Undrained bearing capacity of spatially random clays by finite elements and limit analysis

Capacité portante des argiles non drainées des champs aléatoires par éléments finis et analyse limite

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ABSTRACT: This paper combines the random field methodology with the upper and lower bound finite element limit analysis algorithms (Sloan 1988, 1989) to study the bearing capacity of undrained clays with spatially varying shear strength. The results of the Random Field Limit Analysis (RFLA) analyses are compared with existing results obtained by elastic-plastic Random Finite Element (RFEM) analyses (Griffiths and Fenton 2001). It is shown that RFEM results are bounded by RFLA ones. The difference (Nd) between the upper (Nu) and lower (Nl) bound bearing capacities in random soils is shown to be a lognormally distributed random variable. The effects of spatial correlation length and coefficient of variation of undrained strength on Nu and Nl are also studied.

RÉSUMÉ : Ce document combine la méthode des champs aléatoires avec les limites inférieure et supérieure des algorithmes d'analyse par éléments finis limites (Sloan 1988, 1989) pour étudier la capacité portante des argiles non drainées variant dans l'espace avec la résistance au cisaillement. Les résultats de l'analyse de limiter le champ aléatoire (RFLA) des analyses sont comparés avec les résultats actuels obtenus par élasto-plasticité des éléments finis (Random RFEM) analyses (Griffiths et Fenton, 2001). Il est montré que les résultats RFEM sont délimités par les RFLA. La différence (N_d) entre la tige (N_u) et inférieure (N_l) lié capacités portantes dans les sols aléatoires se révèle être une variable aléatoire une distribution lognormale. Les effets de la longueur de corrélation spatiale et coefficient de variation de la résistance non drainée sur N_u et N_l sont également étudiés.

KEYWORDS: bearing capacity, limit analysis, finite element method, random field.

1 INTRODUCTION

Limit analysis has been used in geotechnical practice for decades as a means of estimating the ultimate strength of structures. Starting from early 80s (e.g., Sloan 1988, 1989), Sloan and his colleagues combined the bound theorems with finite element method and mathematical programming techniques. The resulting methods inherit all the benefits of the finite element approach and are applicable to a wide range of problems involving arbitrary domain geometries, complex loadings and heterogenous material properties. The Random Finite Element Method (RFEM) (Fenton and Griffiths 2008) combines elastoplastic finite elements and random field theory in a Monte-Carlo framework. It has been proved to be able to assess the reliability of a wide range of geotechnical problems including settlement, seepage, consolidation, bearing capacity, earth pressure and slope stability.

In this paper, we combines the finite element limit analysis method developed by Sloan and his colleagues with random field theory. The framework is very similar to RFEM, but three components are combined together, namely, bound theorems, finite element method and random field theory. The finite element limit analysis utilizes recent developments of convex optimization algorithms. The random field is generated by the Local Averaging Subdivision method developed by Fenton and Vanmarcke (1990). The method is then used to investigate the statistical bounds of the bearing capacity of a smooth rigid strip footing (plane strain) at the surface of an undrained clay soil with a shear strength c_u ($\phi_u = 0$) defined by a spatially varying random field.

The study starts with a deterministic analysis which shows the bearing capacity obtained by finite element method is bounded by the ones obtained by limit analysis. By introducing spatial variability, the robustness of finite element limit analysis involving heterogenous soil properties is tested. It is shown that the limit analyses always bounds the finite element analysis no matter how heterogenous the soils are. Although the RFEM always gives estimations lie between the lower and upper bounds, RFLA gives quantitative error estimation which RFEM cannot offer. The probabilistic analysis is then carried out. It is shown that even the mean upper bound bearing capacity factors are lower than the Prandtl solution in all cases. This confirms that using mean soil strength with deterministic analysis or first order probabilistic estimate will be on the unconservative side. In addition, a worst case spatial correlation length is observed where mean bearing capacity is minimized. This suggests that the spatial variability of soil strength has to be taken into account properly.

