

# TRADITIONAL AND ADVANCED PROBABILISTIC SLOPE STABILITY ANALYSIS

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## ***ABSTRACT***

The paper contrasts results obtained by the traditional First Order Reliability Method (FORM) and a more advanced Random Finite Element Method (RFEM) in a benchmark problem of slope stability analysis with random shear strength parameters. The key difference between the methods is that RFEM takes into account spatial correlation in a rigorous way allowing slope failure to occur naturally along the path of least resistance. Both methods lead to predictions of the "probability of slope failure" as opposed to the more traditional "factor of safety" measure of slope safety, however they give significant different results depending on the value of the correlation length. For small correlation lengths FORM is generally conservative, however it is shown that there is a "worst case" correlation length for which FORM leads to *unconservative* predictions of slope reliability.

## ***INTRODUCTION***

Slope stability analysis is one of the main areas of interest to geotechnical designers, and also seems a natural application for probabilistic approaches since the analysis leads to a "probability of failure" as opposed to the more customary "factor of safety". This paper will review a traditional approach to probabilistic slope stability analysis, the first order reliability method (FORM) and then go on to discuss the more advanced random finite element method (RFEM). The methods will be compared on a benchmark slope and conclusions will be drawn regarding the limitations of FORM, in particular, the effect of the spatial correlation length which can be rigorously modeled by RFEM.

## **FIRST ORDER RELIABILITY METHOD**

### **Theory**

The first order reliability method (FORM) is a process which can be used to determine the probability of a failure given the distribution data and limit state function. The method is based on the Hasofer-Lind reliability index (Hasofer and Lind 1964),  $\beta_{HL}$ , which can be described as the distance, in standard deviation units, between the most probable set of values and the most probable set of values that causes a failure. Calculation of this value is an iterative process, finding the minimum value of a matrix calculation subject to the constraint that the values result in a system failure. However, common solver routines found in several software packages (e.g. Excel and Mathematica) can easily arrive at the solution. Once the reliability index has been determined, the probability of failure,  $P_f$ , is a simple calculation.

### **Limit State Function**

Each reliability analysis requires a limit state function, which defines failure or safe performance. Limit states could relate to strength failure, serviceability failure, or anything else that describes unsatisfactory performance. The limit state function,  $g$ , is defined

$$\begin{aligned} g(x_1, \dots, x_N) \geq 0 &\longrightarrow \text{Safe} \\ g(x_1, \dots, x_N) < 0 &\longrightarrow \text{Failure} \end{aligned} \quad (1)$$

where  $N$  is the number of random variables. Often it is sufficient for the limit state function to be the resistance minus the load. Other common forms of the limit state function are the factor of safety minus one and the logarithm of the factor of safety.

The limit state function can be determined from analytical theory for simple systems. For more complex systems, it may need to be approximated numerically with curve fitting.

### **Hasofer-Lind Reliability Index**

The reliability index,  $\beta_{HL}$ , is the distance in standard deviation units between the most probable set of random variables (the means), and the most probable set of random variables that causes a failure. Determination of  $\beta_{HL}$  is an iterative process and it is defined by

$$\beta_{HL} = \min_{g=0} \sqrt{\left\{ \frac{x_i - \mu_i}{\sigma_i} \right\}^T [R]^{-1} \left\{ \frac{x_i - \mu_i}{\sigma_i} \right\}} \quad i = 1, \dots, N \quad (2)$$

where  $\{(x_i - \mu_i)/\sigma_i\}$  is the vector of the random variable values reduced to standard normal space and  $[R]$  is the correlation matrix of the variables.

