Comparison of Slope Reliability Methods of Analysis

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Abstract: Reliability tools have been applied to slope stability analysis more than any other geotechnical application on account of the readily understood concept of "probability of failure" as an alternative or complement to the traditional "factor of safety". Probabilistic slope stability methods in the literature are reviewed. Particular attention is focused on the ability of the methods to correctly model spatially varying soil properties. A benchmark slope is reanalyzed and conclusions reached about their suitability for meaningful and conservative prediction of slope reliability.

Keywords: First Order Reliability Method, Finite element method, Probability of failure, Spatial correlation.

Introduction

Slope stability analysis is a branch of geotechnical engineering that is highly amenable to probabilistic treatment, and has received considerable attention in the literature. Almost all probabilistic methods described in the literature have at some point been applied to slope stability problems. A brief description of the reported probabilistic methods is presented below:

1) Direct integration method

The probability of failure (p_f) is obtained by direct integration of the probability density function of the factor of safety (FS).

$$p_f = \int_{FS<1} f_{FS}(FS)\, d\, FS \tag{1}$$

This method requires that the probability density function of FS , $f_{FS}(FS)$ is known in advance which is rarely the case.

2) Point estimate method (PEM)

The PEM (Rosenblueth 1975, 1981, Griffiths et al. 2002) is an alternative method for approximate estimation of statistical moments of $f_{FS}(FS)$ without needing information about the exact distribution of the input random variables. In this method, probability distributions for input continuous random variables are replaced by discrete or "lumped" equivalent distributions. The mean and variance of FS, where FS depends on n input random variables, can be found from an expressions of the form

$$\mu_{FS} = \sum_{i=1}^{2^n} P_i \left(FS_i \right) \tag{1}$$

$$\sigma_{FS}{}^2 = \sum_{i=1}^{2^n} P_i \left(FS_i \right)^2 - \mu_{FS}{}^2 \tag{2}$$

where each random variable is fixed at a strategic value above and below its mean (Christian and Baecher 2002) and P_i are weighting coefficients $\left(\sum_{i=1}^{2^n} P_i = 1 \right)$.

After the mean and standard deviation of FS are determined, the reliability index can be calculated by

$$\beta = \frac{\mu_{FS} - 1}{\sigma_{FS}} \tag{3}$$

PEM does not require knowledge of the particular form of the probability density functions of the input random variables, however this approximate method it may lead to incorrect interpretations of the reliability if the performance function is highly nonlinear or the random variables asymmetric. The application of PEM requires 2^n evaluations of the performance function. The spatial correlation between the random variables can be accounted for in the weighting coefficients P_i.

3) First Order Second Moment Method (FOSM)

The mean and variance of FS are approximated by a first-order Taylor series expansion about the mean values of random parameters that are characterized by their first two moments. The reliability index is calculated as

$$\beta = \frac{FS\left(\mu_{x_i} \right) - 1}{\sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{\partial FS}{\partial X_i} \right) \left(\frac{\partial FS}{\partial X_j} \right) \mathrm{Cov}\left[X_i, X_j \right]}} \tag{4}$$

where n is the number of random variables, μ_{x_i} are the mean values of the random variables, and $\mathrm{Cov}\left[X_{i,}X_j\right]$ are the covariances between X_i and X_j which can account for spatial correlation.

FOSM does not require knowledge of the particular form of the probability density functions of the input random variables. A serious problem with FOSM is that the reliability index it delivers depends on how the performance function is formulated, thus two people solving the same problem could obtain quite different results.

4) First Order Reliability Method (FORM)

FORM based on the Hasofer-Lind reliability index (Hasofer and Lind 1974), β_{HL}, assumes that the mean values of random variables lie on the safe side of the performance function. The method then obtains the reliability index, which is related to the minimum distance between the mean values and the limit state surface as

$$\beta = \min_{g=0} \sqrt{\left\{\frac{X_i - \mu_i^N}{\sigma_i^N}\right\}^T \left[R\right]^{-1} \left\{\frac{X_i - \mu_i^N}{\sigma_i^N}\right\}} \qquad i=1,2,...,n \tag{5}$$

where X_i is the i^{th} random variable, μ_i^N is the equivalent normal mean of the i^{th} random variable, σ_i^N is the equivalent normal standard deviation of the i^{th} random variable, $\{(X_i - \mu_i^N)/\sigma_i^N\}$ is the vector of n random variables reduced to standard normal space, $\left[R\right]$ is the correlation matrix and g is the limit state function.