2 REVIEW ON FINITE ELEMENT LIMIT ANALYSIS

The lower and upper bound theorems of classical plasticity theory is a powerful tool for analysing the stability of problems in soil mechanics. The theory assumes a perfectly plastic soil model with an associated flow rule. The lower bound theorem states that any statically admissible stress field will furnish a lower bound (or 'safe') estimate of the true limit load.

$$\begin{aligned} &\text{maximize} \quad \alpha \\ &\text{subject to} \quad \mathbf{A}^{\mathrm{T}}\boldsymbol{\sigma} = \alpha\mathbf{p} + \mathbf{p}_0 \\ &\qquad\qquad \mathbf{f}(\boldsymbol{\sigma}) \le 0 \end{aligned} \tag{1}$$

where $\mathbf{A}$ is an equilibrium matrix, $\boldsymbol{\sigma}$ is a vector containing stresses, and the external load consists of a constant part $\mathbf{p}_0$ and a part proportional to a scalar parameter α , $\mathbf{f}$ defines the yield conditions.

A statically admissible stress field is one which satisfies (a) the stress boundary conditions, (b) equilibrium, and (c) the yield condition (the stresses must lie inside or on the yield surface in stress space).

The upper bound theorem states that the load (or the load multiplier), determined by equating the internal power dissipation to the power expended by the external loads in a kinematically admissible velocity field, is not less than the actual collapse load. Based on the duality between the upper and lower bound methods, Krabbenholf et. al (2005) derived a upper bound formulation in terms of stresses rather than velocities and plastic multipliers. This allows for a much simpler implementation and general nonlinear yield conditions can be easily dealt with.

$$\begin{aligned} &\text{maximize} \quad \alpha \\ &\text{subject to} \quad \mathbf{B}^{\mathrm{T}}\boldsymbol{\sigma} = \alpha\mathbf{p} + \mathbf{p}_0 \\ &\qquad\qquad \mathbf{f}(\boldsymbol{\sigma}) \le 0 \end{aligned} \tag{2}$$

where $\mathbf{B} = A\mathbf{LN}$ and A is the area of elements, $\mathbf{N}$ contains the interpolation functions and $\mathbf{L}$ is defined as (for linear triangular elements)

$$\mathbf{L}^{\mathrm{T}} = \begin{bmatrix} \partial/\partial x & 0 & \partial/\partial y \\ 0 & \partial/\partial y & \partial/\partial x \end{bmatrix}$$

It should be mentioned that matrix $\mathbf{B}$ in Eq. (2) can be amended to include kinematically admissible discontinuities, which have previously been shown to be very efficient (e.g., Sloan and Kleeman 1995).

Although both upper and lower bound methods formulated as Eqs. (1) and (2) with a Tresca failure criterion are ready to be solved by public available second order cone programming packages (e.g., SeDumi, Mosek), our in-house limit analysis program (Lyamin and Sloan 2002a and 2002b) is used in this study.

3 DETERMINISTIC ANALYSES

The bearing capacity analyses use an elastic-perfectly plastic stress-strain law with a Tresca failure criterion. Triangular constant stress–linear velocity element is used for both upper and lower bound analysis in this study. A mesh is shown in Fig. 1 consisting of 4000 triangular elements. The strip footing has a width of 10 elements. The bottom of the mesh while the sides are allowed to move only in the vertical direction. Plastic stress redistribution in RFEM analysis is accomplished using a viscoplastic algorithm. For RFEM analysis, 8-node quadrilateral elements and reduced Gaussian integration in both the stiffness and stress redistribution parts of the algorithm (Smith and Griffiths 2004). The mesh for RFEM analysis is not shown but one can easily figure it out by treating four triangular elements as a square 8-node quadrilateral element.

Rather than deal with the actual bearing capacity, this study focuses on the dimensionless bearing capacity factor N_c, defined as

$$N_c = \frac{q_f}{c_u} \tag{3}$$

where q_f is the bearing capacity and c_u is the undrained shear strength of the soil beneath the footing. For a homogeneous soil with a constant undrained shear strength, N_c is given by the Prandtl solution, and equals $2+\pi$ or 5.14.

