Probabilistic Stability Analysis of Slopes by Conditional Random Fields

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Abstract

Slope stability analysis is a branch of geotechnical engineering that is highly amenable to probabilistic treatment. The engineering properties of soils vary spatially, however geotechnical tests can only investigate a small proportion of the site. When random field theory is used to model the spatial variability of soils, the associated statistics are inferred from geotechnical tests, however the random fields do not necessarily account for the specific deterministic properties, albeit limited, as measured from the site investigation data. In this paper, conditional random fields are used to model the spatial variability of soils taking account of the actual site-specific data obtained. Numerical results presented in this paper show that inclusion of this data can be an important factor in the determination of slope reliability.

INTRODUCTION

In the 1970s probabilistic slope stability analysis was first introduced into slope engineering by Alonso (1976). In the last two decades, probabilistic methods are becoming more popular for quantifying uncertainty and incorporating them rationally in designs.

The traditional probabilistic methods such as First Order Second Moment (FOSM) and First Order Reliability Method (FORM) ignore spatial variability by implicitly assuming perfect correlation. A more rigorous method called Random Finite Element Method (RFEM) has been developed by Griffiths and Fenton (1993), in which random fields are used to model spatial variability within the Monte Carlo framework. The RFEM has been applied extensively for geotechnical engineering as seen in Griffiths and Fenton (2004), Griffiths et al. (2009) and Huang et al. (2010). It is noted that the statistics (e.g., mean, variance and spatial correlation length) are inferred from geotechnical tests, however the random fields do not necessarily account for the specific deterministic properties, albeit limited, as measured from the site investigation data. In this paper, conditional random fields are used to model the spatial variability of soils taking into account of site-specific data obtained.

The conditioning of random fields is a process that simulates unmeasured spatial locations conditioned on the known values at measured locations. Both the measurements and their statistics are incorporated in conditional random fields. There are general methods that can be used to transform unconditional simulations into conditional ones. The conditional probability density function method was first suggested to carry out the conditional simulation by Kameda and Morikawa (1992) in relation to their work in earthquake engineering. Using Kriging technique for conditional simulation is first proposed by Journel (1974). The Kriging method has been widely used. Frimpong and Achireko (1998) applied Kriging to simulate ore reserves. Elishakoff et al. (1994) developed conditional simulation in earthquake monitoring engineering. Chiles and Delfiner (2009) applied conditional simulation in reservoir engineering. Van den Eijnden (2010) used Kriging extensively in geo-statistics. However, there are limited studies using conditional simulation in geotechnical engineering.

An undrained slope example is provided to demonstrate the proposed approach. Numerical results show that including site specific data can be an important factor in the estimation of slope reliability. By incorporating more site investigation data, the uncertainties in the estimation can be significantly reduced.

Interpolation by Simple Kriging

Kriging is an interpolation method first formalized by Matheron (1962) into a statistical approach, inspired by the work by Krige (1951) on the evaluation of mineral resources. Simple Kriging (SK) is basically best linear unbiased estimation with the assumption that the mean and standard deviation are constant and known across the entire region of interest. Kriging estimates $\mathrm{X}(\mathrm{x})$ at any unknown location using a weighted linear combination of the values of X at each observation point. Suppose that X_1, X_2, …, X_n are observations of the random field $\mathrm{X}(\mathrm{x})$ at known spatial locations $\mathbf{x_1}$, $\mathbf{x_2}$, …, $\mathbf{x_n}$, that is, $\mathrm{X}_k = \mathrm{X}(\mathrm{x}_k)$. Then the Kriging estimated of $\mathrm{X}(\mathrm{x})$ at x can be expressed as

$$\hat{\mathrm{X}}(\mathrm{x}) = \sum_{k=1}^{n} \beta_k \mathrm{X}_k \qquad (1)$$

where n is the number of observations and β_k are unknown weights determined by the covariance (or correlation) between observations and unknown points.

