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Observations on two- and three-dimensional seepage through a spatially random soil

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ABSTRACT: This paper brings together advanced random field generation and finite element techniques to model steady seepage through a problem of interest to civil engineers. The application is a classical problem of seepage beneath a water retaining structure in which probabilistic issues relating to flow rates are examined. The influence of three-dimensionality is given particular emphasis and contrasted with results that are obtained using an idealized two-dimensional model.

The analyses treat the soil permeability as a spatially random property with specified mean, variance and spatial correlation length. The influence of the spatial correlation or 'scale of fluctuation' is given special consideration, since this aspect is not always included in probabilistic geotechnical analysis. Monte-Carlo simulations are performed to establish statistics relating to the output quantities of interest.

The tool of analysis is the finite element method, which is naturally suited to analyzing materials with properties varying over space. The value of permeability assigned to each element comes from a lognormally distributed random field derived from local averages of a normally distributed random field. The local averaging allows the element dimensions to be rationally accounted for on a statistical basis. For the computationally intensive 3-d studies, strategies are described for optimizing the efficiency of the code in relation to memory and CPU requirements.

1 INTRODUCTION

This work presented in this paper brings together Finite Element Analysis and Random Field Theory in the study of a three-dimensional boundary value problem of steady seepage. The aim of the investigation is to observe the influence of soil variability on the expected value of 'output' quantities such as the flow rate and exit gradient. Smith and Freeze (1979, Pts. 1 and 2) were among the first to study the problem of confined flow through a stochastic medium using finite differences, in which 2-d examples of flow between parallel plates and beneath a single sheet-pile were presented. Recent developments in random field and finite element methodology have led to further studies of steady seepage problems for a range of other two-dimensional boundary value problems (Fenton and Griffiths 1993, Griffiths and Fenton 1993, Griffiths *et al* 1994).

A conference on probabilistic methods in geotechnical engineering (Li and Lo 1993) highlighted some of the recent advances in this field. For example Mostyn and Li (1993) emphasised the importance of taking account of the spatial correlation of soil properties in probabilistic analyses. It was pointed out that the "vast majority of existing models do not do this", and although their particular application was the analysis of slope stability in which the random soil properties in question were the shear strength parameters, the same arguments could be applied to soil permeability in a seepage problem. White (1993) also described how early probabilistic analyses typically represented soil property uncertainty by the use of a single random variable which was varied from one realization to the next.

The use of *random fields* (Vanmarcke 1984, Fenton and Vanmarcke 1990) was considered to be an important refinement, in that the soil property at each location within the soil mass was itself considered to be a random variable. An important feature of the random field approach is that it appropriately takes into account the positive correlation that is observed between soil properties measured at locations that are 'close' together.

The work presented herein extends the previous work of the authors to encompass three-dimensional analysis. The earlier two dimensional model rested on the assumption of perfect correlation in the out-of-plane direction, an assumption no longer necessary with a three-dimensional model.

2 BRIEF DESCRIPTION OF THE FINITE ELEMENT MODEL

In this paper a random field generator known as the Local Average Subdivision Method (LAS) devised by Fenton (1990) is combined with the power of the Finite Element Method for modeling spatially varying soil properties. The problem chosen for study is a simple boundary value problem of steady seepage beneath a single sheet pile wall penetrating a layer of soil. The variable soil property in this case is the soil permeability K.

The overall dimensions of the problem to be solved are shown in Figures 1a, 1b and 1c. Figure 1a shows an isometric view of the 3-d flow regime, and Figures 1b and 1c show elevation and plan views respectively. The two-dimensional studies published previously by the authors (Griffiths *et al* 1994) correspond to the boundary value problem indicated in Figure 1b. In all results presented in this paper, the dimensions L_x and L_y were held constant while the third dimension L_z was gradually increased to monitor the effects of three-dimensionality.

The finite element program used for the solutions of Laplace's equation presented in this paper was obtained by combining Programs 5.9 and 7.0 from the modular code published in the text by Smith and Griffiths (1988). In all analyses presented here, a uniform mesh of cubic 8-node brick elements with a side length of 0.2 was used with 32 elements in the x-direction ($L_x = 6.4$), 16 elements in the y-direction ($L_y = 3.2$) and up to 16 elements in the z-direction ($L_z = 0.8, 1.6$ and 3.2). A time-saving feature of 'brick' elements (i.e. all sides meet at 90^o to each other) such as those used in the present study, is that their conductivity matrices are easily computed explicitly without the need for numerical integration.

Within each mesh, the freedoms were numbered to minimize bandwidths of the global conductivity matrix together with a 'skyline' storage strategy. Cholesky factorization was used to solve the simultaneous equations (see e.g. Griffiths and Smith 1991). The skyline approach was found to run faster than conventional (constant bandwidth) methods as well as giving substantial savings on memory requirement; a particularly important consideration in 3-d analysis .