The lower bound and upper bound bearing capacity factor are found to be 5.02 and 5.19 respectively. The bearing capacity factor obtained by FEM analy[illegible]s 5.12. Although the FEM result lies in between the lo[illegible] and upper bound bearing capacity factors, it lacks error es[illegible]ad.

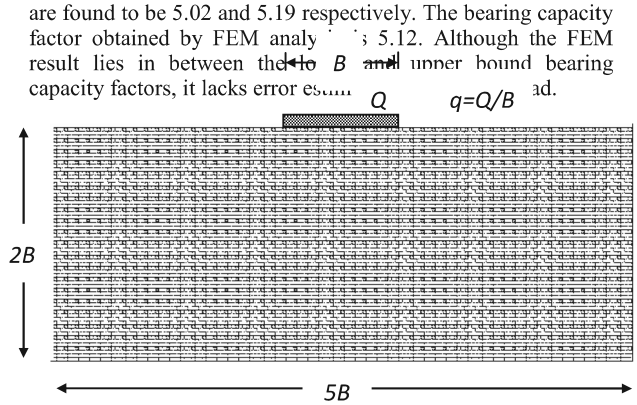

Figure 1. Finite element mesh for limit analysis

4 PROBABILISTIC DESCRIPTIONS OF STRENGTH PARAMETERS

In this study, the dimensionless shear strength parameter c_u is assumed to be a random variable characterized statistically by a lognormal distribution (i.e. the logarithm of the property is normally distributed). The lognormally distributed shear strength c_u has three parameters; the mean, μ_{c_u}, the standard deviation σ_{c_u} and the spatial correlation length $\theta_{\ln c_u}$. The variability of c_u can conveniently be expressed by the dimensionless coefficient of variation defined as

$$V_{c_u} = \frac{\sigma_{c_u}}{\mu_{c_u}} \tag{4}$$

The parameters of the normal distribution (of the logarithm of c_u) can be obtained from the standard deviation and mean of c_u as follows:

$$\sigma_{\ln c_u} = \sqrt{\ln\left\{1 + V_{c_u}^2\right\}} \tag{5}$$

$$\mu_{\ln c_u} = \ln \mu_{c_u} - \frac{1}{2}\sigma_{\ln c_u}^2 \tag{6}$$

A third parameter, the spatial correlation length $\theta_{\ln c_u}$, will also be considered in this study. Since the actual undrained shear strength field is assumed to be lognormally distributed, its logarithm yields an "underlying" normal distribution (or Gaussian) field. The spatial correlation length is measured with respect to $\ln c_u$. In particular, the spatial correlation length ($\theta_{\ln c_u}$) describes the distance over which the spatially random values will tend to be significantly correlated in the underlying Gaussian field. Thus, a large value of $\theta_{\ln c_u}$ will imply a smoothly varying field, while a small value will imply a ragged field.

In the current study, the spatial correlation length has been non-dimensionalized by dividing it by the width of the footing B and will be expressed in the form,

$$\Theta_{c_u} = \theta_{\ln c_u} / B \tag{7}$$

5 RANDOM FINITE ELEMENT LIMIT ANALYSIS

The RFLA involves the generation and mapping of a random field of properties onto a finite element mesh. Full account is taken of local averaging and variance reduction over elements, and an exponentially decaying (Markov) spatial correlation function is incorporated. To be consistent with local averaging procedure, four linear triangluar elements within a square area were assigned a constant property in both lower and upper bound analysis. It should be mentioned that random properties are also assigned to the kinematically admissible discontinuities involved in upper bound analysis. The analysis is repeated numerous times using Monte-Carlo simulations. Each realization of the Monte-Carlo process involves the same underlying mean, standard deviation and spatial correlation length of soil properties, however the spatial distribution of properties varies from one realization to the next. Following a suite of Monte-carlo simulations, the mean and coefficient of variation of the bearing capacity factor can be easily estimated.