FORM has become popular with investigators in recent years because the reliability index it delivers is *not* dependent on the form of the performance function. In cases where no analytical equation exists for the performance function, the Response Surface Method (Box and Wilson 1951) can be introduced to obtain an approximated performance function based on a curve fit.

5) Monte Carlo simulation (MCS)

Monte Carlo simulation samples random variables from their distributions and obtains the probability of failure directly by dividing the number of simulations which failed by the total number of simulations. MCS is usually used to check the results obtained by the methods mentioned previously. If the probability density function of the performance function is known or estimated in advance, “importance sampling” (Harbitz 1983, Shinozuka 1983) can be used to reduce the number of MCS simulations needed.

A deterministic slope stability analysis method such the Limit Equilibrium (LEM) or Finite Element Method (FEM) is needed as the basis of a probabilistic slope analysis. The choice of deterministic slope stability analysis method also determines how spatial variability can be included. Some investigators have combined the LEM with random field theory. Table 1 provides a list of 2D slope reliability publications in the literature that combined the LEM with 1-d random field theory. The inherent nature of LEM is that it leads to a critical failure surface, which in 2D analysis appears as a line which could be non-circular. The influence of the random field is only taken into account along the line and is therefore one-dimensional.

In recent years, the present authors have been pursuing a more rigorous method of probabilistic geotechnical analysis (e.g. Griffiths and Fenton 2004, Griffiths et al. 2009), in which nonlinear finite-element methods are combined with random field generation techniques. This method, called here the ‘‘random finite element method’’ (RFEM), fully accounts for spatial correlation and averaging, and is also a powerful slope stability analysis tool that does not require a priori assumptions related to the shape or location of the failure mechanism. In this study, the limitations of combining LEM with 1D random field are investigated. A benchmark slope problem is used to show that combining LEM with 1D random fields can lead to a lower (unconservative) probability of failure than RFEM.

Table 1. 2D slope reliability analyses by LEM methods and 1D random field

Authors	Probabilistic method	Deterministic method
Catalan and Cornell (1976)	FOSM	Level-crossing method
Alonso (1976)	FOSM	Bishop
Li and Lumb (1987)	FORM	Morgenstern-Price
Mostyn and Soo (1992)	FORM	Morgenstern-Price
El-Ramly et al. (2002)	MCS	Bishop
Low (2003)	FORM	Spence
Babu and Mukesh (2004)	FOSM	Bishop
Cho (2007)	MCS	LEM
Low et al. (2007)	FORM	Spencer
Hong and Roh (2008)	MCS	Chen and Morgenstern

Taking spatial variability into account

Soils are rarely homogeneous in nature and sometimes consist of several different layers of soil/rock (especially in the vertical direction) due to different deposition conditions and different loading histories. Constructed slopes sometimes use different materials to accomplish different functions. For example, compacted clay can serve as a water-proof core of a zoned earth dam. The properties of the clay core are totally different from the rocks or soils that constitute the shoulders of embankment. The first step to model the spatial variability of such slopes is to distinguish the limit of spatial continuity, beyond which essentially no correlation between soil data exists. The second step is to model the spatial correlation structure that describes the variation of soil properties from one point to another in each soil/rock region.

Most numerical solution algorithms require that all continuous parameter fields be discretized. The variance of the strength, spatially averaged over some domain such as a finite element or finite difference zone, is less than the variance at the "point" level. As the size of the domain over which the soil property is being averaged increases, the variance decreases.

In LEM, the soil properties are averaged along the bottom line of each slice as shown in Fig. 1. The variance reduction factor for the i^{th} slice is calculated as

$$\gamma = \frac{2}{L_i^2}\int_0^{L_i}\left(L_i - z_i\right)\exp\left(-2\frac{z_i}{\theta}\right)dz_i \tag{6}$$

and the spatial correlation coefficients between segments are estimated using Eq. (7).

$$\rho\left(c'_{L_i},\ c'_{L_j}\right) = \frac{1}{L_i L_j}\int_0^{L_i}\int_0^{L_j}\exp\left(-2\frac{z}{\theta}\right)dz_i dz_j \tag{7}$$

where $c'_{L_i},\ c'_{L_j}$ are the locally averaged soil properties.