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Essentially, the permeability field is obtained through the transformation

$$K_i = \exp\{\mu_{\ln k} + \sigma_{\ln k}\, G_i\} \tag{1}$$

in which K_i is the permeability assigned to the i^{th} element, G_i is the local average of a standard Gaussian random field, G, over the domain of the i^{th} element, and $\mu_{\ln k}$ and $\sigma_{\ln k}$ are the mean and standard deviation of the logarithm of K (obtained from the 'target' mean and standard deviation μ_k and σ_k).

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$$\rho(\tau) = \exp\left\{-\frac{2}{\theta_k}|\tau|\right\} \tag{2}$$

where τ is the distance between points in the field and θ_k is called the scale of fluctuation. Loosely speaking, the scale of fluctuation is the distance over which points in the field are significantly correlated. As the scale of fluctuation goes to infinity, G_i becomes equal to G_j for all elements i and j – that is the field of permeabilities tends to become uniform on each realization (but each realization can still be quite different). At the other extreme, as the scale of fluctuation goes to zero, G_i and G_j become independent for all $i \neq j$ – the soil permeability changes rapidly from point to point.

In the three-dimensional analyses presented in this paper, the scales of fluctuation in all directions are taken to be equal (isotropic) for simplicity. Although beyond the scope of this paper, it should be noted that for a layered soil mass the horizontal scales of fluctuation are generally larger than the vertical scale due to the natural stratification of many soil deposits. A limitation of the 2-d models considered previously was that the out-of-plane scale of fluctuation was assumed infinite – soil properties are constant in this direction – which is equivalent to specifying that the streamlines must remain in the plane of the analysis. This was clearly a deficiency and motivated the present work in which no such assumptions are made.

4 SUMMARY OF THE RESULTS FROM SEEPAGE ANALYSES

A Monte-Carlo approach to the seepage problem was adopted in which, for each set of input statistics ($\mu_k, \sigma_k, \theta_k$) and mesh geometry ($L_z$), 1000 realizations were performed. Experience from 2-d analyses and other reproducibility studies performed by Paice (1994), have indicated that 1000 will usually be a sufficient number of realizations to obtain meaningful output statistics for steady seepage problems with moderate input variance. The main output quantities of interest from each realization in this problem are the total flow rate through the system Q and the exit gradient i_e. The rather more complicated issues relating to analysis of the exit gradient are beyond the scope of the present work and will be discussed in a future publication.

In this paper therefore we focus on the flow rate. Following Monte-Carlo simulations, the mean and standard deviation of Q were computed and presented in non-dimensional form by representing it in terms of a normalized flow rate $\bar{Q}$ thus:

$$\bar{Q} = Q/(H\mu_k L_z) \tag{3}$$

where H is the total head loss across the wall, typically set to unity. Division by L_z has the effect of expressing the average flow rate over one unit of thickness in the z-direction enabling a direct comparison to be made with the 2-d results.

The following parametric variations were implemented for fixed $\mu_k = 1 \times 10^{-5}$, $L_x = 6.4$, and $L_y = 3.2$;

$$\sigma_k/\mu_k = 0.125, 0.25, 0.51, 2, 4 \text{ and } 8$$

$$\theta = 1, 2, 4, 8 \text{ and } \infty(\text{analytical})$$

$$L_z = 0.8, 1.6 \text{ and } 3.2$$

and a selection of results will be presented here.

As the coefficient of variation of the input permeability ($CV_k = \sigma_k/\mu_k$) was increased, a consistent fall in the expected value of the flow rate from its deterministic value of $\bar{Q}_{det} \approx 0.46$ was observed as shown in Figure 2(a) for the case where $L_z/L_y = 1$. This was especially true for small values of the scale of fluctuation θ_k, however as θ_k is increased the value of $\mu_{\bar{Q}}$ is tending towards the deterministic result that would be expected for a strongly correlated permeability field ($\theta \rightarrow \infty$).

Figure 2(b) shows the standard deviation of the normalized flow rate $\sigma_{\bar{Q}}$ for the same geometry. For small θ_k very little variation in $\bar{Q}$ was observed, even for high coefficients of variation. This is understandable if one thinks of the total flow through the domain as effectively an averaging process – high flow rates in some regions are offset by lower flow rates in other regions. It is well known in statistics that the variance of an average decreases linearly with the number of independent samples used in the average. In the random field context, the 'effective' number of independent samples increases as the scale of fluctuation decreases, thus the decrease in variance in flow rate is to be expected. Conversely, when the scale of fluctuation is large, the variance in the flow rate is also expected to be larger – there is less 'averaging' within each realization. The maximum flow rate variance is obtained when the field becomes completely correlated, $\theta_k = \infty$, as given by

$$\sigma_{\bar{Q}} = \frac{\sigma_k}{\mu_k}\bar{Q}_{det} \tag{4}$$

By the same reasoning, the variance of the estimate of $\sigma_{\bar{Q}}$ will increase as the scale of fluctuation increases. This can be seen in both Figures 2(a) and 2(b) where the curves for larger CV_k and θ_k shows some erratic behaviour. In these cases, more than 1000 realizations may be required to obtain accurate results.

Figures 3(a) and 3(b) show the influence of three-dimensionality on the mean and standard deviation of Q by comparing results with gradually increasing numbers of elements in the z-direction.