Figure 2 shows a typical deformed mesh at failure by lower bound limit analysis with a superimposed greyscale corresponding to $\Theta_{c_u} = 1$, in which lighter regions indicated weaker soil and darker regions indicated stronger soil. In this case the dark zones and the light zones are roughly the width of the footing itself, and it appears that the weak (light) region near the ground surface to the left of the footing has triggered a quite non-symmetric failure mechanism.

Figure 3 compares RFLA and RFEM for ten typical simulations. It can be seen from Figure 2 that the RFEM is always bounded by the RFLA results.

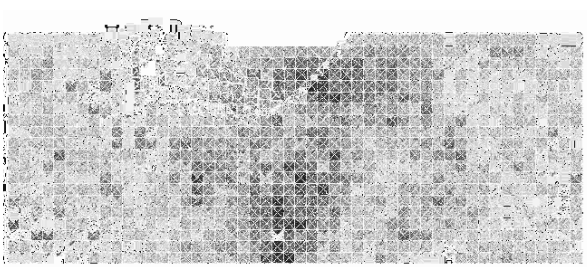

Figure 2. Typical deformed mesh and greyscale at failure with $\Theta_{c_u} = 1$. (the darker zones indicate stronger soil)

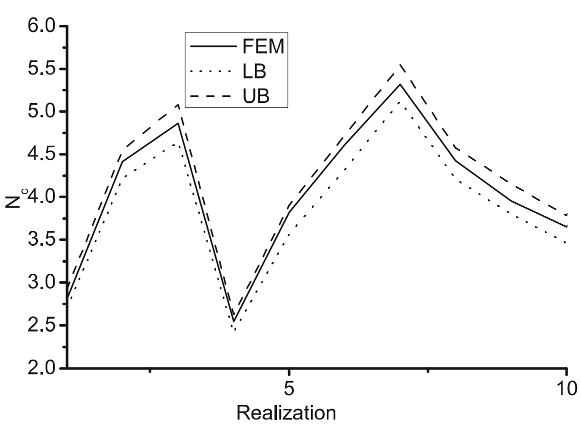

Figure 3. Comparison of lower and upper bounds with finite element analysis for ten typical simulations.

6 PROBABILISTIC ANALYSES

Analyses were performed using the input parameters in the range $0.125 \le \Theta_{c_u} \le 4$ and $0.125 \le V_{c_u} \le 4$. For each combination of Θ_{c_u} and V_{c_u}, 1000 realizations of the Monte Carlo process were performed, and the estimated mean and standard deviation of the resulting 1000 bearing capacity factors were computed. Figure 4 shows how the estimated mean bearing capacity factor, μ_{N_l} and μ_{N_u}, varies with Θ_{c_u} and V_{c_u} (The RFEM results were omitted due to length limit). The plot confirms that, for low values of V_{c_u}, μ_{N_l} and μ_{N_u} tend to the deterministic Prandtl value of 5.14. For higher values of V_{c_u}, however, the mean bearing capacity factors fall steeply, especially for lower values of Θ_{c_u}.What this implies from a design standpoint is that the bearing capacity of a heterogeneous soil will on average be less than the Prandtl solution that would be predicted assuming the soil is homogeneous with its strength given by the mean value. The influence of Θ_{c_u} is also pronounced with the greatest reduction from the Prandtl solution being observed with values around $\Theta_{c_u} = 0.5$. Figure 6 shows the influence of Θ_{c_u} and V_{c_u} on the estimated coefficient of variation of the bearing capacity factor . The plots indicate that μ_{N_l} and μ_{N_u} are positively correlated with both Θ_{c_u} and V_{c_u}. It is also interesting to note that there are essential no difference between V_{N_l} and V_{N_u}.