The critical probabilistic slip surface is determined by searching all possible slip surfaces (e.g. Bhattacharya et al. 2003). Usually, the existing deterministic slope stability program is modified such that the factor of safety is replaced with the reliability index as the objective function. The critical deterministic surface is used as the starting slip surface for this search. The critical probabilistic slip surface is typically in a different location but close to the critical deterministic slip surface (Hassan and Wolff 1999). It should be noted that locally averaged properties (mean, variance and correlation coefficients) need to be recalculated for each slip surface because the bottom secant-line of each slice is varying.

If 2D RFEM is used, the soil properties are averaged over the area of each element as shown in Fig. 1. The variance reduction factor is calculated as

$$\gamma = \frac{4}{L^2}\int_0^{L}\int_0^{L}\left(L-x\right)\left(L-y\right)\exp\left(-2\frac{\sqrt{x^2+y^2}}{\theta}\right)dxdy \tag{8}$$

Full account is taken of local averaging, variance reduction and cross correlation over each element by the Local Average Subdivision Method (Fenton and Vanmarcke 1990). The random field is initially generated and properties assigned to the elements. After application of gravity loads, if the algorithm is unable to converge within a user-defined iteration ceiling (see e.g. Griffiths and Lane 1999), the implication is that no stress distribution can be found that is simultaneously able to satisfy both the Mohr-Coulomb failure criterion and global equilibrium. If the algorithm is unable to satisfy these criteria, failure is said to have occurred. The analysis is repeated numerous times using Monte-Carlo simulations. Each realization of the Monte-Carlo process involves the same mean, standard deviation and spatial correlation length of soil properties, however the spatial distribution of properties varies from one realization to the next. Following a "sufficient" number of realizations, the p_f can be easily estimated by dividing the number of failures by the total number of simulations. The analysis has the option of including cross correlation between properties and anisotropic spatial correlation lengths (e.g. the spatial correlation length in a naturally occurring stratum of soil is often higher in the horizontal direction). Further details of RFEM can be found in Griffiths and Fenton (2004) and Fenton and Griffiths (2008).

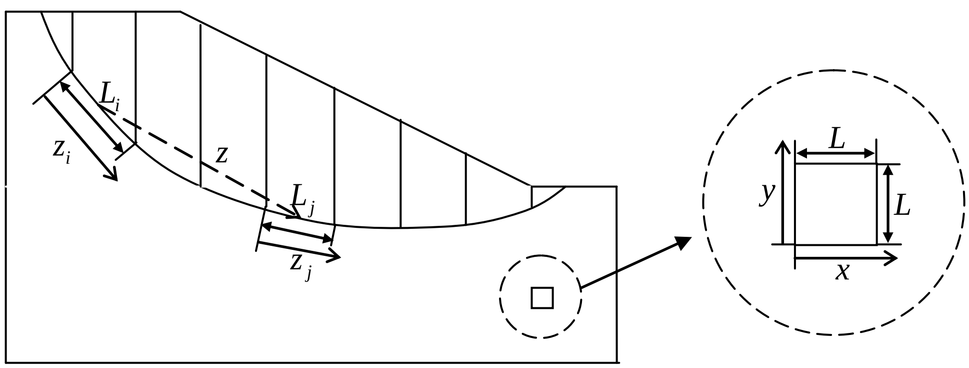

Fig. 1. 1D and 2D random fields in Limit Equilibrium Method and RFEM

Numerical example

A benchmark two-layered slope that has been used by several investigators (e.g., Hassan and Wolff 1999 and Cho 2007) has been reanalyzed in the current paper. The slope section with height $H = 10.0$ m is shown in Fig. 2. The soil parameters (unit weight, friction angle, and cohesion), are modeled as lognormally distributed random variables with parameters given in Table 3.