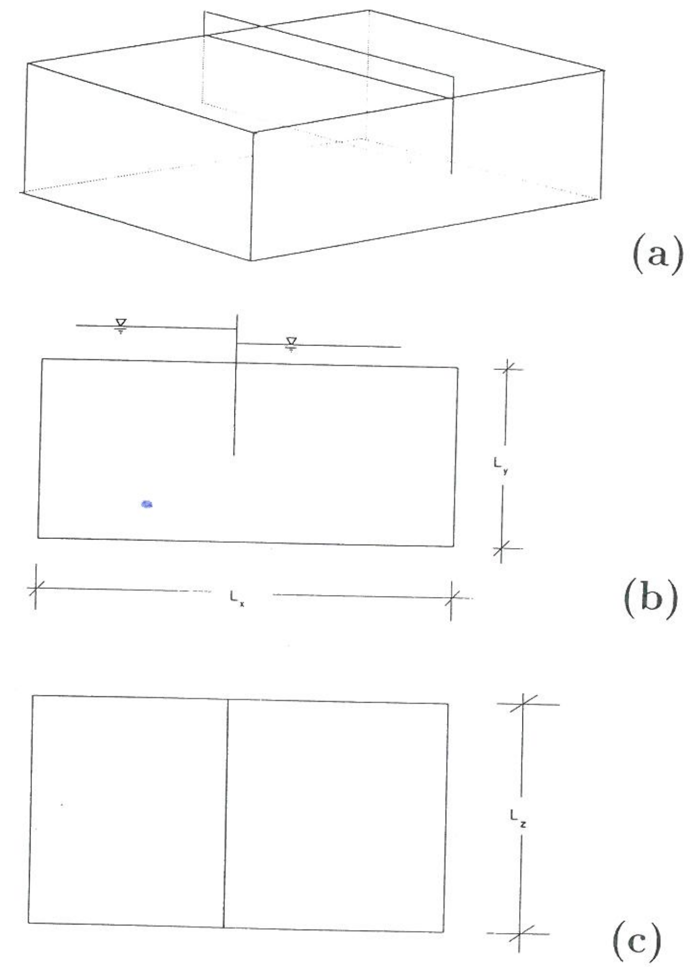

Figure 1(a) Isometric view of 3-d seepage problem, (b) Elevation (c) Plan

Also included in these figures is the 2-d result which implies an infinite scale of fluctuation in the z-direction and allows no flow out of the plane of the analysis. The particular cases shown correspond to a fixed scale of fluctuation $\theta = 1$.

Compared with two-dimensional analysis, three-dimensions allows the flow greater freedom to 'avoid' the low permeability zones. This results in a less steep reduction in the expected flow rate with increasing CV_k as shown in Figure 3(a). There is also a corresponding reduction in the variance of the expected flow rate as the 3rd dimension is elongated as shown in Figure 3(b). In summary, the effect of allowing flow in three-dimensions is to increase the averaging effect discussed above within each realization. The difference between the two-and three-dimensional results are not that great, and it could be argued that a 2-d analysis is a reasonable first approximation to the 'true' behavior. It should be noted however, that the 2-d approximation will tend to underestimate the expected flow through the system which is an unconservative result from the point of view of engineering design.

4.1 Comment on computer timings

The results presented in this paper were run on a DEC 3000 Model 400 workstation. A summary of the total CPU time consumed by the various analyses is presented in the table below. The 'Timing' column is expresses in non-dimensional form with respect to an equivalent 2-d analysis with the same mesh density in the xy-plane.

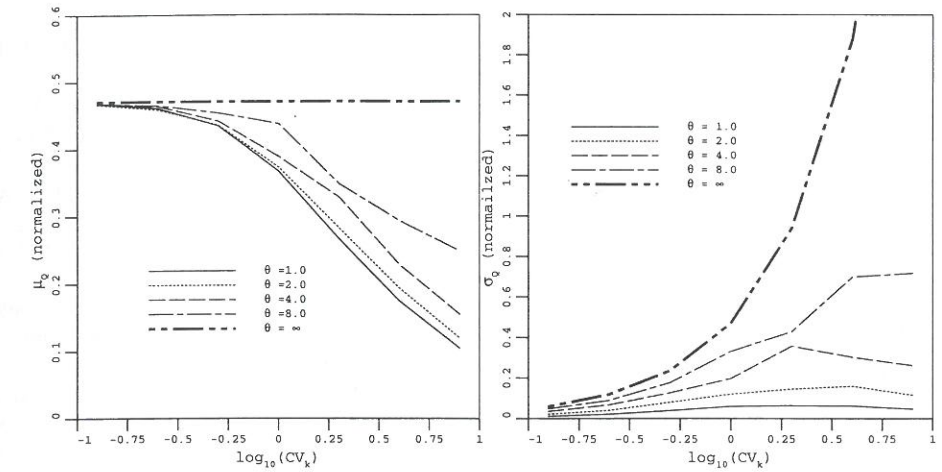

Figure 2: Influence of θ_k on statistics of normalized flow rate ($L_z/L_y = 1$), (a) mean (b) standard deviation