In Kriging, it is assumed that the mean $\mu_X(\mathrm{x})$ in a regression analysis is given by,

$$\mu_X(\mathrm{x}) = \sum_{i=1}^{m} a_i g_i(\mathrm{x}) \tag{2}$$

where a_i denotes unknown coefficients and $g_i(\mathrm{x})$ denotes a specified function of how the mean varies with position. SK assumes stationary data which has constant known mean value μ_X and

$$\mu_X(\mathrm{x}) = \mu_X \tag{3}$$

For the Kriging estimate to be unbiased, the mean difference between the estimate and the true (but random) value should be zero,

$$\mathrm{E}\left[\hat{\mathrm{X}}(\mathrm{x}) - \mathrm{X}(\mathrm{x})\right] = \mathrm{E}\left[\hat{\mathrm{X}}(\mathrm{x})\right] \text{-} \mathrm{E}\left[\mathrm{X}(\mathrm{x})\right] = \sum_{k=1}^{n} \beta_k \mu_X - \mu_X = 0 \tag{4}$$

Since this must be true for any mean values μ_X, the unbiased condition reduces to

$$\sum_{k=1}^{n} \beta_k = 1 \tag{5}$$

The unknown Kriging weights $\boldsymbol{\beta}$ are obtained by minimizing the estimation variance,

$$\sigma_E^2 = \mathrm{E}\left[\left(\hat{\mathrm{X}}(\mathrm{x}) - \mathrm{X}(\mathrm{x})\right)^2\right] \tag{6}$$

The Lagrange method (Wackernagel, 2013) is used to reduce the solution to the matrix equation

$$\mathbf{K}\boldsymbol{\beta} = \mathbf{M} \tag{7}$$

where $\mathbf{K}$ and $\mathbf{M}$ depend on the covariance structure,

$$\begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} & 1 \\ 1 & & \cdots & 1 & 0 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \\ -\eta \end{pmatrix} = \begin{pmatrix} C_{1x} \\ \vdots \\ C_{nx} \\ 1 \end{pmatrix} \tag{8}$$

in which C_{ij} is the covariance between X_i and X_j, η is a Lagrangian parameter used to solve the variance minimization problem subjected to the unbiased condition. The covariance C_{ix} appearing in the vector on the right-hand side, $\mathbf{M}$, is the covariance between the ith observation point and the point at x where the best estimate is to be calculated. Note that the Kriging matrix **K** only depends on the location of observations and their covariance. Thus Kriging matrix can be used to obtain kriging weights by inverting **K** only once and then Eqs.(7) and (1) can be used repeatedly for different spatial locations.

Conditional Random Fields

The Local Average Subdivision (LAS) method proposed by Fenton and Vanmarcke (1990) is used to generate unconditional random fields in this paper. Kriging technique is used for generate conditional simulation. Consider the decomposition of the process into the simple kriging predictor and the residual:

$$\mathrm{X}(\mathrm{x}) = \hat{\mathrm{X}}(\mathrm{x}) + \left(\mathrm{X}(\mathrm{x}) - \hat{\mathrm{X}}(\mathrm{x}) \right) \tag{9}$$

Replace the second component in Eq.(9) with $(\mathrm{X}_{us}(\mathrm{x}) - \hat{\mathrm{X}}_{us}(\mathrm{x}))$, which is based on an unconditional simulation of a process with mean μ_X and covariance C_{ij}. That is, define the conditional simulation $\mathrm{X}_{cs}(\mathrm{x})$ by Journel (1974):

$$\mathrm{X}_{cs}(\mathrm{x}) = \hat{\mathrm{X}}(\mathrm{x}) + \left(\mathrm{X}_{us}(\mathrm{x}) - \hat{\mathrm{X}}_{us}(\mathrm{x}) \right) \tag{10}$$

where $\mathrm{X}_{cs}(\mathrm{x})$ is the conditionally simulated random field, $\mathrm{X}_{us}(\mathrm{x})$ is the unconditional random field, $\hat{\mathrm{X}}_{us}(\mathrm{x})$ is the interpolated field by Simple Kriging based on unconditionally simulated values at the same measurement locations.