Table 3. Statistical properties of soil parameters

	γ_1 kN/m^3	c'_1 kN/m^2	ϕ'_1	γ_2 kN/m^3	c'_2 kN/m^2	ϕ'_2
μ	18	38.31	0°	18	23.94	12°
v	0.05	0.4	-	0.05	0.2	0.1

Cho (2007) obtained a deterministic factor of safety of 1.59 based on mean property values using Spencer's (LEM) method which can be compared with 1.61 using the authors' finite element method. The deterministic slip surface by finite elements is shown in Fig. 3, which is very close to that obtained by Cho (2007). Assuming perfect spatial correlation, Cho (2007) obtained a probability of failure of 0.11 using FORM.

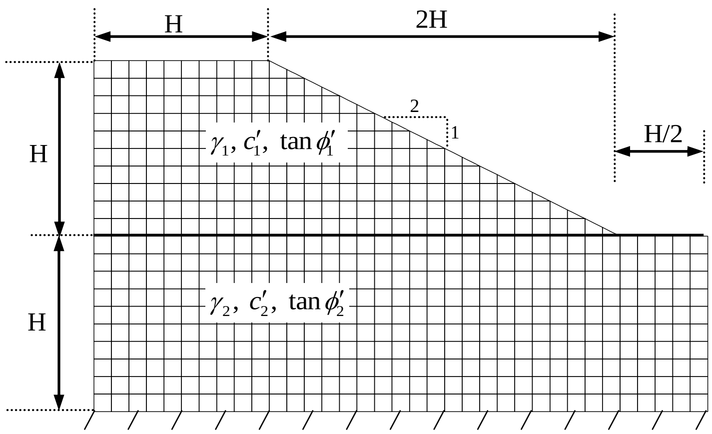

Fig. 2 Two-layered slope finite element mesh

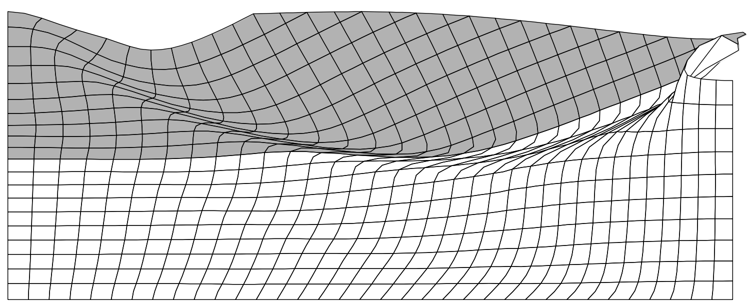

Fig. 3 Critical deterministic slip surface

Cho (2007) further considered the influence of a finite spatial correlation length on the slope reliability. This involved local averaging over the base of each slice (or group of slices) as described previously using Eq.(6) for variance reduction and Eq.(7) for correlation. It should be noted that the correlation coefficients between slices in different material regions should be zero.

Assuming lognormal distributions and a spatial correlation length of 2m and 20m for embankment and foundation, two different 1D random fields were generated as shown in Fig. 4. The 1D random fields shown in Fig. 4 have different means, standard deviations and correlation lengths. There is an obvious discontinuity between the two random fields.

There is an obvious discontinuity at the boundary between the materials. The materials in the embankment and foundation could have different spatial correlation lengths however, but as far as we are aware, no LEM probabilistic study has considered this possibility. In the RFEM, different materials can be readily modeled with different random fields. Fig. 5 shows two independent 2D random fields generated by RFEM according to the parameters shown in Table 3. The embankment and the foundation also have different spatial correlation lengths of 2m and 20m respectively.

