Comparison of Slope Reliability Methods of Analysis

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Abstract: Reliability tools have been applied to slope stability analysis more than any other geotechnical application on account of the readily understood concept of “probability of failure” as an alternative or complement to the traditional “factor of safety”. Probabilistic slope stability methods in the literature are reviewed. Particular attention is focused on the ability of the methods to correctly model spatially varying soil properties. A benchmark slope is reanalyzed and conclusions reached about their suitability for meaningful and conservative prediction of slope reliability.

Keywords: First Order Reliability Method, Finite element method, Probability of failure, Spatial correlation.

Introduction

Slope stability analysis is a branch of geotechnical engineering that is highly amenable to probabilistic treatment, and has received considerable attention in the literature. Almost all probabilistic methods described in the literature have at some point been applied to slope stability problems. A brief description of the reported probabilistic methods is presented below:

1) Direct integration method

The probability of failure \( p_f \) is obtained by direct integration of the probability density function of the factor of safety \( FS \).

\[
p_f = \int_{FS < 1} f_{FS}(FS) d FS
\]  

(1)

This method requires that the probability density function of \( FS \), \( f_{FS}(FS) \) is known in advance which is rarely the case.
2) Point estimate method (PEM)

The PEM (Rosenblueth 1975, 1981, Griffiths et al. 2002) is an alternative method for approximate estimation of statistical moments of \( f_{FS}(FS) \) without needing information about the exact distribution of the input random variables. In this method, probability distributions for input continuous random variables are replaced by discrete or “lumped” equivalent distributions. The mean and variance of \( FS \), where \( FS \) depends on \( n \) input random variables, can be found from an expressions of the form

\[
\mu_{FS} = \sum_{i=1}^{2^n} P_i (FS_i)
\]

(1)

\[
\sigma_{FS}^2 = \sum_{i=1}^{2^n} P_i (FS_i)^2 - \mu_{FS}^2
\]

(2)

where each random variable is fixed at a strategic value above and below its mean (Christian and Baecher 2002) and \( P_i \) are weighting coefficients \( \left( \sum_{i=1}^{2^n} P_i = 1 \right) \).

After the mean and standard deviation of \( FS \) are determined, the reliability index can be calculated by

\[
\beta = \frac{\mu_{FS} - 1}{\sigma_{FS}}
\]

(3)

PEM does not require knowledge of the particular form of the probability density functions of the input random variables, however this approximate method it may lead to incorrect interpretations of the reliability if the performance function is highly nonlinear or the random variables asymmetric. The application of PEM requires \( 2^n \) evaluations of the performance function. The spatial correlation between the random variables can be accounted for in the weighting coefficients \( P_i \).

3) First Order Second Moment Method (FOSM)

The mean and variance of \( FS \) are approximated by a first-order Taylor series expansion about the mean values of random parameters that are characterized by their first two moments. The reliability index is calculated as

\[
\beta = \frac{FS(\mu_i) - 1}{\sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} \left( \frac{\partial FS}{\partial X_i} \right) \left( \frac{\partial FS}{\partial X_j} \right) \text{Cov}(X_i, X_j)}}
\]

(4)
where \( n \) is the number of random variables, \( \mu_i \) are the mean values of the random variables, and \( \text{Cov}[X_i, X_j] \) are the covariances between \( X_i \) and \( X_j \) which can account for spatial correlation.

FOSM does not require knowledge of the particular form of the probability density functions of the input random variables. A serious problem with FOSM is that the reliability index it delivers depends on how the performance function is formulated, thus two people solving the same problem could obtain quite different results.

4) First Order Reliability Method (FORM)

FORM based on the Hasofer-Lind reliability index (Hasofer and Lind 1974), \( \beta_{HL} \), assumes that the mean values of random variables lie on the safe side of the performance function. The method then obtains the reliability index, which is related to the minimum distance between the mean values and the limit state surface as

\[
\beta = \min_{g=0} \sqrt{\left( \frac{X_i - \mu_i^N}{\sigma_i^N} \right)^T \left[ R \right] \left( \frac{X_i - \mu_i^N}{\sigma_i^N} \right)} \quad i=1,2,...,n
\]

where \( X_i \) is the \( i^{th} \) random variable, \( \mu_i^N \) is the equivalent normal mean of the \( i^{th} \) random variable, \( \sigma_i^N \) is the equivalent normal standard deviation of the \( i^{th} \) random variable, \( \{(X_i - \mu_i^N)/\sigma_i^N\} \) is the vector of \( n \) random variables reduced to standard normal space, \( \left[ R \right] \) is the correlation matrix and \( g \) is the limit state function.

FORM has become popular with investigators in recent years because the reliability index it delivers is not dependent on the form of the performance function. In cases where no analytical equation exists for the performance function, the Response Surface Method (Box and Wilson 1951) can be introduced to obtain an approximated performance function based on a curve fit.