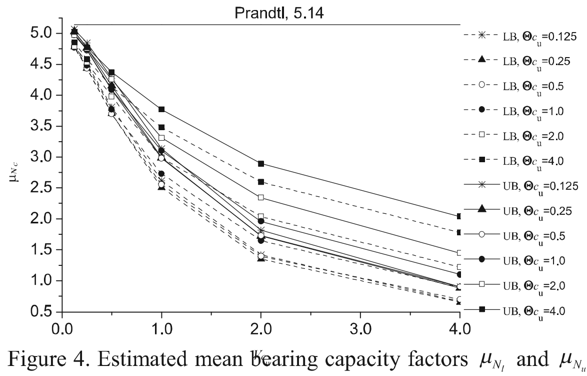

Figure 4. Estimated mean bearing capacity factors μ_{N_l} and μ_{N_u} verse the coefficient of variation of undrained shear strength

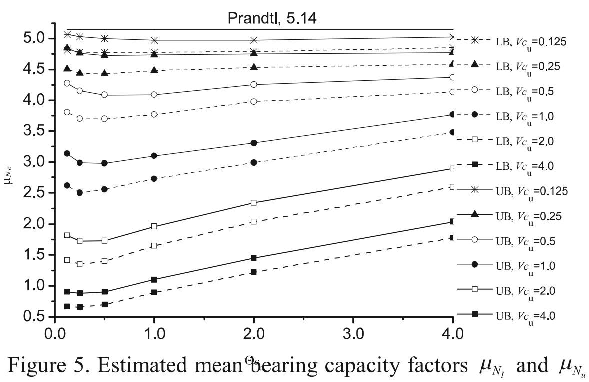

Figure 5. Estimated mean bearing capacity factors μ_{N_l} and μ_{N_u} verse the spatial correlation length of undrained shear strength

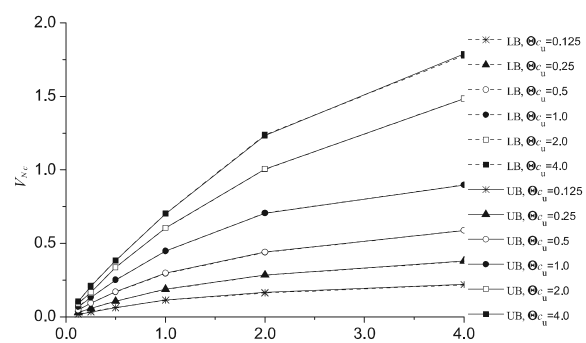

Figure 6. Estimated coefficient of variation of the bearing capacity factors μ_{N_l} and μ_{N_u}

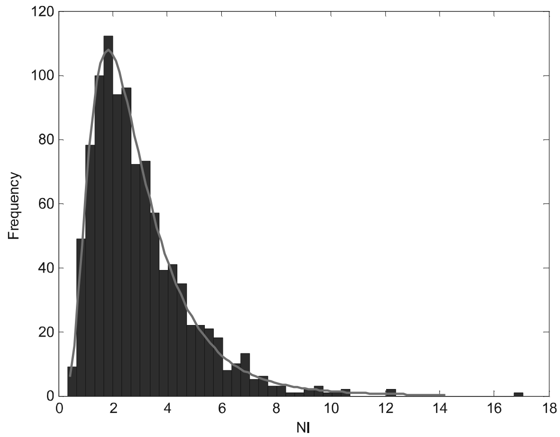

Figure 7. Histogram and log-normal fit for the computed lower bound bearing capacity factors when $\Theta_{c_u} = 2$ and $V_{c_u} = 1$

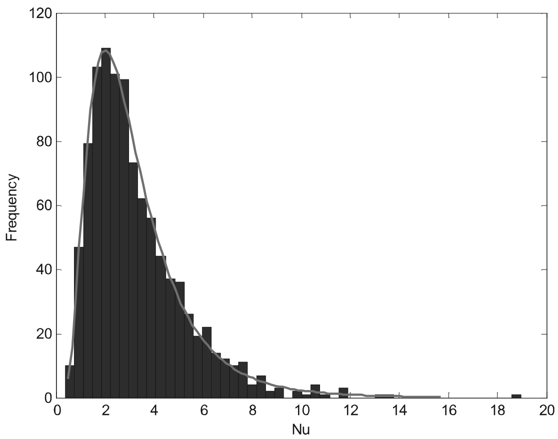

Figure 8. Histogram and log-normal fit for the computed upper bound bearing capacity factors when $\Theta_{c_u} = 2$ and $V_{c_u} = 1$

