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LETTERS TO THE EDITOR

ELASTO-PLASTIC ANALYSES OF DEEP FOUNDATIONS IN COHESIVE SOIL

by D. V. Griffiths, *Int. j. numer. anal. methods geomech.*, **6**, 211–218 (1982)

Griffiths[1] has used the finite element method to investigate the behaviour of deep foundations in a cohesive soil. Particular attention has been focused on the problem of predicting collapse loads accurately using the displacement type of formulation. This question is obviously of practical significance, and has received attention on several occasions.[2-5] If, for a given soil model, finite elements can be used to predict deformations accurately right up to incipient collapse, then it is no longer necessary to make the traditional distinction between settlement and stability analysis.

In his paper, Griffiths[1] has used the 8-noded quadrilateral element with 2×2 (reduced) integration for undrained plasticity analysis. As discussed by Sloan and Randolph,[4] reduced integration has the beneficial effect of decreasing the total number of incompressibility constraints on the nodal degrees of freedom, thus avoiding the well-known phenomenon of 'locking'. Moreover, since the number of integration points for each element is reduced, the cost of solution is also reduced. There are, however, certain aspects of this approach which warrant further attention. Recently, Nagtegaal and De Jong[6] have noted that, for large strain analysis under conditions of axial symmetry, the use of reduced integration with the 8-noded element may lead to the development of curious deformation patterns. In this note, it will be shown that similar problems may arise in small strain applications where the material is modelled as elastic perfectly plastic.

Figure 1 illustrates the initial and deformed meshes for undrained analysis of a smooth flexible strip footing on an elastic perfectly-plastic Tresca material. In this analysis, with 25 elements and 192 degrees of freedom, 8-noded quadrilateral elements were employed with reduced integration. The total number of integration points is equal to 100. The Euler integration procedure, with an equilibrium correction at each of the 50 load steps, was used to solve the governing non-linear equations, and all computations were conducted in double precision on an IBM 370/165. An equilibrium check, based on the ratio of the norm of unbalanced forces to the norm of applied forces, indicated that equilibrium was satisfied to within 1 per cent for all load steps until collapse occurred at a pressure of approximately $5{\cdot}22\ C_u$. As collapse is approached, the elements in the vicinity of the footing deform in a peculiar pattern which is similar to that observed by Nagtegaal and De Jong.[6] This behaviour is due to the dominance of zero energy modes as a large region of the continuum becomes plastic. Figure 2 illustrates the initial and deformed meshes for the same problem, but analysed using the cubic strain triangle as discussed by Sloan and Randolph[4] (8 elements, 162 degrees of freedom, and a total of 96 integration points). This element requires 12 integration points to evaluate the element stiffness matrices exactly under plane strain conditions. Because the stiffnesses are exact, the deformed mesh does not display the 'barrelling' phenomenon associated with the reduced integration results. The collapse pressure obtained from this analysis, using the same solution algorithm and load steps described previously, was again in the vicinity of $5{\cdot}22\ C_u$. By way of interest, the CPU times required for the two meshes were identical, taking a total of 37 seconds.

To illustrate that difficulties may also arise when reduced integration is used for other classes of problems, Figure 3 shows the initial and deformed meshes for an embankment analysis with a non-linear elastic soil model. This plot has been taken from Reference 7. The element with the heavy outline appears to be deforming in a pattern which is very close to a zero strain energy mode.

There are a number of potential advantages in using the cubic strain triangle with exact integration for plastic collapse calculations. These are:

(1) No problems are encountered with barrelling or zero strain energy modes, since the element stiffness matrices are evaluated exactly. As emphasized by Bathe,[8] reliability is an extremely important quality in finite element analysis, particularly in large scale computations.