5) Monte Carlo simulation (MCS)

Monte Carlo simulation samples random variables from their distributions and obtains the probability of failure directly by dividing the number of simulations which failed by the total number of simulations. MCS is usually used to check the results obtained by the methods mentioned previously. If the probability density function of the performance function is known or estimated in advance, “importance sampling” (Harbitz 1983, Shinozuka 1983) can be used to reduce the number of MCS simulations needed.

A deterministic slope stability analysis method such the Limit Equilibrium (LEM) or Finite Element Method (FEM) is needed as the basis of a probabilistic slope analysis. The choice of deterministic slope stability analysis method also determines how spatial variability can be included. Some investigators have
combined the LEM with random field theory. Table 1 provides a list of 2D slope reliability publications in the literature that combined the LEM with 1-d random field theory. The inherent nature of LEM is that it leads to a critical failure surface, which in 2D analysis appears as a line which could be non-circular. The influence of the random field is only taken into account along the line and is therefore one-dimensional.

In recent years, the present authors have been pursuing a more rigorous method of probabilistic geotechnical analysis (e.g. Griffiths and Fenton 2004, Griffiths et al. 2009), in which nonlinear finite-element methods are combined with random field generation techniques. This method, called here the ‘‘random finite element method’’ (RFEM), fully accounts for spatial correlation and averaging, and is also a powerful slope stability analysis tool that does not require a priori assumptions related to the shape or location of the failure mechanism. In this study, the limitations of combining LEM with 1D random field are investigated. A benchmark slope problem is used to show that combining LEM with 1D random fields can lead to a lower (unconservative) probability of failure than RFEM.

Table 1. 2D slope reliability analyses by LEM methods and 1D random field

<table>
<thead>
<tr>
<th>Authors</th>
<th>Probabilistic method</th>
<th>Deterministic method</th>
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<tbody>
<tr>
<td>Catalan and Cornell (1976)</td>
<td>FOSM</td>
<td>Level-crossing method</td>
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<tr>
<td>Alonso (1976)</td>
<td>FOSM</td>
<td>Bishop</td>
</tr>
<tr>
<td>Li and Lumb (1987)</td>
<td>FORM</td>
<td>Morgenstern-Price</td>
</tr>
<tr>
<td>Mostyn and Soo (1992)</td>
<td>FORM</td>
<td>Morgenstern-Price</td>
</tr>
<tr>
<td>El-Ramly et al. (2002)</td>
<td>MCS</td>
<td>Bishop</td>
</tr>
<tr>
<td>Low (2003)</td>
<td>FORM</td>
<td>Spence</td>
</tr>
<tr>
<td>Babu and Mukesh (2004)</td>
<td>FOSM</td>
<td>Bishop</td>
</tr>
<tr>
<td>Cho (2007)</td>
<td>MCS</td>
<td>LEM</td>
</tr>
<tr>
<td>Low et al. (2007)</td>
<td>FORM</td>
<td>Spencer</td>
</tr>
<tr>
<td>Hong and Roh (2008)</td>
<td>MCS</td>
<td>Chen and Morgenstern</td>
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**Taking spatial variability into account**

Soils are rarely homogeneous in nature and sometimes consist of several different layers of soil/rock (especially in the vertical direction) due to different deposition conditions and different loading histories. Constructed slopes sometimes use different materials to accomplish different functions. For example, compacted clay can serve as a water-proof core of a zoned earth dam. The properties of the clay core are totally different from the rocks or soils that constitute the shoulders of embankment. The first step to model the spatial variability of such slopes is to distinguish the limit of spatial continuity, beyond which essentially no correlation between soil data exists. The second step is to model the spatial correlation structure...
that describes the variation of soil properties from one point to another in each soil/rock region.

Most numerical solution algorithms require that all continuous parameter fields be discretized. The variance of the strength, spatially averaged over some domain such as a finite element or finite difference zone, is less than the variance at the “point” level. As the size of the domain over which the soil property is being averaged increases, the variance decreases.

In LEM, the soil properties are averaged along the bottom line of each slice as shown in Fig. 1. The variance reduction factor for the \(i\)th slice is calculated as

\[
\gamma = \frac{2}{L_i} \int_0^{L_i} (L_i - z_i) \exp \left(-\frac{z_i}{\theta}\right) dz_i
\]

and the spatial correlation coefficients between segments are estimated using Eq. (7).

\[
\rho(c_{ij}, c_{ij}') = \frac{1}{L_i L_j} \int_0^{L_i} \int_0^{L_j} \exp \left(-\frac{z_i}{\theta}\right) dz_i dz_j
\]

where \(c_{ij}, c_{ij}'\) are the locally averaged soil properties.

The critical probabilistic slip surface is determined by searching all possible slip surfaces (e.g. Bhattacharya et al. 2003). Usually, the existing deterministic slope stability program is modified such that the factor of safety is replaced with the reliability index as the objective function. The critical deterministic surface is used as the starting slip surface for this search. The critical probabilistic slip surface is typically in a different location but close to the critical deterministic slip surface (Hassan and Wolff 1999). It should be noted that locally averaged properties (mean, variance and correlation coefficients) need to be recalculated for each slip surface because the bottom secant-line of each slice is varying.