Because engineering properties are generally non-negative, lognormal distribution are usually adopted. However Eq.(10) cannot be used directly for lognormal random fields. Elishakoff et al. (1994) and Chilès and Delfiner (2012) proposed a conditional simulation method for non-Gaussian fields by transforming actual data to Gaussian data as shown in Figure 1.

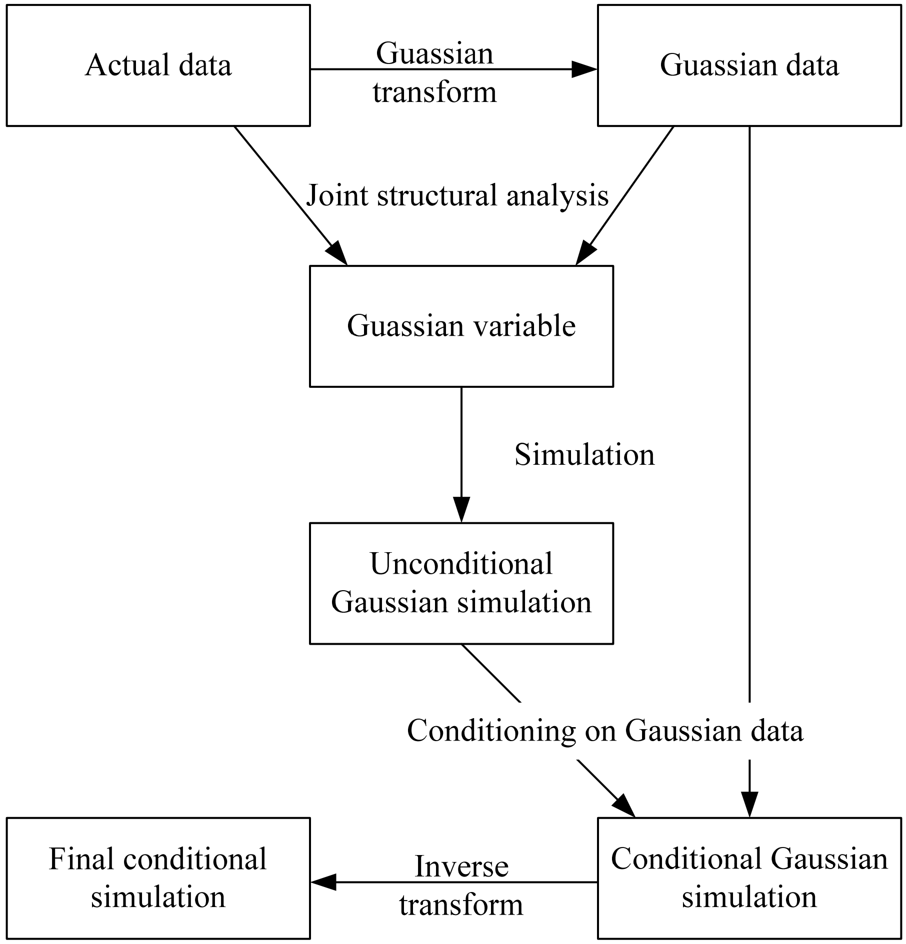


Figure 1 Flowchart of conditional simulation

An undrained slope is considered with the finite element mesh shown in Figure 2 . The slope is inclined to the horizontal at angle $\alpha = 26.6^{\circ}$ (2:1slope), with height $H = 10m$, and depth ratio to a lower firm layer $D = 2$, and soil unit weight $\gamma_{sat}\left(or\ \gamma\right) = 20.0\text{kN/m}^3$, which are all held constant. The undrained shear strength is assumed to be lognormally distributed with the mean $\mu_{C_u} = 50\text{kPa}$ and the standard deviation $\sigma_{C_u} = 25\text{kPa}$. Using strength reduction method with an iteration ceiling of 1000, the FS of the slope was found to be 1.47 based on the mean. Two thousand unconditional RFEM simulations have been conducted. It can also be noted that the spatial correlation length is fix at 10m and assumed isotropic in this paper. The probability of failure (p_f) is found to be 0.17. Two typical failures are shown in Figure 4 (a) and (b). When the spatial correlation length is relatively high, 1000m, the p_f is found to be 0.28.

