A note on load and resistance factors in slopes and foundations

Resultados recientes sobre factores de carga y factores de resistencia en taludes y fundaciones

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Abstract

The widely differing factors of safety traditionally used for ultimate limit state design in slope stability and bearing capacity have been re-examined using numerical and analytical methods. The results show that for typical slope and bearing capacity systems, geotechnical design outcomes are much more sensitive to resistance factors than load factors.

Resumen

Los factores de seguridad ampliamente diferentes usados tradicionalmente en cálculos de estado límite último en el diseño de estabilidad de taludes y capacidad de carga han sido re-examinados usando métodos numéricos y analíticos. Los resultados muestran que para sistemas típicos de taludes y capacidad de carga, los resultados del diseño geotécnico son mucho mas sensibles a los factores de resistencia que a los factores de carga.

1 INTRODUCTION

It is well known that slope stability and bearing capacity analyses for design typically target factors of safety of around 1.5 and 3.0 respectively. Both applications involve ultimate limit state conditions, so there is an implication that the bearing capacity design is twice as safe as that for slope stability. Can this be true? The question in not easily answered, because the factor of safety used in slopes is a strength factor, while that for bearing capacity is a loads factor. Atkinson (2007) suggested that the high factor of safety applied to the bearing capacity equation may have originated as an empirical means of limiting settlements, i.e. a load about three times smaller than the ultimate value would typically give acceptable settlements. Clearly in current practice, serviceability limit states are dealt with quite separately to ultimate limit states. A general discussion of the difference between factors of safety based on resistance and load for slope and bearing capacity problems, and the disparity between them, was presented by Duncan and Wright (2005), and there are numerous other references in which load and resistance factors have been discussed (e.g. Terzaghi *et al.* 1996, Salgado 2008). While strength reduction is the traditional way of finding the factor of safety of slope, the idea has been applied to other geotechnical stability problems such as excavations and retaining walls (e.g. Dawson *et al*. 2000). In this paper, factors of safety related to strength reduction and loads increase have been compared for slope stability and bearing capacity examples. In the slope stability example, factors of safety have been computed using both Bishop's method and elastic-plastic finite elements. In the case of bearing capacity, a direct analytical comparison between load and strength factors has been facilitated by a novel identity for the passive earth pressure coefficient K_P .

2 A SLOPE STABILITY EXAMPLE

The slope shown in Fig. 1 has a slope angle of β=33.7° (1.5h:1v), a slope height of $H = 6$ m, and a depth ratio of $D = 1.5$. The cohesion and unit weight are set to $c' = 20.08$ kPa and $\gamma = 20$ kN/m³ respectively, and the friction angle is varied in the range $0 \le \phi' < \beta$ (i.e. the soil friction angle is always less than the slope angle). Effective stress analyses are performed throughout, except for the special case of $\phi_u = 0$, when a total stress analysis can be assumed with an undrained shear strength of $c_u = 20.08$ kPa and saturated unit weight of $\gamma_{sat} = 20$ kN/m³. In this study, the total stress case is calibrated to give a factor of safety of $FS = 1$.

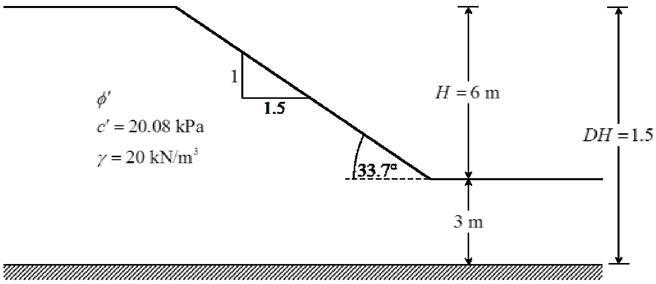

Figure 1. Test slope

Table 1 gives a list of several ways in which a slope can be brought to failure as discussed by Pantelidis and Griffiths (2014). Method 1 represents the classical approach, in which $\tan\phi'$ and c' are factored simultaneously and by the same amount. In this slope stability example, Methods 1 and 2 will be considered, although it may be noted that unit weight increase gives the same factor of safety as cohesion (only) strength reduction, i.e. Methods 2 and 3 are the same, thus