If 2D RFEM is used, the soil properties are averaged over the area of each element as shown in Fig. 1. The variance reduction factor is calculated as

\[
\gamma = \frac{4}{L^2} \int_0^L \int_0^L (L - x)(L - y) \exp \left(-\frac{\sqrt{x^2 + y^2}}{\theta}\right) dx dy
\]

Full account is taken of local averaging, variance reduction and cross correlation over each element by the Local Average Subdivision Method (Fenton and Vanmarcke 1990). The random field is initially generated and properties assigned to the elements. After application of gravity loads, if the algorithm is unable to converge within a user-defined iteration ceiling (see e.g. Griffiths and Lane 1999), the implication is that no stress distribution can be found that is simultaneously able to satisfy both the Mohr-Coulomb failure criterion and global equilibrium. If the algorithm is unable to satisfy these criteria, failure is said to have occurred. The
analysis is repeated numerous times using Monte-Carlo simulations. Each realization of the Monte-Carlo process involves the same mean, standard deviation and spatial correlation length of soil properties, however the spatial distribution of properties varies from one realization to the next. Following a “sufficient” number of realizations, the $p_f$ can be easily estimated by dividing the number of failures by the total number of simulations. The analysis has the option of including cross correlation between properties and anisotropic spatial correlation lengths (e.g. the spatial correlation length in a naturally occurring stratum of soil is often higher in the horizontal direction). Further details of RFEM can be found in Griffiths and Fenton (2004) and Fenton and Griffiths (2008).

![Fig. 1. 1D and 2D random fields in Limit Equilibrium Method and RFEM](image)

**Numerical example**

A benchmark two-layered slope that has been used by several investigators (e.g., Hassan and Wolff 1999 and Cho 2007) has been reanalyzed in the current paper. The slope section with height $H = 10.0$ m is shown in Fig. 2. The soil parameters (unit weight, friction angle, and cohesion), are modeled as lognormally distributed random variables with parameters given in Table 3.

<table>
<thead>
<tr>
<th>Table 3. Statistical properties of soil parameters</th>
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<tr>
<td>$\gamma_1$, kN/m$^3$</td>
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<tr>
<td>$\mu$</td>
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<tr>
<td>$\nu$</td>
</tr>
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</table>

Cho (2007) obtained a deterministic factor of safety of 1.59 based on mean property values using Spencer’s (LEM) method which can be compared with 1.61 using the authors’ finite element method. The deterministic slip surface by finite
elements is shown in Fig. 3, which is very close to that obtained by Cho (2007).
Assuming perfect spatial correlation, Cho (2007) obtained a probability of failure of
0.11 using FORM.

Assuming lognormal distributions and a spatial correlation length of 2m and
20m for embankment and foundation, two different 1D random fields were generated
as shown in Fig. 4. The 1D random fields shown in Fig. 4 have different means,
standard deviations and correlation lengths. There is an obvious discontinuity
between the two random fields.

There is an obvious discontinuity at the boundary between the materials. The
materials in the embankment and foundation could have different spatial correlation
lengths however, but as far as we are aware, no LEM probabilistic study has
considered this possibility. In the RFEM, different materials can be readily modeled
with different random fields. Fig. 5 shows two independent 2D random fields
generated by RFEM according to the parameters shown in Table 3. The embankment and the foundation also have different spatial correlation lengths of 2m and 20m respectively.

\[ \mu_{c1} = 38.31 \text{kN/m}^2, \quad \nu_{c1} = 0.4 \]
\[ \mu_{c2} = 23.94 \text{kN/m}^2, \quad \nu_{c2} = 0.2 \]
\[ \sigma_{\ln c1} = 0.39 \text{kN/m}^2 \]
\[ \sigma_{\ln c2} = 0.20 \text{kN/m}^2 \]

The results obtained by Cho (2007) and RFEM are contrasted in Fig. 6 (A dimensionless spatial correlation length \( \Theta = \theta/H \) is used). It can be seen that combining LEM with 1D random fields gives lower probabilities of failure than RFEM. This is because the LEM method fixes the failure surface using deterministic methods (in this example, using Spencer’s method), while the RFEM allows the failure mechanism to develop wherever the weakest path through the soil layers happens to lie in a particular Monte-Carlo simulation.
Fig. 6 Influence of spatial correlation on slope reliability

Concluding remarks

Probabilistic slope stability methods are reviewed, and their ability to take spatial variability into account is analyzed in a benchmark problem. It is shown that LEM combined with 1D random fields can give lower probabilities of failure than RFEM which uses 2D random fields. The reason for this is that RFEM does not require a priori assumptions related to the shape or location of the failure mechanism. In an RFEM analysis, the failure mechanism has more freedom to “seek out” the weakest path through the random soil, which is in contrast to the LEM approach, where the failure surface location is fixed before the random field can be accounted for.

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