(2) The element is efficient. Timing runs reported by Sloan[5] indicate that the cubic strain triangle solutions cost no more than

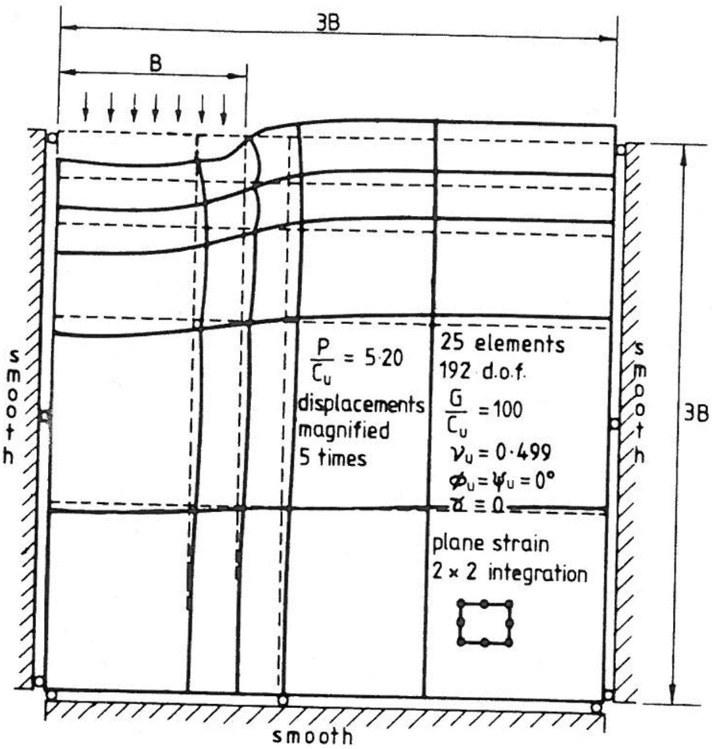

Figure 1. Deformation response for smooth flexible strip footing on a purely cohesive layer; 8-noded quadrilateral element results with reduced (2 × 2) integration

equivalent solutions obtained using the 8 noded quadrilateral with reduced integration. The integration rules that are available for triangles are very efficient (only 12 points are required to evaluate the stiffness matrix exactly for an element with a quartic displacement expansion).

(3) No substantial modifications to a traditional stiffness program are necessary, it is simply a matter of incorporating another shape function subroutine. The information necessary for implementing the cubic strain triangle may be found in Reference 4. Contrary to the claim of Griffiths[1] that the cubic strain triangle is 'extremely complicated', the additional coding required to implement this element is minimal (an additional 30 lines or so compared with the 8-noded element). Moreover, for manual data preparation, there is a positive advantage in using high order formulations as fewer elements need to be specified (compare the 8 elements of Figure 2 with the 25 elements of Figure 1). In cases where all element boundaries are straight, the co-ordinates for all non-vertex nodes may be generated automatically by linear interpolation.

(4) Recent mathematical studies by Babuska and Szabo[9] on covergence of hierarchic displacement elements have shown that, for incompressible elasticity, high order formulations are more accurate than low order formulations (for a given number of degrees of freedom). Indeed, based on their investigations, these authors recommend that elements with at least a cubic displacement expansion should be employed for plane strain analysis of incompressible materials.

(5) The cubic strain triangle is also suitable for analysis of dilatant-frictional behaviour (e.g. soil masses which are modelled using a Mohr–Coulomb yield function). Numerical experiments by Sloan[5] indicate that accurate estimates of collapse may be achieved for both plane strain and axisymmetric conditions.

below the lower bounds given by Gunn.[10] This may imply that the use of reduced integration can lead to collapse loads which are below the lower bounds furnished from plasticity theory. In using high order elements with exact integration of the stiffness matrices, the writer has never observed this type of behaviour.

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AUTHOR'S REPLY TO SLOAN'S DISCUSSION*

As far as element integration is concerned, there is no argument over which integration scheme (exact or reduced) is more rigorous in a mathematical sense. It should not be forgotten, however, that finite element techniques are approximate numerical methods for the solution of differential equations, irrespective of the integration scheme used.