## Visualization

To better understand and visualize this method, consider the following arbitrary problem. Two random variables,  $x_1$  and  $x_2$ , are normally distributed and have the following parameters:

$$\begin{aligned} \mu_{x_1} &= 6.0 & \sigma_{x_1} &= 1.0 \\ \mu_{x_2} &= 7.0 & \sigma_{x_2} &= 0.75 & \rho_{x_1, x_2} &= -0.35 \end{aligned} \quad (3)$$

Failure of the system is given by the limit state function:

$$g(x_1, x_2) = -0.03x_1^3 - 0.25x_2^2 + 29.16 \quad (4)$$

The probability density function governing two normal random variables correlated by  $\rho$  can be written as (e.g., Fenton and Griffiths 2006):

$$f_{x_1, x_2}(x_1, x_2) = \frac{1}{2\pi\sigma_{x_1}\sigma_{x_2}\sqrt{1-\rho^2}} e^{-\frac{\beta(x_1, x_2)^2}{2}} \quad (5)$$

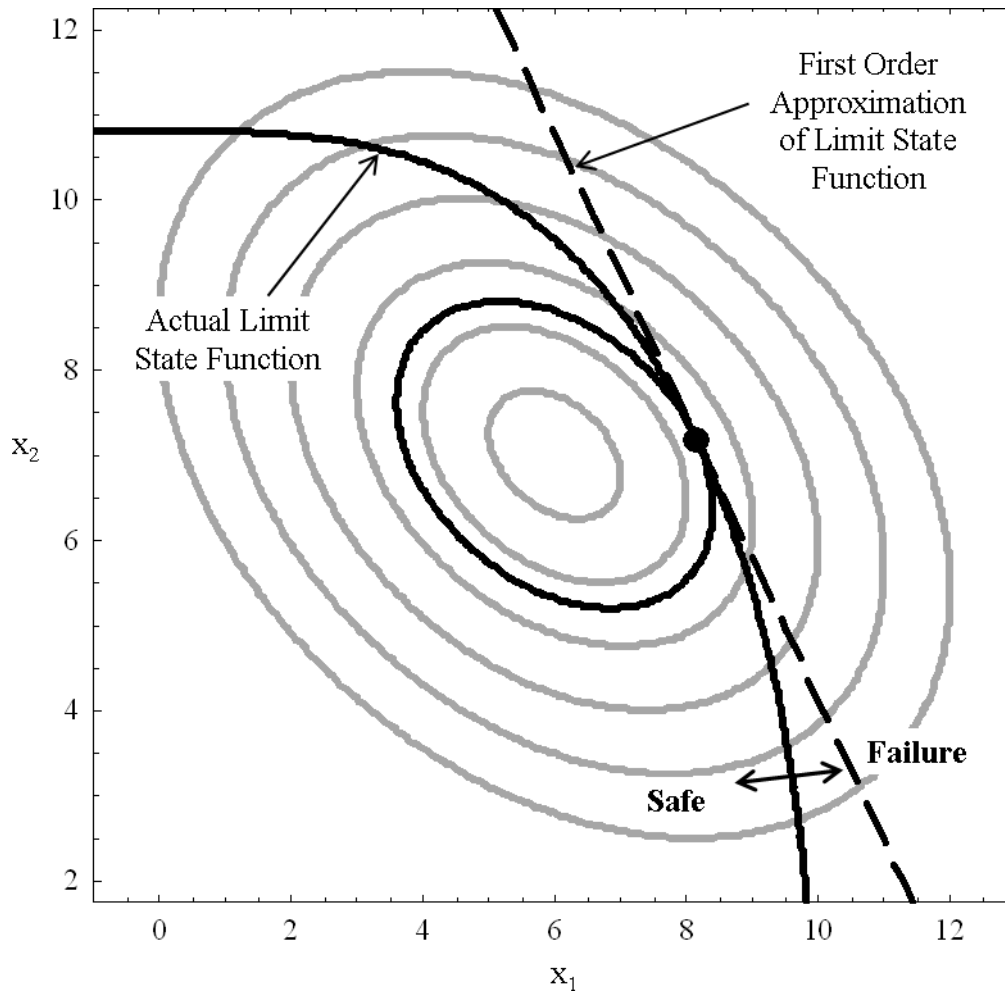
where

$$\beta(x_1, x_2) = \sqrt{\left\{ \frac{x_i - \mu_{x_i}}{\sigma_{x_i}} \right\}^T [R]^{-1} \left\{ \frac{x_i - \mu_{x_i}}{\sigma_{x_i}} \right\}} \quad i = 1, 2 \quad (6)$$

Note that the minimum value of  $\beta(x_1, x_2)$ , given that the limit state function is zero, is the Hasofer-Lind reliability index,  $\beta_{HL}$ .