$$FS_{c'} = FS_{\gamma} \tag{1}$$

Table 1. Different factoring strategies to failure

Method	Factoring strategy
1	Reduce c' and $\tan\phi'$ $\left(FS_{c',\tan\phi'}\right)$
2	Increase γ $\left(FS_{\gamma}\right)$
3	Reduce c' $\left(FS_{c'}\right)$
4	Reduce $\tan\phi'$ $\left(FS_{\tan\phi'}\right)$
5	Increase k_h $\left(FS_{k_h}\right)$
6	Increase r_u $\left(FS_{r_u}\right)$

In the current work, for each friction angle selected in the range mentioned above, two factors of safety were computed using the following methods:

1. $FS_{c',\tan\phi'}$ by strength reduction, i.e. the factor by which the shear strength parameters c' and $\tan\phi'$ must be reduced to bring the slope to the point of failure. This is the method used in conventional geotechnical engineering practice.
2. FS_{γ} by gravity increase, i.e. the factor by which the unit weight γ must be increased to bring the slope to the point of failure

Two methods were used to compute the factors of safety mentioned above:

a) A limit equilibrium program using Bishop's method was used to find the conventional factor of safety $\left(FS_{c',\tan\phi'}\right)$ by Method 1. To find the factor of safety based on gravity increase $\left(FS_{\gamma}\right)$ by Method 2, the soil unit weight was gradually increased by trial and error, keeping the strength constant, until the program gave a factor of safety of unity. Thus in the example under consideration where the actual soil unit weight is $\gamma = 20\ \text{kN/m}^3$, if a unit weight of $100\ \text{kN/m}^3$ was needed to generate failure, the factor of safety based on gravity increase was calculated as $FS_{\gamma} = 100/20 = 5$.

b) An elastic-plastic finite element program[1] was developed that gives users a wide choice of ways in which to bring a slope to the point of failure as indicated in Table 1. All FE slope analyses were run with a non-associated (no plastic volume change) flow rule.

Fig. 2a, shows a gradually increasing $FS_{c',\tan\phi'}$ with increasing ϕ' with both FE and Bishop giving essentially identical results. On the other hand, Fig. 2b indicates a steeply increasing FS_{γ} with increasing ϕ'. At lower values of ϕ' in Fig. 2b, FE and Bishop are in close agreement, but as ϕ' increases, they start to diverge which might be expected given the different assumptions made by the two methods, i.e. Bishop's method searches for a circular failure mechanism, whereas the FE approach has no such constraint.

[1] The program is freely available for download at the author's web site
www.mines.edu/~vgriffit/5th_ed/Software/5th_ed.exe

