | $F$ |
|---|---|
| 0.15 | 0.88 |
| 0.17 | 1.00 |
| 0.20 | 1.18 |
| 0.25 | 1.47 |
| 0.30 | 1.77 |

For each set of assumed statistical properties given by $C.O.V._{c_u}$ and $\theta_{\ln c_u}/H$, Monte-Carlo simulations were performed, typically involving 1000 repetitions or "realizations" of the shear strength random field and the subsequent finite element analysis. Each realization of the random field, while having the same underlying statistics, led to a quite different spatial pattern of shear strength values within the slope.

Figure 3 shows two typical random field realizations and associated failure mechanisms for slopes with $\theta_{\ln c_u}/H = 0.5$ and $\theta_{\ln c_u}/H = 2$. Notice how the higher $\theta_{\ln c_u}/H$ gives a more slowly varying shear strength over space and a smoother failure surface.

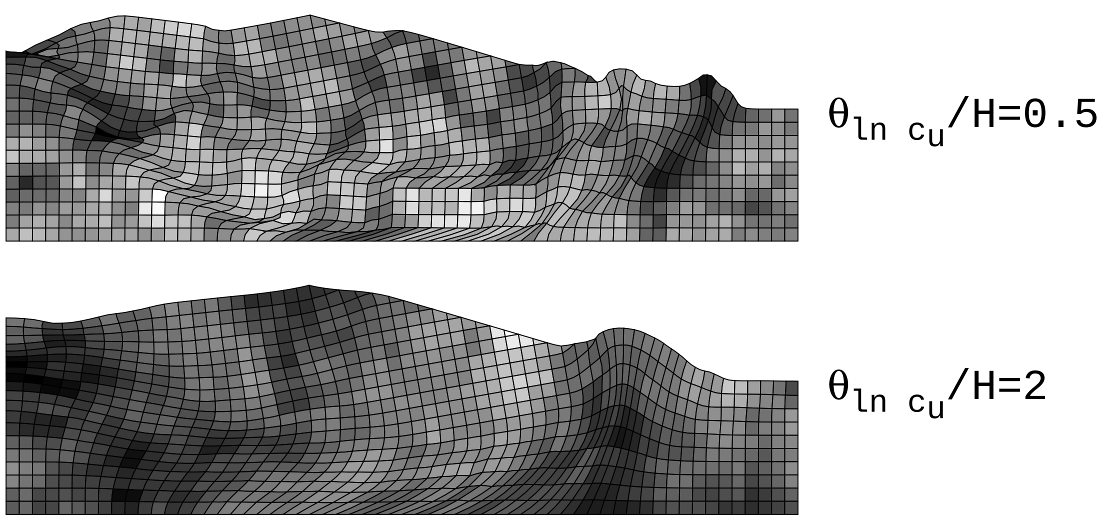


**Fig 3. Typical random field realizations. Darker zones indicate weaker soil.**

*Single random variable approach*

It is instructive to consider the special case of $\theta_{\ln c_u} = \infty$, which implies that each realization of the Monte-Carlo process gives a uniform strength, the same everywhere, but with the strength varying from one realization to the next according to the governing lognormal distribution. The probability of failure in such a case is simply equal to the probability that the Stability Number $N_s$ will be below 0.17, the value that would give a Factor of Safety of unity.

For example, if $\mu_{c_u} = 0.25\gamma H$ and $\sigma_{c_u} = 0.125\gamma H$, corresponding to $C.O.V._{c_u} = 0.5$, the statistics of the Stability Number are therefore given by $\mu_{N_s} = 0.25$, $\sigma_{N_s} = 0.125$ and $C.O.V._{N_s} = 0.5$.

From standard relationships, the mean and standard deviation of the underlying *normal* distribution of the Stability Number are given by:

$$\sigma_{\ln N_s} = \sqrt{\ln\left\{1 + \left(\frac{\sigma_{N_s}}{\mu_{N_s}}\right)^2\right\}} \tag{4}$$

$$\mu_{\ln N_s} = \ln \mu_{N_s} - \frac{1}{2}\sigma^2_{\ln N_s} \tag{5}$$

hence $\mu_{\ln N_s} = -1.498$ and $\sigma_{\ln N_s} = 0.472$.

The probability of failure is therefore given by:

$$p(N_s < 0.17) = \Phi\left(\frac{\ln 0.17 - \mu_{\ln N_s}}{\sigma_{\ln N_s}}\right) \tag{6}$$

$$= 0.281 \tag{7}$$

where $\Phi$ is the cumulative normal distribution function. The relationship between the Factor of Safety (assuming a constant shear strength equal to $\mu_{c_u}$) and the probability of failure assuming a Single Random Variable ($\theta_{\ln c_u} = \infty$) is summarized in Figure 4 for a range of $C.O.V._{c_u}$ values.