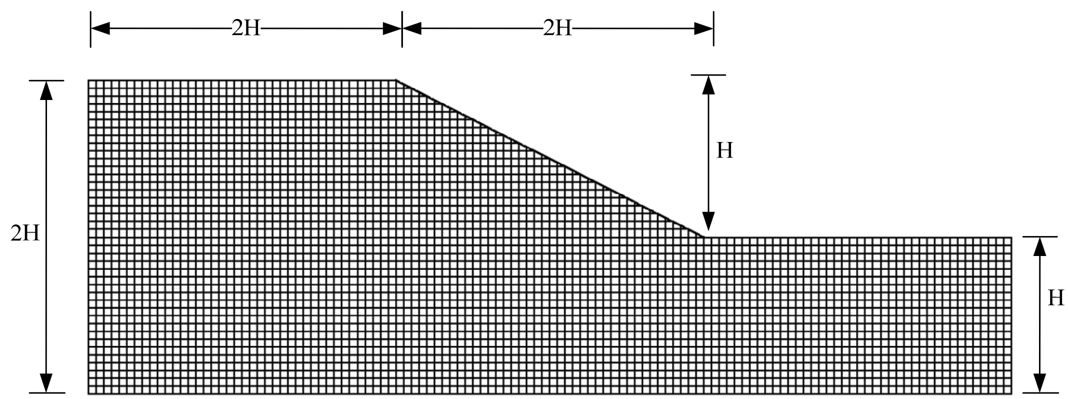


Figure 2 Finite element mesh

The above calculations do not take site investigation data directly into account. Suppose three CPTs have been conducted at the locations shown in Figure 3. The measurements of undrain shear strength by the CPTs are extracted from an unconditional simulation with the same statistics and the spatial correlation length fixed at 10m.

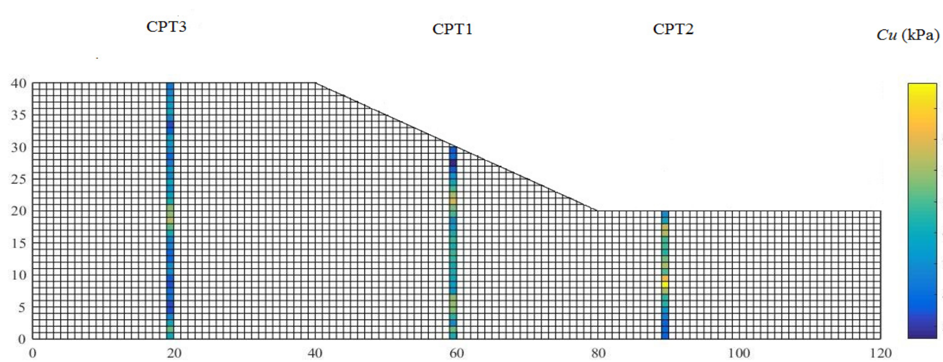
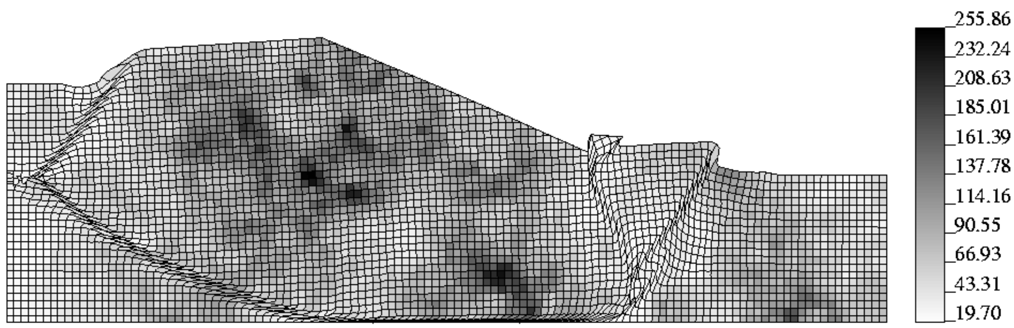