Plotting the probability density function in three dimensions would result in a surface in the shape of a bell. By definition, the volume under the surface is unity. The limit state function divides the volume into a failure region and a safe region. The probability of failure is defined as the volume under the probability density function in the failure region. FORM uses a first order approximation of the limit state function and therefore the calculated probability of failure is also approximate. Numerical integration of the probability distribution function in the failure region leads to more accurate results and is discussed later.

In plan view, the probability density function can be visualized as a contour plot involving a series of ellipses, and the limit state function can be seen as a line separating the failure and safe regions, see Figure 1. The contours in Figure 1 are actually contours of  $\beta(x_1, x_2)$  (i.e.  $\beta(x_1, x_2) = 1, 2, 3, 4, \dots$ ), nevertheless, each contour represents a constant value of the probability density function.



**Figure 1. Plan View of the Probability Density Function.**

The solid curved line represents the actual limit state function. The smallest ellipse that the limit state function touches is the contour of  $\beta = \beta_{HL}$ , represented above by the darker ellipse. The point where they meet represents the most probable failure point. The dashed straight line that also passes through that point is the first order approximation of the limit state function.

The first order approximation assumed in FORM could lead to an underestimate of the probability of failure if the actual limit state function curves towards the mean values as seen in Figure 1. A more accurate, yet more time consuming, method to determine the probability is to numerically integrate the probability distribution function in the region of failure. A relatively simple algorithm involving the repeated mid-point rule (e.g., Griffiths and Smith 2006) can be devised to accomplish this task.

## **FORM software**

### *Excel*

The limit state function and properties described in Eq. 3 and Eq. 4 have been run through an Excel spreadsheet using the solver add-in (e.g., Low and Tang 1997, Denavit 2006) in which the FORM algorithm has been implemented. The Hasofer-

Lind reliability index is given as  $\beta_{HL} = 2.40$ , corresponding to a probability of failure of  $P_f = 0.814\%$

### Mathematica

Using Mathematica, the same calculations can be performed. The following shows the lines which must be executed:

```

μx1 = 6.0; σx1 = 1.0; μx2 = 7.0; σx2 = 0.75; ρ = -0.35;
g[x1_, x2_] = -0.03 x1 ^3 - 0.25 x2 ^2 + 29.16;
Z1[x1_] = (x1 - μx1) / σx1; Z2[x2_] = (x2 - μx2) / σx2;
Z[x1_, x2_] = {Z1[x1], Z2[x2]}; R = {{1, ρ}, {ρ, 1}};
β[x1_, x2_] = Sqrt[Z[x1, x2].Inverse[R].Z[x1, x2]];
sol = Minimize[β[x1, x2], g[x1, x2] == 0, {x1, x2}]
{2.40244, {x1 -> 8.1466, x2 -> 7.19444}}
βHL = sol[[1]]; Dx1 = x1 /. sol[[2]]; Dx2 = x2 /. sol[[2]];
PfFORM = 1 - CDF[NormalDistribution[0, 1], βHL]
0.00814306

```

Again, the probability of failure is 0.814%, with a reliability index of 2.40, corresponding to a most probable failure point of  $x_1 = 8.15$  and  $x_2 = 7.19$ . Both the reliability index and the most probable failure point can be graphically checked using Figure 1.