The debate here is about whether the use of a high order element with exact integration (15-noded triangle with 12 integrating points) can justify its use in preference to a lower order element with reduced integration (e.g. 8-noded quadrilateral with 4 integrating points) in the prediction of collapse loads for geotechnical problems.

In order to probe this point, it must be demonstrated that the higher order element gives significantly better collapse predictions than the lower order method, especially for the purposes of engineers. Widespread acceptance of finite element techniques in industry has been slow and the use of constant strain triangles is still common. The writer's concern is that the profession might be deterred from finite element collapse predictions if the impression is given that high order elements *must* be used for this type of problem.

Consideration of the evidence of available results indicates that the surface footing problem does not justify the use of high order elements. For example, Griffiths[1] has produced results for surface footings on a wide range of soil types, including cohesionless soils, using 8-noded quadrilaterals with reduced integration which were in close agreement with published closed form and approximate solutions. It may be that 'difficult problems' will come to light in which the higher order elements are an advantage, but the lower order techniques with reduced integration have been used with success[2] in a wide range of geotechnical collapse problems including slope stability and earth pressures.

Referring to Sloan's discussion, the barrelling shown in Figure 1 does not appear particularly serious in view of the crude mesh used. A finer mesh would reduce this effect, but in any case this phenomenon does not appear to have had an influence on the computed collapse load. Indeed, Sloan achieved the same collapse load using both element types (Figures 1 and 2) which fully supports one of the main conclusions of the paper under discussion, namely that provided collapse

* Preceding letter.

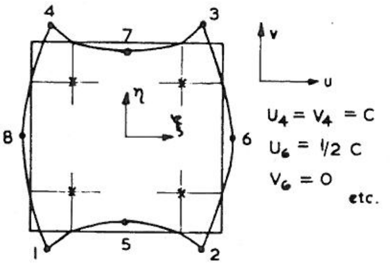

Figure A. Zero energy deformation for 8-noded quadrilateral element with reduced integration

loads are the subject of the analysis, the lower order methods with reduced integration perform adequately.

The barrelling elements in the upper part of the mesh are not, in fact, deforming with zero energy. Zero energy modes only occur when a pattern of nodal displacements produces a strain field that is zero at all quadrature points (Figure A). This definition cannot be applied to the elements considered here because at least two sides of these elements remain approximately straight, and if anything curve away from the zero energy mode (Figure B).

A further point raised by Cooke,[3] is that even if a quadratic element did deform with zero energy, its neighbours could not. This implies that the zero energy mode should not be of major

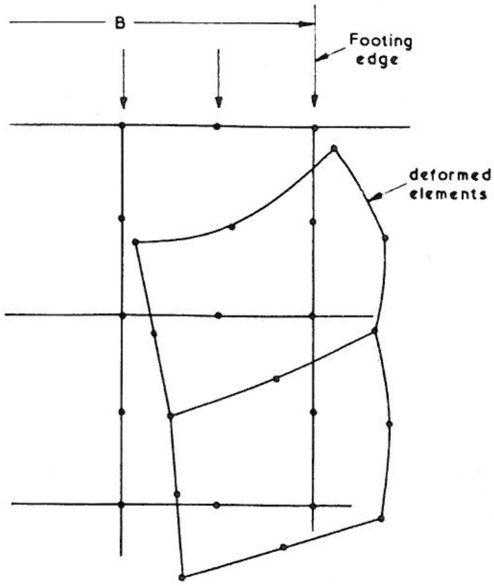

Figure B. Deformation response near footing edge. 8-noded quadrilateral element with reduced integration

consequence in meshes of quadratic elements using reduced integration.