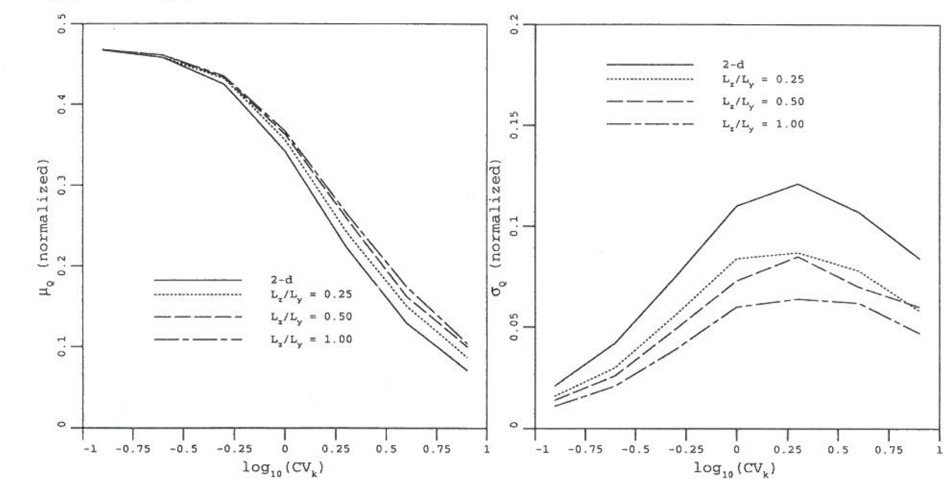

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Table 1. Comparison of timings from 2-d and 3-d analyses

Dimension	Timing
2-d	1
$L_z/L_y = 0.25$	49
$L_z/L_y = 0.50$	239
$L_z/L_y = 1.00$	1463

5 CONCLUDING REMARKS

The paper has presented results which form part of a broad study conducted by the authors into the influence of random soil properties on geotechnical design. In this paper, three-dimensional random field and finite element methods have been combined in the study of steady seepage beneath a single sheet-pile wall embedded in a layer of random soil. A quite extensive parametric study has been performed in which the coefficient of variation and the spatial correlation of soil permeability has been systematically varied to observe their influence on the mean and standard deviation of the flow rate through the system. In addition, the influence of three-dimensionality has been investigated by gradually increasing the width of the model in the direction of the wall. In all cases,

results were compared with those that would have been obtained using a two-dimensional analysis. For moderate values of the scale of fluctuation, the expected value of the flow rate was found to fall consistently as the coefficient of variation of the input permeability was increased. For higher values of the scale of fluctuation, the normalized flow rate mean tends to the deterministic value. The standard deviation of the flow rate was shown to consistently increase with the scale of fluctuation, but within the bounds defined analytically for the limiting case of perfect correlation.

The influence of three-dimensionality was to reduce the overall 'randomness' of the results observed from one realization to the next. This had the effect of increasing the expected flow rate and reducing the variance of the flow rate over those values observed from a two-dimensional analysis with the same input statistics. Although unconservative in the estimation of flow rates, there was not a great difference between the two- and three-dimensional results presented suggesting that the simpler and less expensive two-dimensional approach may give acceptable accuracy for engineering purposes.

The influence of three-dimensionality will continue to be investigated as part of this program of research. One of the main objectives will be to present results in a general form that enables the statistics of flow relating to a range of boundary value problems to be investigated without resort to a specific analysis in each case.

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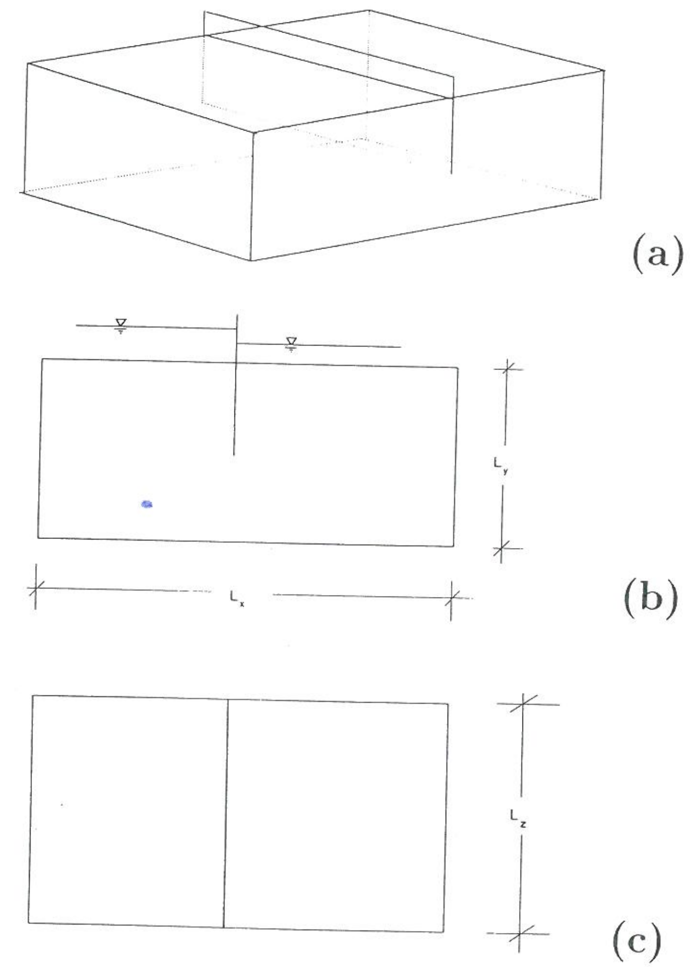