As discussed earlier, numerical integration can determine the probability of failure directly but more slowly. Below is a set of commands which will perform the numerical integration:

```

nsize = 8; ndivs = 1000; PfNInt = 0;
x1min = μx1 - nsize σx1; x1max = μx1 + nsize σx1;
x2min = μx2 - nsize σx2; x2max = μx2 + nsize σx2;
width = {(x1max - x1min) / ndivs, (x2max - x2min) / ndivs};
Area = width[[1]] * width[[2]];
means = {μx1, μx2}; cov = {{σx1^2, ρ σx1 σx2}, {ρ σx1 σx2, σx2^2}};
dist = MultinormalDistribution[means, cov];
For[i = 0, i < ndivs, i++, For[j = 0, j < numdivs, j++,
  value = {x1min + width[[1]] * (i + 0.5), x2min + width[[2]] * (j + 0.5)};
  If[g[value[[1]], value[[2]]] < 0, PfNInt = PfNInt + PDF[dist, value] * Area,
  1];
PfNInt
0.00963503

```

Numerical integration of the volume of the probability density function corresponding to  $g(x_1, x_2) < 0$  gave the probability of failure 0.964%, relatively 16% higher than given by FORM.

## PROBABILISTIC SLOPE STABILITY ANALYSIS

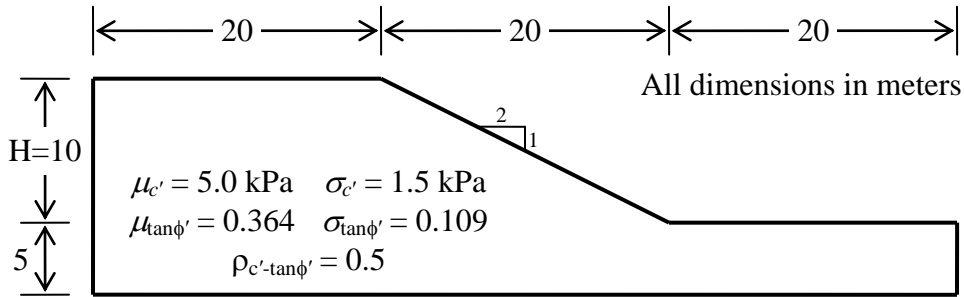
For slope stability, no analytical equation exists which can serve as a limit state function. In this case, a numerical approximation will need to be formulated to use as the limit state function. This can be accomplished by fitting a curve to the results from several finite element analyses using the strength reduction method (e.g., Griffiths and Lane 1999). This method involves applying gravity loads to the finite element mesh and systematically weakening the soil until a sufficient number of element have yielded to allow the formation of a failure mechanism.

For example, with two ( $N=2$ ) random variables ( $c'$ ,  $\tan \phi'$ ), a quadratic surface without cross-terms with five ( $2N + 1 = 5$ ) constants of the form

$$FS(c', \tan \phi') = a_1 + a_2 c' + a_3 \tan \phi' + a_4 c'^2 + a_5 \tan^2 \phi' \quad (7)$$

could be used to approximate the factor of safety function.

Figure 2 shows the dimensions and properties of a hypothetical sample slope which was analyzed using this method.



**Figure 2. Slope Dimensions and Properties.**

The following parameters were taken as deterministic: unit weight ( $\gamma = 20 \text{ kN/m}^3$ ), modulus of elasticity ( $E = 100,000 \text{ kPa}$ ), Poisson's ratio ( $\nu = 0.3$ ), and dilation angle ( $\psi = 0$ ). It was assumed that there was no correlation between the variables and that the variables were normally distributed. The limit state function will then be the factor of safety function minus one, thus

$$g(c', \tan \phi') = FS(c', \tan \phi') - 1 \quad (8)$$

In order to find the constants in Eq. 7, five finite element analyses were run with the following input and results.