Sloan also refers to the prediction of deformation up to incipient collapse. In view of this it is interesting to observe from Figures 1 and 2 that,

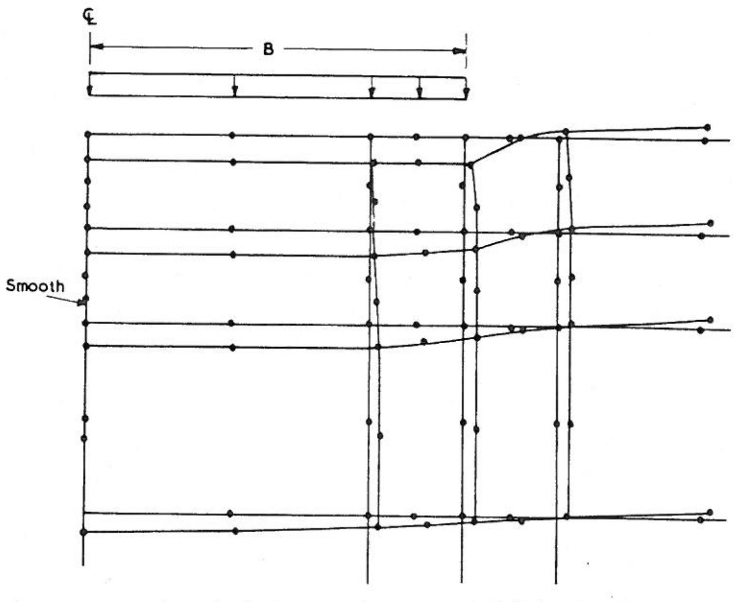

Figure C. Deformation response of a cohesive layer under a smooth rigid footing. 8-noded quadrilateral elements with reduced integration in spite of the barrelling and the crude meshes, the predicted surface displacements beneath and adjacent to the footing are in close agreement.

Finally, using the same mesh as in Figures 1 and 2, a displacement control solution was attempted. This involves prescribing equal vertical displacements at the footing nodes and back-calculating the stresses by either integrating the stresses over the element to give equivalent nodal forces, or simply averaging the vertical stress component in the first row of integrating points beneath the displaced nodes. Physically this method models a rigid footing, and is favoured by the present writer for the solution of bearing capacity problems. It is interesting to note in Figure C that by forcing the displaced nodes to remain in a horizontal line, the barrelling phenomenon occurring in the upper elements is less than when the footing is assumed to be flexible under load control.

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INEXPENSIVE BUT TECHNICALLY SOUND MINE PILLAR DESIGN ANALYSIS

by W. G. Pariseau, *Int. j. numer. anal. methods geomech.*, **5**, 429–447 (1981)

I should like to discuss a few points which cropped up in my mind as regards this interesting paper.

It is indeed surprising that a three-dimensional situation can thus be reduced to a two-dimensional one by merely changing the pillar loading to an 'equivalent' value. The author has modelled relatively flat pillars with a fixed working height of 8 ft (2·43 m). It would perhaps be interesting to study this equivalence between 2-D and 3-D solutions on slender pillars, say 5 × 5 × 5 m high or 3 × 3 × 2 m high, which do occur in practice. The behaviour of the corners of such pillars becomes critical. I have a feeling that the stresses at the corners will differ by a magnitude which will depend on the slenderness of pillars.

The 3-D and 2-D finite element pillar models of Figures 3 and 4 could be reduced significantly in size if the middle horizontal plane of the pillar is assumed to remain plane after excavation—an assumption which is quite logical and common in pillar design, unless the roof and floor have widely different properties. Since the purpose of the paper appears to be to compare two methods, considerable effort would be saved if a symmetric horizontal midplane (zero vertical displacements) were introduced.

One thing that is not very clear from the paper is how virgin or pre-excavation stresses were accounted for and what was the stress field assumed. This, I think, is important since the total stresses in the pillar will depend greatly on the pre-excavation stresses. Will it not be necessary to apply some sort of an equivalence to the virgin stress field as well?

I would also like to know the reason for using the Drucker–Prager failure criterion rather than the Mohr or Coulomb–Navier criterion. Can the spot safety factor be used for predicting the strength of a pillar as a whole? Unless this is possible, we shall be required again to fall back upon empirical pillar strength formulae in spite of exhaustive finite element stress analyses.

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