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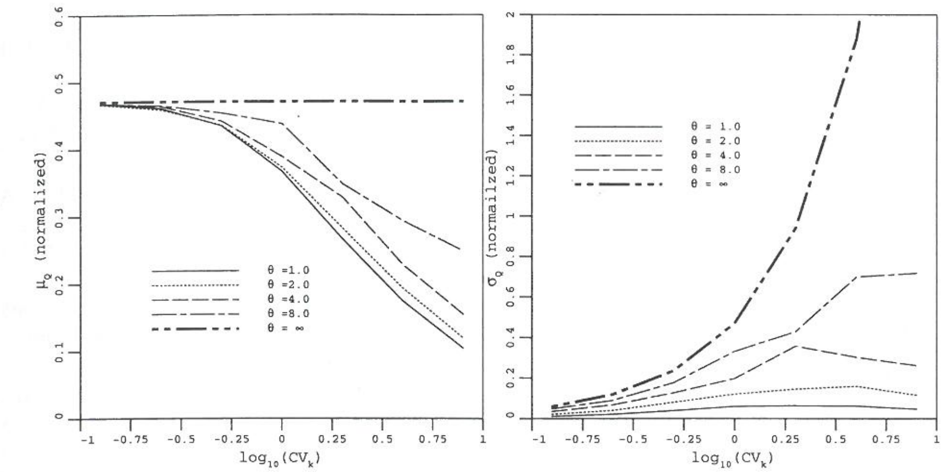

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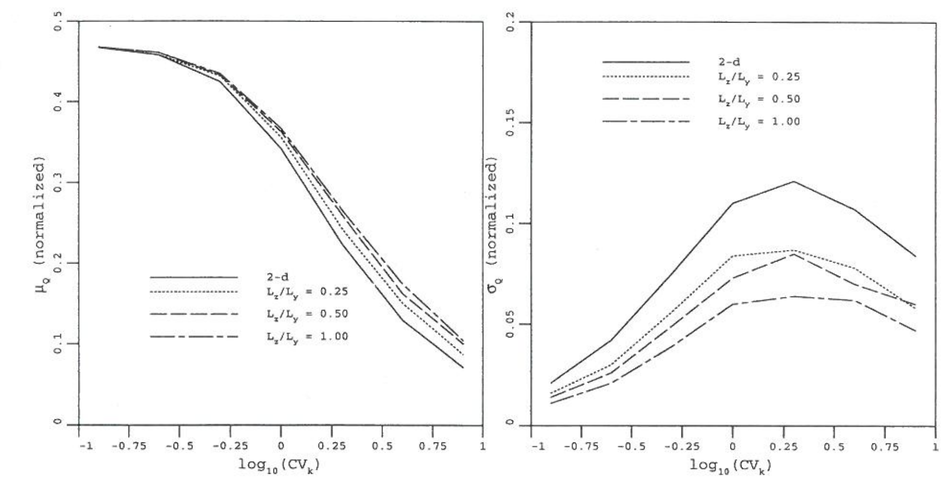

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In the three-dimensional analyses presented in this paper, the scales of fluctuation in all directions are taken to be equal (isotropic) for simplicity. Although beyond the scope of this paper, it should be noted that for a layered soil mass the horizontal scales of fluctuation are generally larger than the vertical scale due to the natural stratification of many soil deposits. A limitation of the 2-d models considered previously was that the out-of-plane scale of fluctuation was assumed infinite – soil properties are constant in this direction – which is equivalent to specifying that the streamlines must remain in the plane of the analysis. This was clearly a deficiency and motivated the present work in which no such assumptions are made.

4 SUMMARY OF THE RESULTS FROM SEEPAGE ANALYSES

A Monte-Carlo approach to the seepage problem was adopted in which, for each set of input statistics $(\mu_k, \sigma_k, \theta_k)$ and mesh geometry (L_z), 1000 realizations were performed. Experience from 2-d analyses and other reproducibility studies performed by Paice (1994), have indicated that 1000 will usually be a sufficient number of realizations to obtain meaningful output statistics for steady seepage problems with moderate input variance. The main output quantities of interest from each realization in this problem are the total flow rate through the system Q and the exit gradient i_e. The rather more complicated issues relating to analysis of the exit gradient are beyond the scope of the present work and will be discussed in a future publication.