**Table 1. Sample Points for the Approximate Limit State Function.**

Sample Point	Value of $c'$	$c'$ (kPa)	Value of $\tan\phi'$	$\tan\phi'$	Factor of Safety
1	$\mu_{c'}$	5.00	$\mu_{\tan\phi'}$	0.364	1.09
2	$\mu_{c'} + \sigma_{c'}$	6.50	$\mu_{\tan\phi'}$	0.364	1.17
3	$\mu_{c'} - \sigma_{c'}$	3.50	$\mu_{\tan\phi'}$	0.364	1.00
4	$\mu_{c'}$	5.00	$\mu_{\tan\phi'} + \sigma_{\tan\phi'}$	0.473	1.33
5	$\mu_{c'}$	5.00	$\mu_{\tan\phi'} - \sigma_{\tan\phi'}$	0.255	0.84

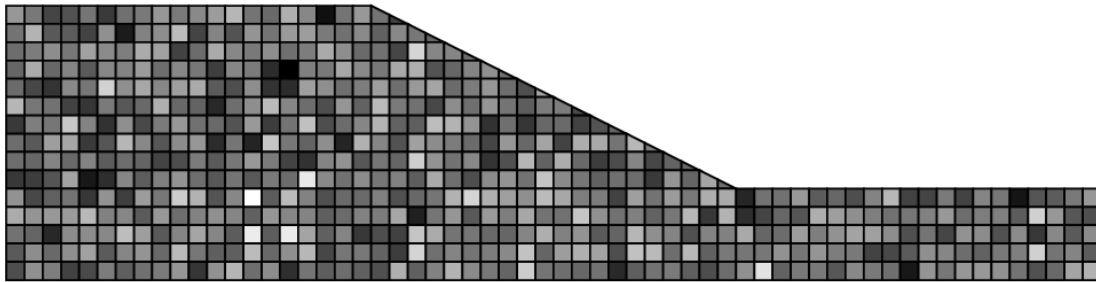
Solving for the five constants yields the limit state function:

$$g(c', \tan \phi') = -0.123 + 0.079c' + 2.554 \tan \phi' - 0.002c'^2 - 0.421 \tan^2 \phi' - 1 \quad (10)$$

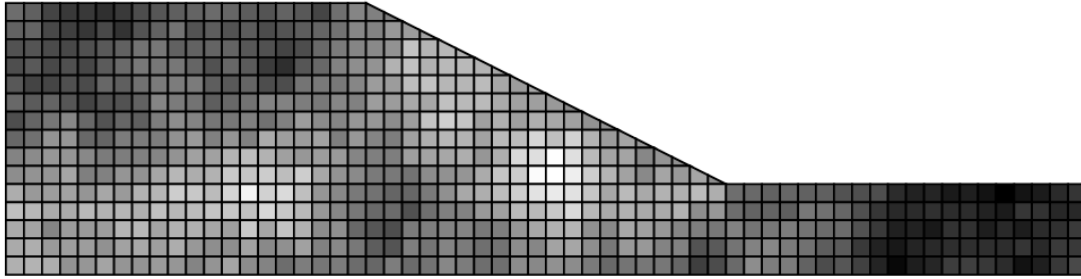
Implementing this function along with the soil properties into Excel or Mathematica as described earlier will yield a Hasofer-Lind reliability index,  $\beta_{HL}$ , of 0.301, corresponding to a probability of failure,  $P_f$ , of 38.2%.

### Random Finite Element Method

The random finite element method (RFEM) is an entirely separate method for determining the probability of failure. This method involves a Monte Carlo simulation with many different realizations of the soil properties. Each realization of the soil properties involves overlaying a random field onto a finite element mesh, essentially resulting in each element being a random variable. In doing this, a new parameter becomes evident, the spatial correlation length. This parameter describes the tendency of elements spatially near each other to be correlated. For slope stability problems, it is described in the dimensionless form,  $\Theta$ , which is the correlation length divided by the height of the slope,  $H$ . This parameter can be clearly seen in the following two figures of slopes with the same mean and standard deviation where the darker elements represent higher strength.