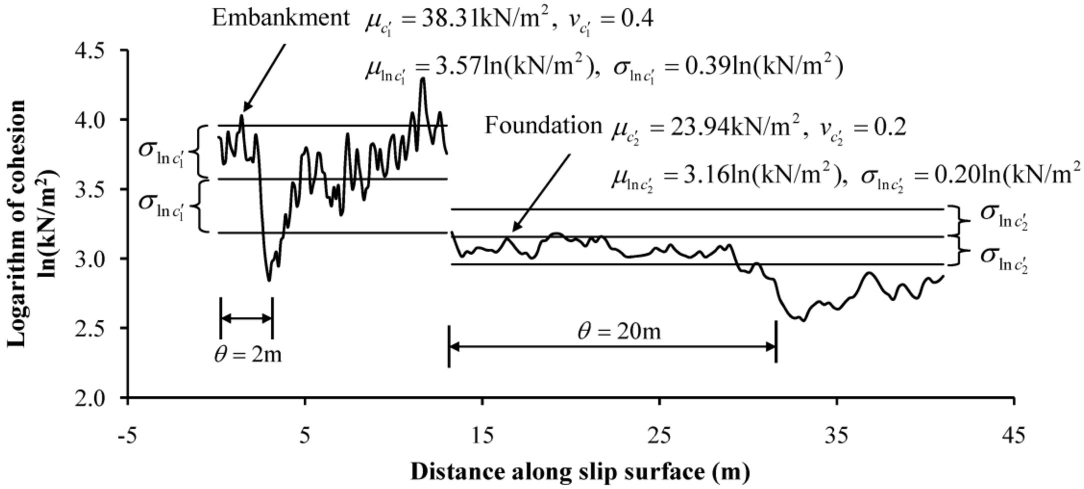

Fig. 4 Two independent 1D random fields for embankment and foundation

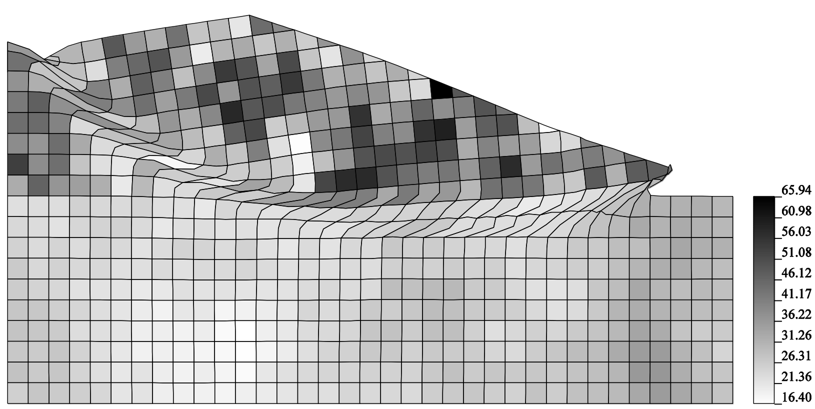

Fig. 5 Slope at failure involving independent random fields for the embankment and foundation

The results obtained by Cho (2007) and RFEM are contrasted in Fig. 6 (A dimensionless spatial correlation length $\Theta = \theta/H$ is used). It can be seen that combining LEM with 1D random fields gives lower probabilities of failure than RFEM. This is because the LEM method fixes the failure surface using deterministic methods (in this example, using Spencer's method), while the RFEM allows the failure mechanism to develop wherever the weakest path through the soil layers happens to lie in a particular Monte-Carlo simulation.

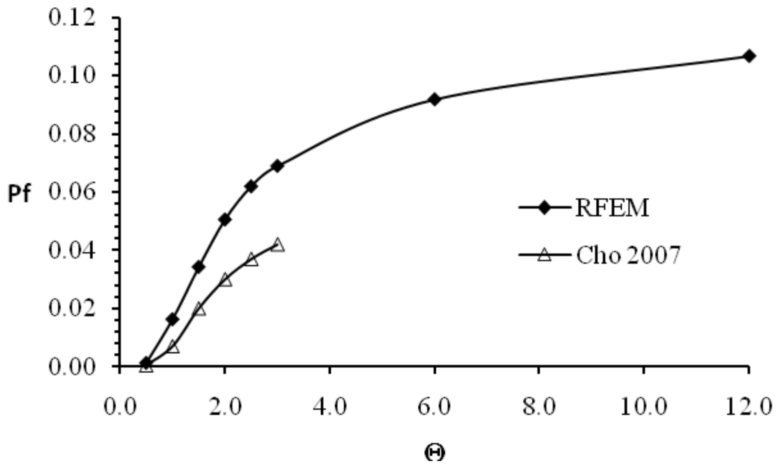

Fig. 6 Influnce of spatial correlation on slope reliability

Concluding remarks

Probabilistic slope stability methods are reviewed, and their ability to take spatial variability into account is analyzed in a benchmark problem. It is shown that LEM combined with 1D random fields can give lower probabilities of failure than RFEM which uses 2D random fields. The reason for this is that RFEM does not require *a priori* assumptions related to the shape or location of the failure mechanism. In an RFEM analysis, the failure mechanism has more freedom to "seek out" the weakest path through the random soil, which is in contrast to the LEM approach, where the failure surface location is fixed before the random field can be accounted for.