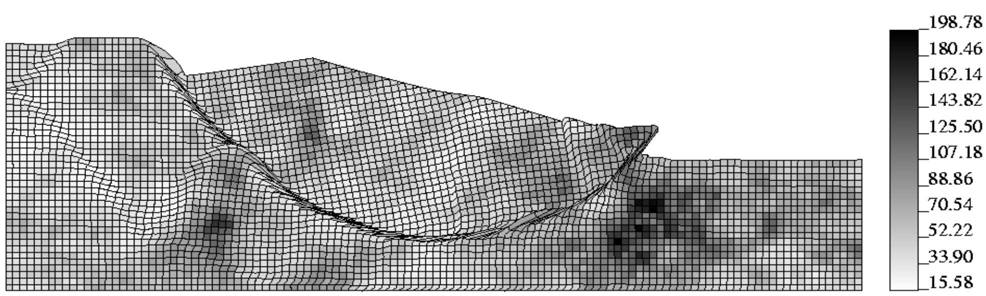
Figure 3. Undrained shear strength measured by CPT

Using the measurements of the CPTs, two thousand simulations are carried out. When CPT1 is incorporated in the conditional simulations, it can be seen from Figure 4 (c) and (d) that the measurements at the CPT1 remain constant from simulation to simulation. However, as shown in Figure 4 (a) and (b), the shear strengths at the same place change from simulation to simulation in unconditional simulations. The CPU time depends on p_f and runs to about 2 hours if $p_f = 0$ and 6 hours if $p_f = 1$ (every simulation hits the iteration ceiling) on a X5675@3.07GHz laptop for 2,000 simulations.

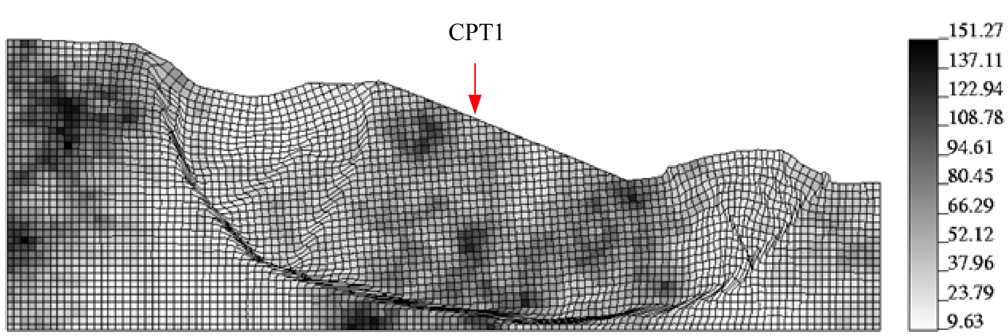
The p_f when CPT1 is incorporated is found to be 0.06, which is lower than the one estimated from unconditional simulations. The same calculations have been conducted when CPT1 and CPT2 are incorporated. The estimated p_f is found to be 0.007. When all three CPTs are incorporated, the p_f is 0. These results show that incorporating site investigation data have significant influence on the estimated probability of failure.



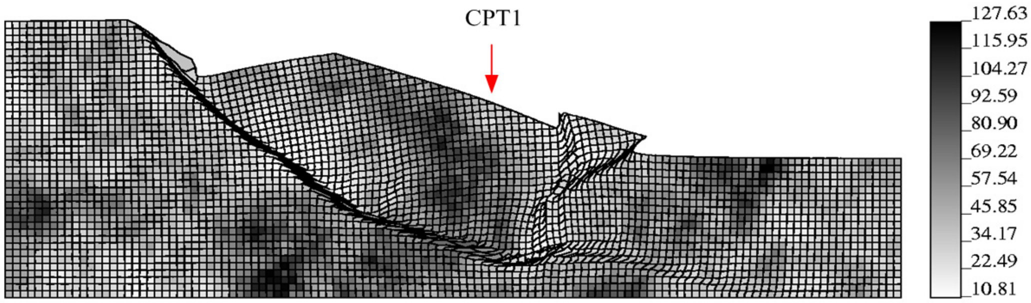
(a) A typical failure using unconditional random field