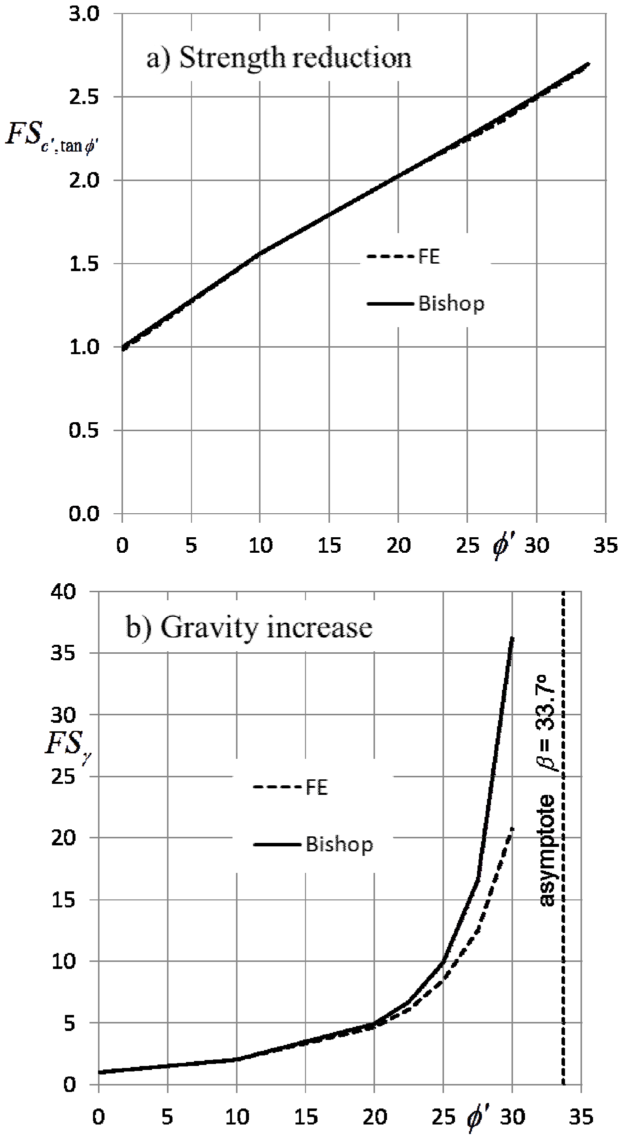

Figure 2. FS vs. ϕ' for the test slope with a) Strength reduction and b) Gravity increase.

Also shown on Fig. 2b is a vertical line corresponding to the slope angle β=33.7° which is the asymptotic limit of FS_γ. This limiting value is most easily explained by considering the equivalence of the factor of safety obtained through gravity increase and cohesion strength reduction as indicated in Eq.(1). As c' is reduced $\left(\text{keeping } \tan\phi' \text{ constant}\right)$ the slope tends to become purely frictional with a factor of safety in the limit given by the “angle of repose” equation.

$$FS = \frac{\tan\phi'}{\tan\beta} \qquad (2)$$

This means that if $\phi' \geq \beta$, it will never be possible to fail a slope through cohesion reduction (or gravity increase) since the frictional component of strength is more than enough to support the weight of the slope. On the other hand, if $\phi' < \beta$, it can be stated that

$$\text{as} \quad \phi' \to \beta, \quad \text{then} \quad FS_{c'} = FS_\gamma \to \infty \qquad (3)$$

The contrasting shapes of the failure mechanisms obtained by strength reduction and gravity increase from FE analysis for the case of $\phi' = 30°$ are clearly shown in Figs. 3a and b. The strength reduction case gives a circular shape while the gravity increase case is essentially translational.

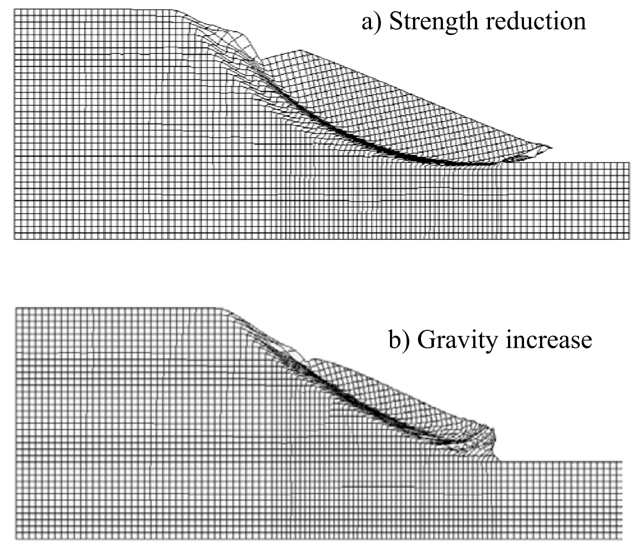

Figure 3. Finite element failure mechanisms for the test slope with $\phi' = 30°$ with a) Strength reduction and b) Gravity increase.