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References

Alonso, E. E., (1976). "Risk analysis of slopes and its application to slopes in Canadian sensitive clays." *Géotechnique*, **26**(453-472).

Babu, G. L. S. and Mukesh M. D., (2004), "Effect of soil variability on reliability of soil slopes." *Géotechnique*, **54**(5):335–337.

Bhattacharya, G. Jana, D. Ojha, S. and Chakraborty, S. (2003). "Direct search for minimum reliability index of earth slopes." *Comput. Geotech.*, **30**(6): 455–462.

Box, G. E. P. and Wilson, K.B. (1951) "On the Experimental Attainment of Optimum Conditions (with discussion)." *Journal of the Royal Statistical Society Series B* 13(1):1–45.

Catalan, J. M. and Cornell, C. A. (1976). "Earth slope reliability by a level-crossing method", ASCE Journal of the Geotechnical Engineering Division, 102(GT6):691-604

Cho, S. E. (2007). “Effects of spatial variability of soil properties on slope stability.” Engineering Geology, **92** (3-4): 97–109

Christian, J.T., Baecher, G.B., (2002). “The point-estimate method with large numbers of variables.” *International Journal for Numerical and Analytical Methods in Geomechanics* 26 (15), 1515-1529.

El-Ramly, H., Morgenstern, N. R., and Cruden, D. M. (2002). “Probabilistic slope stability analysis for practice. ” *Can Geotech J*, **39**:665–683.

Fenton, G. A., and Griffiths, D. V., (2008). “Risk Assessment in Geotechnical Engineering.” John Wiley & Sons, New York.

Griffiths, D. V., and Fenton, G. A. (2004). “Probabilistic slope stability analysis by finite elements.” *ASCE Journal of Geotechnical and Geoenvironmental Engineering*, **130**(5): 507-518.

Griffiths, D.V., Huang, J. and Fenton G.A. “Influence of spatial variability on slope reliability using 2-d random fields.” To appear in the J Geotech Geoenviron, (2009)

Griffiths, D. V. and Lane, P. A. (1999). “Slope stability analysis by finite elements.” *Géotechnique*, **49**(3):387–403.

Harbitz, A. (1983). “Efficient and accurate probability of failure calculation by use of the importance sampling technique.” In: Proc. of the 4th Int. Conf. on App. of Statist. and Prob. in Soils and Struct. Eng., ICASP-4, Pitagora Editrice Bologna, p. 825-836.

Hasofer, A. M. and Lind, N. C. (1974). “Exact and invariant second moment code format.” *J. Engrg. Mech. Div.*, **100**(1):111–121.

Hassan, A. M., and Wolff, T. F. (1999). “Search algorithm for minimum reliability index of earth slopes.” *J Geotech Geoenv Eng, ASCE*, **125**(4):301–308

Hong, H. P. and Roh, G. (2008). “Reliability Evaluation of Earth Slopes.” Journal of Geotechnical and Geoenvironmental Engineering, 134(12):1700-1705

Li, K. S., and Lumb, P. (1987). “Probabilistic design of slopes.” *Can Geotech J*, **24**:520–531.

Low, B. K., Lacasse, S. and Nadim, F. (2007). “Slope reliability analysis accounting for spatial variation”, Georisk,Taylor & Francis, London, **1**(4), 177-189

Mostyn, G. R., and Soo, S. (1992). “The effect of autocorrelation on the probability of failure of slopes.” In 6th Australia, New Zealand Conference on Geomechanics: Geotechnical Risk, pages 542–546.

Rosenblueth E. (1975) “Point estimates for probability moments.” *Proceedings of the National Academy of Science*, 72(10):3812-3814.

Rosenblueth E. (1981) “Two-point estimates in probabilities.” *Applied Mathematical Modeling*, 5(2):329-335.

Shinozuka M. (1984). “Basic analysis of structural safety.” *J Struct Eng ASCE*, 109(3):721-40.