Apart from the rather obvious conclusion that the probability of failure goes up as the Factor of Safety goes down, it is also clear that for the majority of cases, the probability of failure also goes up as the $C.O.V._{c_u}$ of the shear strength increases. This result is not necessarily intuitive, since soil with a higher $C.O.V._{c_u}$ contains elements that are much weaker *and* much stronger than the mean. The result indicates however, that the weaker elements dominate the stability calculation.

The only exception to this trend occurs when the mean strength indicates a Factor of Safety of *less* than unity. As shown in Figure 4, the probability of failure in such cases is understandably high, however the role of $C.O.V._{c_u}$ has the opposite effect to that described above, with lowest values of $C.O.V._{c_u}$ tending to give the highest values of the probability of failure.

It is interesting to note that using this approach, a slope with a Factor of Safety of 1.50, based on the mean strength, would have a probability of failure as high as 27% if $C.O.V._{c_u} = 0.5$, the upper limit of the recommended range of Lee *et al* 1983 and others.

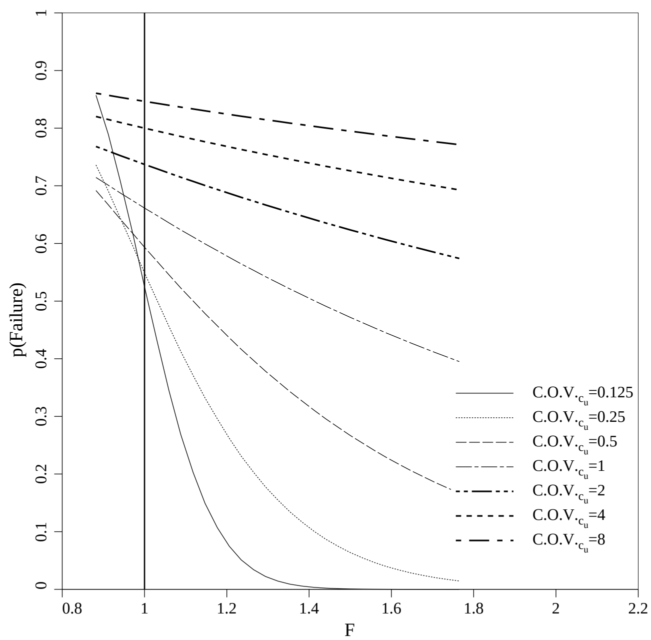


**Fig 4. Factor of Safety vs. Probability of Failure.**
**Single random variable approach, $\theta_{\ln c_u} = \infty$**

*Random field approach*

The code developed by the authors enables a random field of shear strength values to be generated and subsequently mapped onto the finite element mesh. In a random field, the value assigned to each cell (or finite element in this case) is itself a random variable, thus the mesh of Figure 1 which has 910 finite elements consists of 910 random variables. The random variables can be correlated to one another by controlling the spatial correlation length $\theta_{\ln c_u}$ as described previously, hence the single random variable approach discussed in the previous section can now be viewed as just a special case of a much more powerful analytical tool.

Figures 5 and 6 show the effect of the spatial correlation length $\theta_{\ln c_u}/H$ on a soil with a Factor of Safety of 1.47 (based on the mean strength) for a range of $C.O.V._{c_u}$ values. Figure 5 clearly indicates two branches relating to the value of $C.O.V._{c_u}$. For low values of $0 < C.O.V._{c_u} < 0.5$, the probability of failure increases as $\theta_{\ln c_u}/H$ increases, indicating that the Single Random Variable approach in which $\theta_{\ln c_u} = \infty$ is conservative. For high values of $1 < C.O.V._{c_u}$ quite the reverse trend is apparent, with the higher values of $\theta_{\ln c_u}$ tending to *underestimate* the probability of failure.