In this paper therefore we focus on the flow rate. Following Monte-Carlo simulations, the mean and standard deviation of Q were computed and presented in non-dimensional form by representing it in terms of a normalized flow rate $\bar{Q}$ thus:

$$\bar{Q} = Q/(H\mu_k L_z) \qquad (3)$$

where H is the total head loss across the wall, typically set to unity. Division by L_z has the effect of expressing the average flow rate over one unit of thickness in the z-direction enabling a direct comparison to be made with the 2-d results.
The following parametric variations were implemented for fixed $\mu_k = 1 \times 10^{-5}$, $L_x = 6.4$, and $L_y = 3.2$;

$$\sigma_k/\mu_k = 0.125, 0.25, 0.51, 2, 4 \text{ and } 8$$

$$\theta = 1, 2, 4, 8 \text{ and } \infty(\text{analytical})$$

$$L_z = 0.8, 1.6 \text{ and } 3.2$$

and a selection of results will be presented here.

As the coefficient of variation of the input permeability ($CV_k = \sigma_k/\mu_k$) was increased, a consistent fall in the expected value of the flow rate from its deterministic value of $\bar{Q}_{det} \approx 0.46$ was observed as shown in Figure 2(a) for the case where $L_z/L_y = 1$. This was especially true for small values of the scale of fluctuation θ_k, however as θ_k is increased the value of $\mu_{\bar{Q}}$ is tending towards the deterministic result that would be expected for a strongly correlated permeability field ($\theta \rightarrow \infty$).

Figure 2(b) shows the standard deviation of the normalized flow rate $\sigma_{\bar{Q}}$ for the same geometry. For small θ_k very little variation in $\bar{Q}$ was observed, even for high coefficients of variation. This is understandable if one thinks of the total flow through the domain as effectively an averaging process – high flow rates in some regions are offset by lower flow rates in other regions. It is well known in statistics that the variance of an average decreases linearly with the number of independent samples used in the average. In the random field context, the 'effective' number of independent samples increases as the scale of fluctuation decreases, thus the decrease in variance in flow rate is to be expected. Conversely, when the scale of fluctuation is large, the variance in the flow rate is also expected to be larger – there is less 'averaging' within each realization. The maximum flow rate variance is obtained when the field becomes completely correlated, $\theta_k = \infty$, as given by

$$\sigma_{\bar{Q}} = \frac{\sigma_k}{\mu_k}\bar{Q}_{det} \qquad (4)$$

By the same reasoning, the variance of the estimate of $\sigma_{\bar{Q}}$ will increase as the scale of fluctuation increases. This can be seen in both Figures 2(a) and 2(b) where the curves for larger CV_k and θ_k shows some erratic behaviour. In these cases, more than 1000 realizations may be required to obtain accurate results.

Figures 3(a) and 3(b) show the influence of three-dimensionality on the mean and standard deviation of Q by comparing results with gradually increasing numbers of elements in the z-direction.

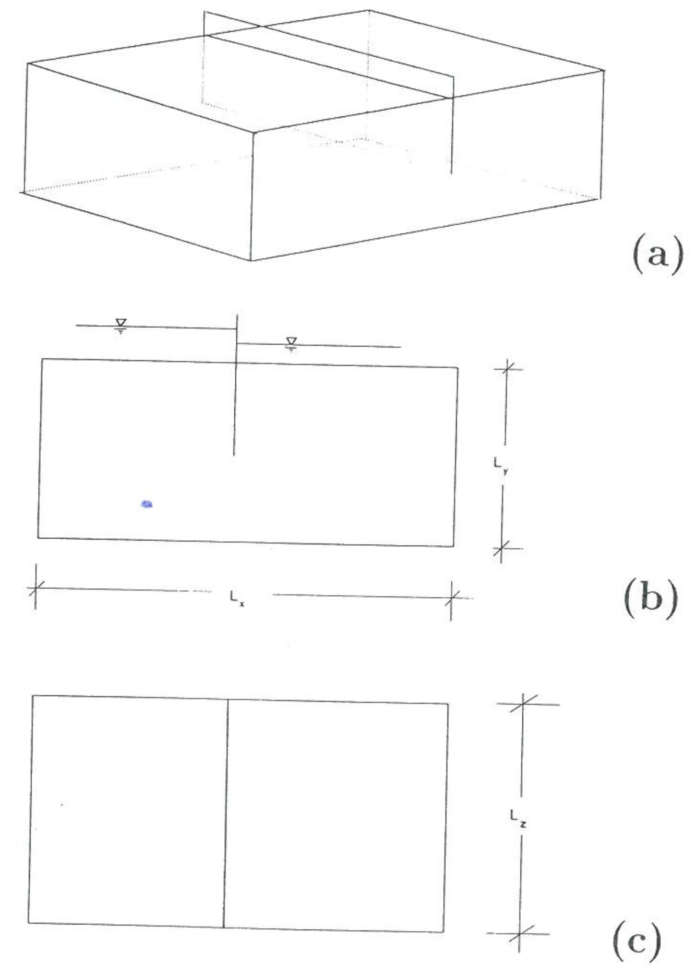

Figure 1(a) Isometric view of 3-d seepage problem, (b) Elevation (c) Plan

Also included in these figures is the 2-d result which implies an infinite scale of fluctuation in the z-direction and allows no flow out of the plane of the analysis. The particular cases shown correspond to a fixed scale of fluctuation $\theta = 1$.