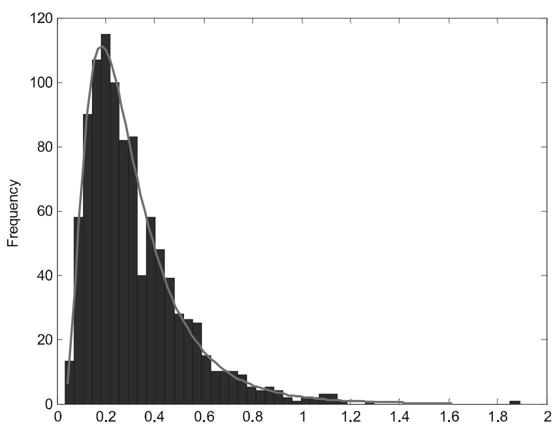

Figure 9. Histogram and log-normal fit for the difference between the computed lower and upper bound bearing capacity factors when $\Theta_{c_u} = 2$ and $V_{c_u} = 1$

Figures 7 and 8 show histograms of lower and upper bounds of bearing capacity factors with best-fit lognormal distributions for the case where $\Theta_{c_u} = 2$ and $V_{c_u} = 1$. The Pearson's coefficient of correlation between the lower and upper bounds was found to be 0.9995, which implies a strong correlation. This is expected since the same random field was used for both lower and upper bound analyses. The histogram of the difference between the lower and upper bounds is plotted in Figure 9, which is also well fitted by a lognormal distribution. Although nothing is known in elementary probability theory about the distribution of the difference of two lognormals, Figure 9 suggests the difference is a lognormally distributed random variable, at least when the two lognormals are strong correlated.

7 CONCLUDING REMARKS

The paper has investigated the bearing capacity factor using both lower and upper bound limit anlysis combined with random field theory. The mean upper bound bearing capacity factors are always lower than the Prandtl solution using mean soil strength. The main conclusion is that by implicitly assuming an infinite spatial correlation in traditional first order probabilistic analysis (e.g., First Order Second Moment and First Order Reliability Method) may overestimate the mean bearing capacity factor. When performing probabilistic analysis of bearing capacity of strip footings, spatial variability must be properly considered to avoid unconservative designs.

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REFERENCES

Fenton, G. A., and Griffiths, D. V., 2008. Risk Assessment in Geotechnical Engineering. John Wiley & Sons, New York.

Fenton, G. A., and Vanmarcke, E. H., (1990). Simulation of random fields via local average subdivision. J Eng Mech, ASCE, 116(8):1733–1749

Griffiths, D.V., and Fenton, G.A., Bearing capacity of spatially random soil: The undrained clay Prandtl problem revisited, Géotechnique, 54(4), 351--359, 2001.

Krabbenhøft, K., Lyamin, A.V., Hjiaj, M., and Sloan, S.W. 2005. A new discontinuous upper bound limit analysis formulation. International Journal for Numerical Methods in Engineering, 63: 1069-1088.

Lyamin, A.V., and Sloan, S.W. 2002a. Lower bound limit analysis using non-linear programming. International Journal for Numerical Methods in Engineering, 55: 573-611.

Lyamin AV, Sloan SW. Upper bound limit analysis using linear finite elements and non-linear programming. International Journal for Numerical and Analytical Methods in Geomechanics 2002b; 26:181– 216.

Sloan, SW (1988) "Lower bound limit analysis using finite elements and linear programming", Int J Numer Anal Meths Geomech, 12(1), pp.61-77

Sloan, SW (1989) Upper bound limit analysis using finite elements and linear programming. International Journal of Numerical and Analytical Methods in Geomechanics, 13(3), 263-282.

Sloan SW, Kleeman PW. Upper bound limit analysis using discontinuous velocity fields. Computer Methods in Applied Mechanics and Engineering 1995; 127:293–314.

Smith IM, Griffiths DV. Programming the finite element method. 4th edition, John Wiley & Sons: Chichester, 2004.