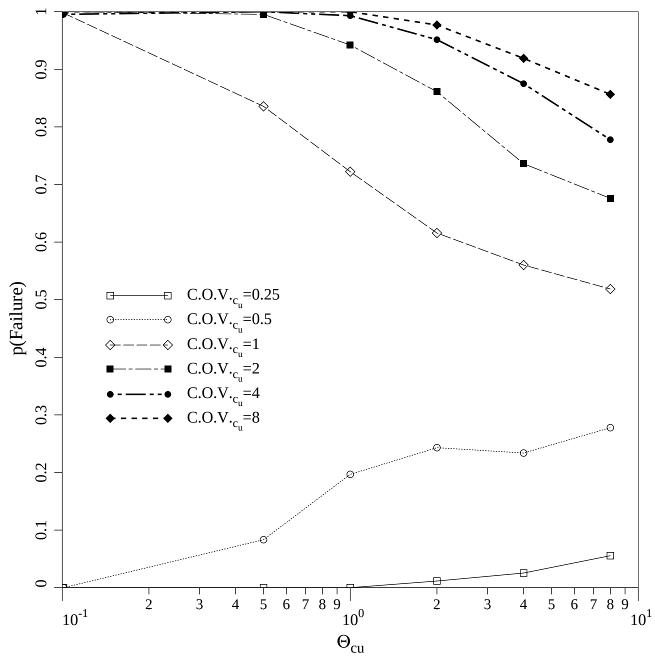


**Fig 5. Influence of $\theta_{\ln c_u}/H$ on a slope with F=1.47.**

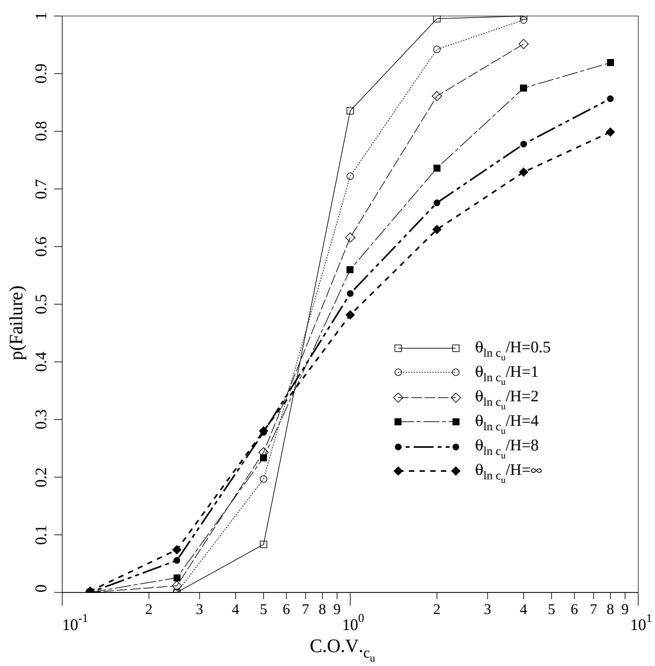


**Fig 6. Influence of $C.O.V._{c_u}$ on a slope with F=1.47.**

Figure 6 shows an alternative representation of the same data with $C.O.V._{c_u}$ plotted along the abscissa. This figure shows more clearly how $\theta_{\ln c_u} = \infty$ will tend to overestimate the probability of failure for low $C.O.V._{c_u}$ values and underestimate it for high values. It is also of interest to note the sensitivity of the probability of failure to the value of $C.O.V._{c_u}$

for low levels of correlation. For example, the line corresponding to $\theta_{\ln c_u}/H = 0.5$ rises steeply from zero to 100% probability of failure within the relatively narrow band of $0.25 < C.O.V._{c_u} < 2$. For even smaller values of $\theta_{\ln c_u}/H$ the rise was observed to be even more dramatic, although these results are not presented here. A further point of interest from Figure 6 is that all the lines appear to coincide at approximately the same value of $C.O.V._{c_u} \approx 0.65$, implying that at this level of shear strength variance, the probability of failure is independent of $\theta_{\ln c_u}/H$. This result and others are currently under further investigation by the authors.

The observations made with respect to Figures 5 and 6 were for the particular case of a mean shear strength that would have given a Factor of Safety of 1.47. The results from further analyses of a range of mean shear strength values corresponding to the Stability Numbers in Table 1 are shown in Figure 7. In order to reduce the number of variables, only the results assuming $C.O.V._{c_u} = 0.5$ are shown.