Compared with two-dimensional analysis, three-dimensions allows the flow greater freedom to 'avoid' the low permeability zones. This results in a less steep reduction in the expected flow rate with increasing CV_k as shown in Figure 3(a). There is also a corresponding reduction in the variance of the expected flow rate as the 3rd dimension is elongated as shown in Figure 3(b). In summary, the effect of allowing flow in three-dimensions is to increase the averaging effect discussed above within each realization. The difference between the two-and three-dimensional results are not that great, and it could be argued that a 2-d analysis is a reasonable first approximation to the 'true' behavior. It should be noted however, that the 2-d approximation will tend to underestimate the expected flow through the system which is an unconservative result from the point of view of engineering design.

4.1 Comment on computer timings

The results presented in this paper were run on a DEC 3000 Model 400 workstation. A summary of the total CPU time consumed by the various analyses is presented in the table below. The 'Timing' column is expresses in non-dimensional form with respect to an equivalent 2-d analysis with the same mesh density in the xy-plane.

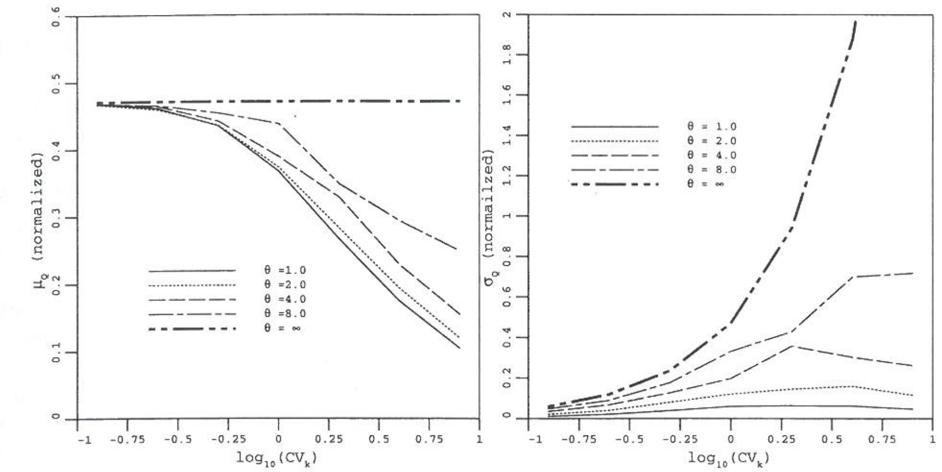

Figure 2: Influence of θ_k on statistics of normalized flow rate ($L_z/L_y = 1$), (a) mean (b) standard deviation

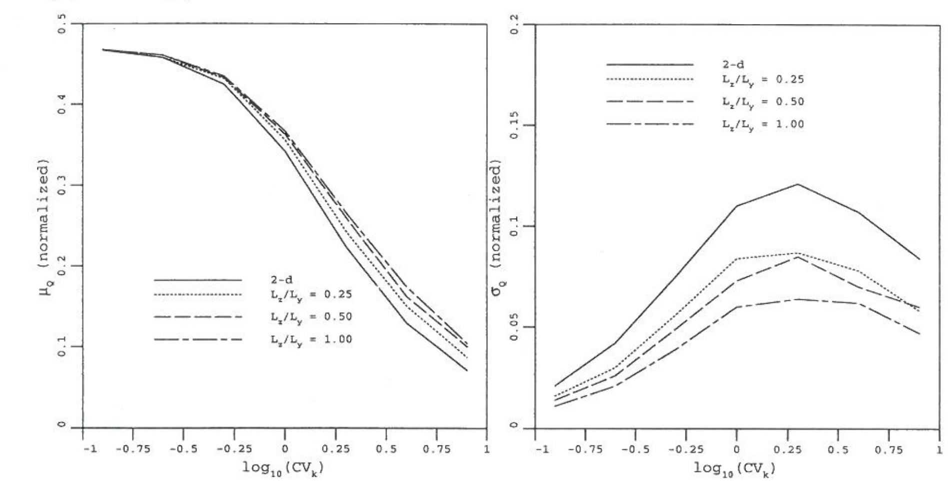

Figure 3: Influence of L_z/L_y on statistics of normalized flow rate ($\theta = 1$), (a) mean (b) standard deviation

Table 1. Comparison of timings from 2-d and 3-d analyses

Dimension	Timing
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$L_z/L_y = 0.25$	49
$L_z/L_y = 0.50$	239
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5 CONCLUDING REMARKS

The paper has presented results which form part of a broad study conducted by the authors into the influence of random soil properties on geotechnical design. In this paper, three-dimensional random field and finite element methods have been combined in the study of steady seepage beneath a single sheet-pile wall embedded in a layer of random soil. A quite extensive parametric study has been performed in which the coefficient of variation and the spatial correlation of soil permeability has been systematically varied to observe their influence on the mean and standard deviation of the flow rate through the system. In addition, the influence of three-dimensionality has been investigated by gradually increasing the width of the model in the direction of the wall. In all cases,