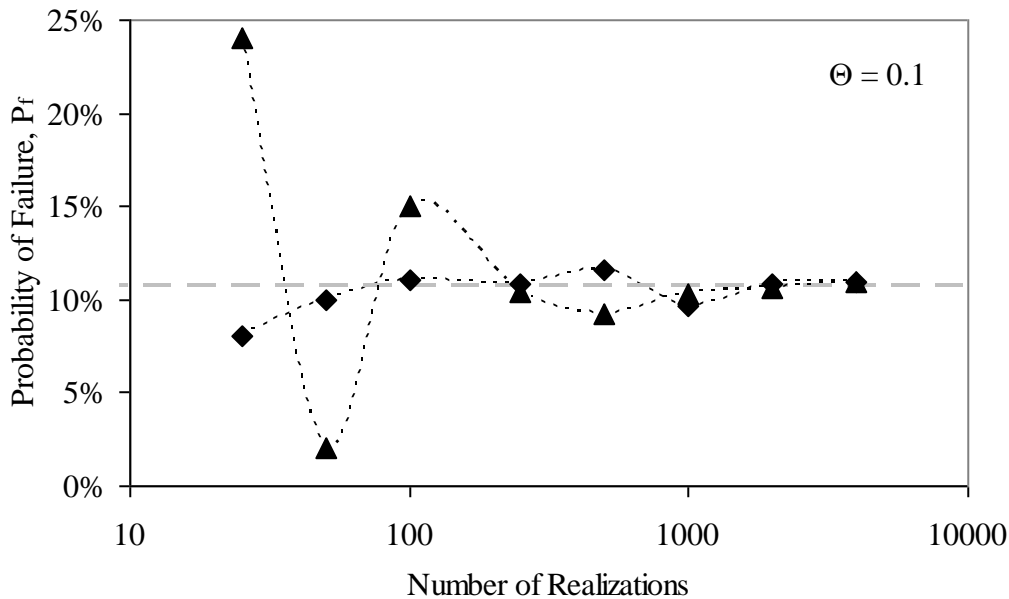
**Figure 3. Slope with Low Correlation Length.**



**Figure 4. Slope with High Correlation Length.**

Once the properties are assigned, a finite element analysis determines whether or not failure occurs, and the process is repeated. The nature of RFEM can lead to quite time-consuming calculations compared with FORM, however the latter method does not explicitly incorporate the correlation length.

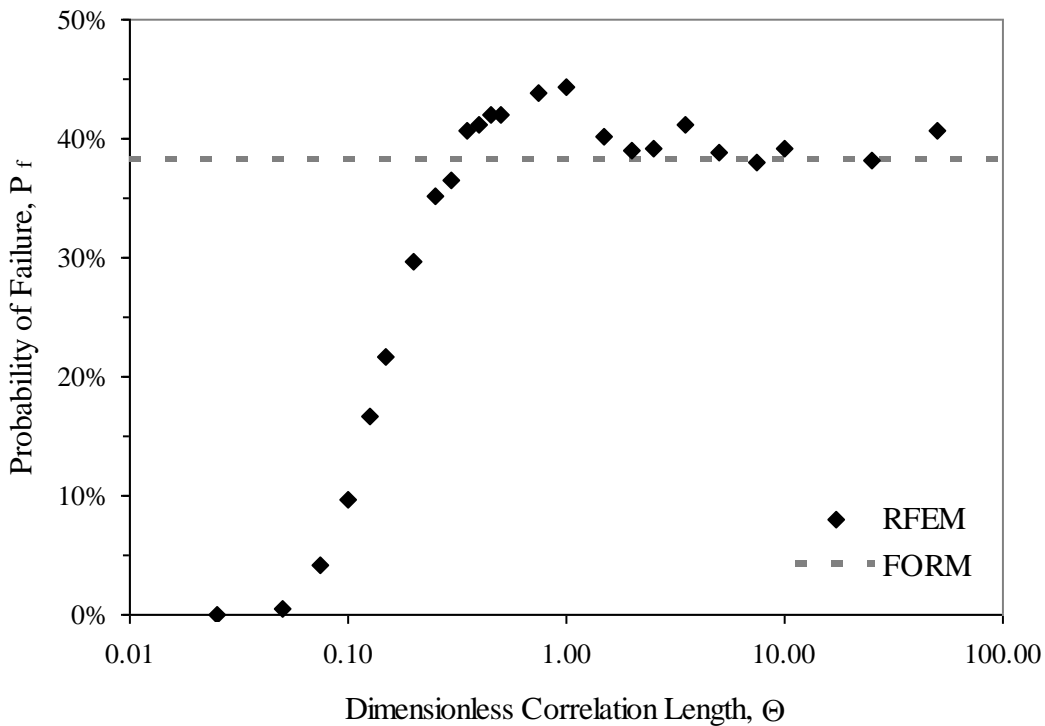
Since RFEM is based on Monte Carlo simulation, it is important to ensure that the number of realizations is sufficient to provide accurate and repeatable results. To check this, the slope was analyzed multiple times with increasing numbers of realizations. The results of two such runs are shown in Figure 5. Note that the dimensionless correlation length,  $\Theta$ , was set at 0.1 for this analysis. In both cases the probability of failure converges to the same constant value as the number of realizations increases. It can be seen that 1000 realizations yields sufficiently repeatable results and this value has been used in the subsequent analyses.