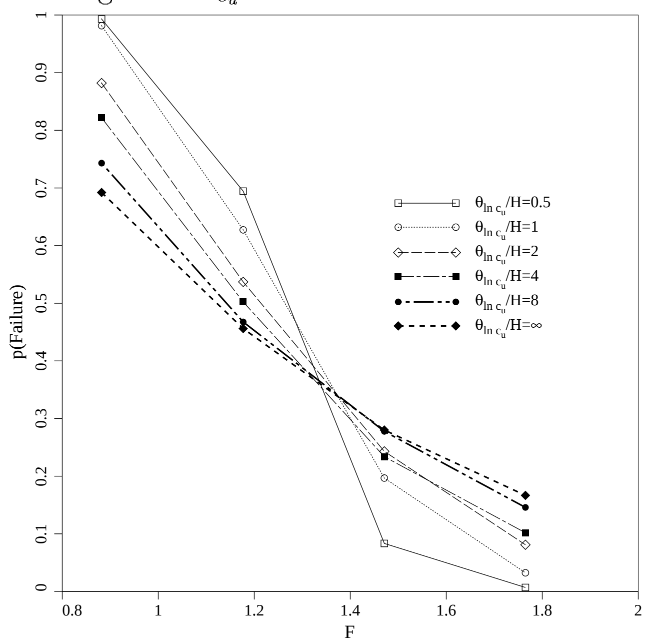


**Fig 7. Influence of $\theta_{\ln c_u}/H$ on the Probability of Failure for a range of deterministic Factors of Safety ($C.O.V._{c_u} = 0.5$).**

Figure 7 indicates another type of "cross over" with respect to the Factor of Safety. For the given value of $C.O.V._{c_u} = 0.5$, the Single Random Variable approach corresponding to $\theta_{\ln c_u} = \infty$ appears to overestimate the probability of failure for slopes with relatively high deterministic Factors of Safety ($F > 1.4$) and underestimate it for lower Factors of Safety ($F < 1.4$).

*Concluding remarks*

The paper has shown that soil strength heterogeneity in the form of a spatially varying lognormal distribution can significantly affect the stability of a slope of undrained clay when viewed in a probabilistic context. In this paper, particular attention was paid to the validity of treating the heterogeneity as a Single Random Variable which was shown to be a special case of the authors' formulation corresponding to an infinite correlation length of $\theta_{\ln c_u} = \infty$.

The following more specific observations can be made from the results presented in this paper:

1. For the slope considered in this study with a Factor of Safety of 1.47 based on the mean strength, the Single Random Variable approach gave conservative estimates of the probability of failure for Coefficient of Variation values in the "typical" range of $0 < C.O.V._{c_u} < 0.5$. For higher values of $C.O.V._{c_u}$ however, the Single Random Variable approach gave unconservative estimates.

2. For the slope considered in this study with $C.O.V._{c_u} = 0.5$, the Single Random Variable approach gave conservative estimates of the probability of failure for higher Factors of Safety in the approximate range $F > 1.4$ and unconservative estimates for lower Factors of Safety when $F < 1.4$.

More work remains to be done in this area, but the implications of this study are that the Single Random Variable approach is an acceptable guide to probabilistic slope stability providing the mean strength indicates a relatively high Factor of Safety. For more critical cases, in which the mean strength indicates a Factor of Safety closer to unity, the Single Random Variable approach can give an unconservative estimate of the probability of failure, i.e. *lower* than the "true" value.

A final comment relates to the influence of the Coefficient of Variation of the soil shear strength. While increasing the value of $C.O.V._{c_u}$ introduces both stronger and weaker zones of soil into the slope, the weaker soil always dominates the overall performance leading to a *less* stable slope.

**ACKNOWLEDGEMENT**

The results shown in this paper form part of a much broader study into the influence of soil heterogeneity on stability problems in geotechnical engineering. The writers wish to acknowledge the support of NSF Grant No. CMS-9877189

# References

[1] J.M. Duncan. Factors of safety and reliability in geotechnical engineering. *J Geotech Geoenv Eng, ASCE*, 126(4):307–316, 2000.

[2] G.A. Fenton. *Simulation and analysis of random fields.* PhD thesis, Department of Civil Engineering and Operations Research, Princeton University, 1990.

[3] D.V. Griffiths and P.A. Lane. Slope stability analysis by finite elements. *Géotechnique*, 49(3):387–403, 1999.

[4] M.E. Harr. *Reliability based design in civil engineering.* McGraw Hill, London, New York, 1987.

[5] F.H. Kulhawy, M.J.S. Roth, and M.D. Grigoriu. Some statistical evaluations of geotechnical properties. In *Proc. ICASP6, 6th Int. Conf. Appl. Stats. Prob. Civ. Eng.* 1991.

[6] I.K. Lee, W. White, and O.G. Ingles. *Geotechnical Engineering.* Pitman, London, 1983.

[7] G.M. Paice and D.V. Griffiths. Reliability of an undrained clay slope formed from spatially random soil. In J-X. Yuan, editor, *IACMAG 97*, pages 1205–1209. A.A. Balkema, Rotterdam, 1997.

[8] I.M. Smith and D.V. Griffiths. *Programming the Finite Element Method.* John Wiley and Sons, Chichester, New York, 2nd edition, 1988.

[9] E.H. Vanmarcke. *Random fields: Analysis and synthesis.* The MIT Press, Cambridge, Mass., 1984.