(b) A typical failure using unconditional random field



(c) A typical failure using conditional random field



(b) A typical failure using conditional random field

Figure 4 Typical failure mechanisms($\theta = 10\text{m}$ **)**

In order to investigate the confidence in the estimated p_f, the probability density function (PDF) of FS are simulated by two thousand simulations. Figure 5 shows the PDF of FS calculated by conditional RFEM simulations. The associate statistics of FS are shown in

Table ***1***. Incorporating only CPT1, the coefficient of variation (COV) of FS is 25.92%. The COV of FS is reduced to 11.30% when incorporating all three CPTS. It can be seen from

Table ***1*** that incorporating more site investigation data, the uncertainty in the estimation can be significantly reduced.

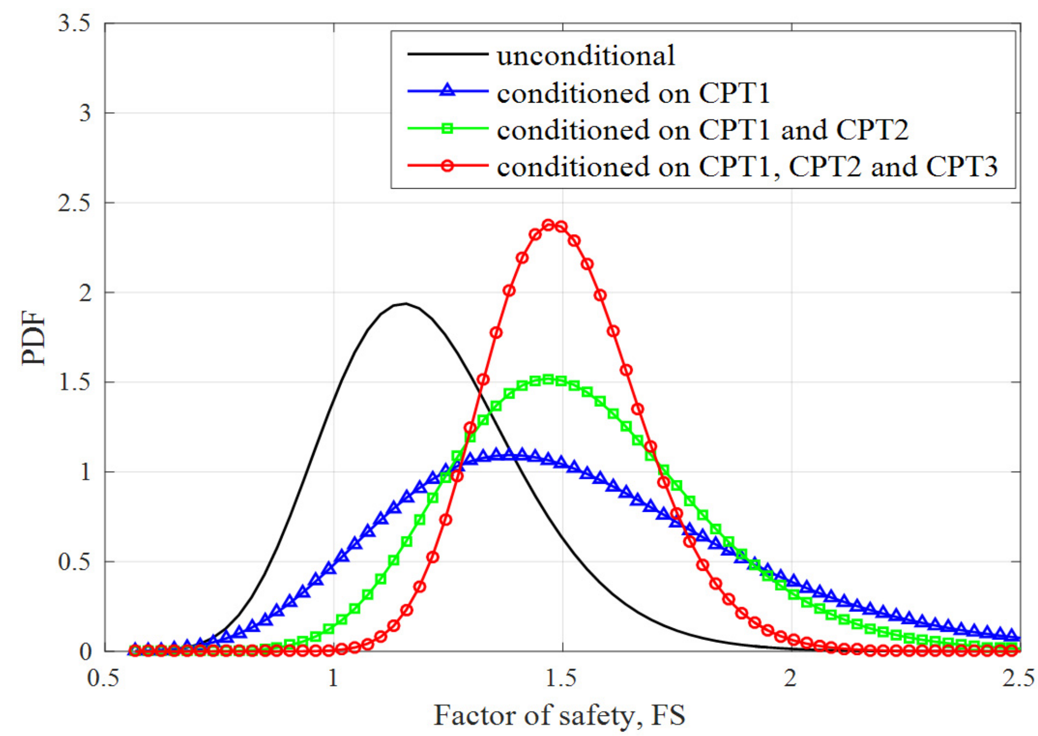


Figure 5 Probability density functions of factor of safety ($\theta = 10\text{m}$ **)**

Table 1 Probability of failure and the statistics of FS

case	p_f	Variance of FS	Mean of FS	COV (%)
Unconditional	0.1705	0.046	1.21	17.74
Conditioned on CPT1	0.06	0.157	1.53	25.92
Conditioned on CPT1 and CPT 2	0.007	0.075	1.54	17.75
Conditioned on CPT1, CPT 2 and CPT3	0	0.029	1.51	11.30

Conclusion

The conditional random field method enables the site investigation data be incorporated directly in probabilistic analysis. The numerical study shows that constraining the random field by the investigation data significantly affects the result of probability of failure. The coefficient of variation of factor of safety can be reduced by incorporating more site investigation data. The method proposed in this paper can increase the confidence in probabilistic designs.

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