**Figure 5. Probability of Failure versus Number of Realizations.**

To examine the effect of the correlation length on the probability of failure, a parametric study was performed. The results of RFEM compared with FORM for the slope shown in Figure 2 are shown in Figure 6.





**Figure 6. Probability of Failure versus Correlation Length.**

For small correlation lengths, the probability of failure is essentially zero, for intermediate correlation lengths, the probability of failure increases rapidly, and for large correlation lengths, the probability of failure is essentially constant and similar to that found by FORM.

Consider the limits of zero and infinity for correlation length. As the correlation length approaches zero, the soil will vary rapidly between any two points and become essentially homogeneous, with the soil properties tending to their mean values. Assuming the mean values provide a safe design ( $FS > 1$ ), the probability of failure will always be zero. As the correlation length approaches infinity, however, the soil across the slope is highly correlated and will not vary. It becomes essentially homogeneous within each realization although different from one realization to the next. Use of the FORM is therefore equivalent to a system with a correlation length tending to infinity. For intermediate values of correlation length, the slope is not homogeneous and anomalies, such as locations of weak areas, control the probability of failure since the finite element analysis is able to “seek out” the weakest path through the slope.

### **CONCLUSION**

The first order reliability method is a powerful tool in probabilistic geotechnical analysis; however, it fails to explicitly account for the spatial correlation length. In the example considered, when the correlation length was small, results produced by

FORM were inaccurate and conservative. At high correlation lengths, the FORM results tended to agree with RFEM. While traditional methods like FORM or FOSM can account for spatial correlation indirectly by including variance reduction, this is inevitably subjective, since the local averaging zone cannot be known *a priori*. A number of investigators have attempted to include the effects of spatial correlation by locally averaging the random properties over the circular failure surface that would be predicted by a classical slope stability method (e.g. Bishop). The RFEM studies described in this paper are more realistic and conservative, in that they allow the critical mechanism to “seek out” the weakest path through the soil without any *a priori* assumption about the shape or location of the critical failure surface.

Of particular interest for designers is the case when FORM gave *unconservative* predictions. This is due to observation of a “worst case” correlation length ( $\Theta \approx 1$ ) which gave higher probabilities of failure than FORM. At this intermediate correlation length, the failure mechanism is able to “seek out” the optimal path through the weaker zones of soil and is a phenomenon that has been documented by the authors in other geotechnical failure analyses by RFEM (e.g. Griffiths and Fenton 2001, Fenton and Griffiths 2003)

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