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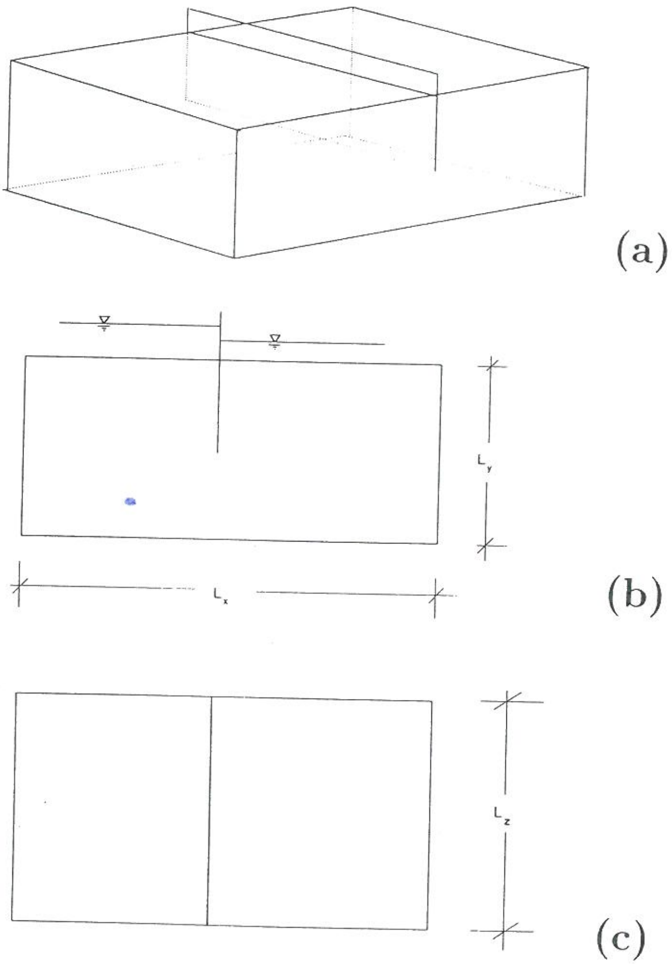

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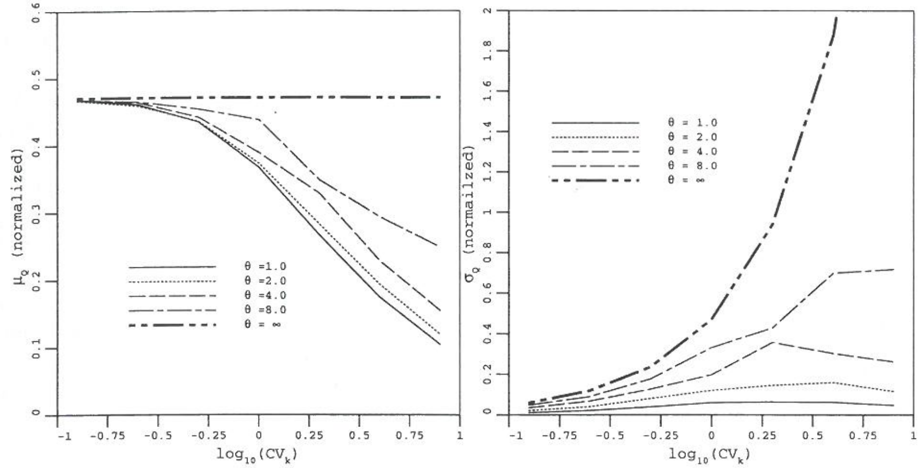

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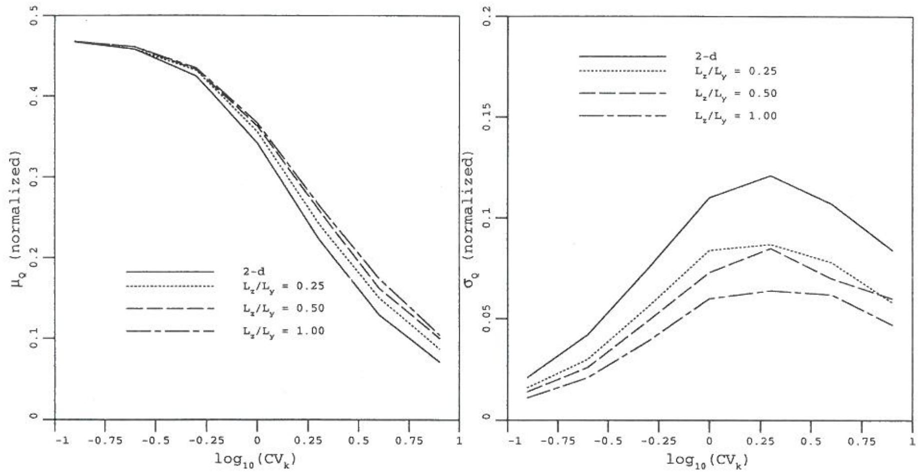

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