It may also be noted that when comparing the results from strength reduction and gravity increase, $FS_\gamma \geq FS_{c',\tan\phi'}$ as shown in Fig. 4 based on FE results. The factors of safety are equal only for the special case of $\phi_u = 0$, but the difference between them increases rapidly as the frictional component of strength increases, reaching a ratio of $FS_\gamma / FS_{c',\tan\phi'} \approx 8$ when $\phi' = 30°$.

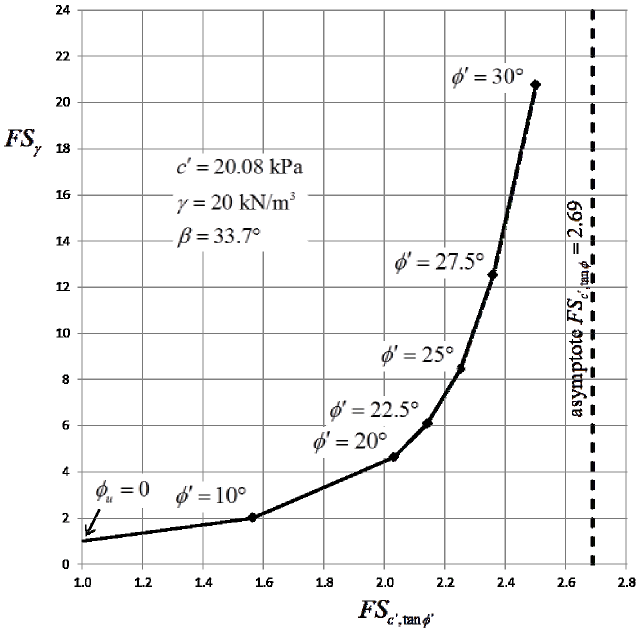

Figure 4. Factors of safety by strength reduction vs. gravity increase for the test slope for a range of ϕ' values from FE analysis

3 A BEARING CAPACITY EXAMPLE

A bearing capacity analysis is now considered as shown in Fig. 5, involving a rough strip footing of width B supported on a $c' - \phi'$ soil of unit weight γ with a surface surcharge of q.

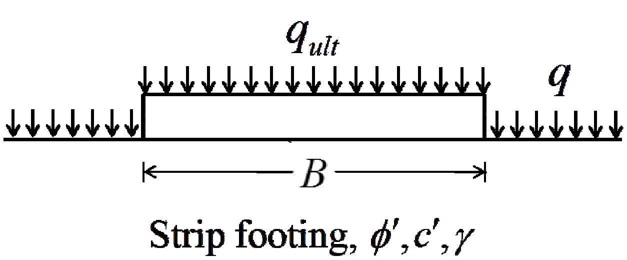

Figure 5. Bearing capacity of a strip footing

The ultimate bearing capacity q_{ult} of the footing can be given by Terzaghi's bearing capacity equation

$$q_{ult} = c'N_c + qN_q + \frac{\gamma B}{2} N_\gamma \tag{4}$$

where N_c, N_q and N_γ are the bearing capacity factors.

The factor of safety in bearing capacity analysis is typically based on loads, and given by

$$FS_q = \frac{q_{ult}}{q_{all}} \tag{5}$$

where q_{all} is the allowable design pressure that the footing can safely support. A target factor of safety against bearing failure of about $FS_q \approx 3$ would typically be required in design.

An alternative approach in now proposed, using Method 1 in Table 1, where the factor of safety against bearing failure is instead based on strength reduction. In this case, the strength reduction factor of safety is defined as $FS_{c',\tan\phi'}$, namely the factor by which c' and $\tan\phi'$ must be reduced in Eq.(4) to cause failure, i.e. $q_{ult} = q_{all}$.

In order to compare FS_q and $FS_{c',\tan\phi'}$ analytically, the following bearing capacity factors will be used

$$N_c = \left(K_p e^{\pi \tan\phi'} - 1\right)\cot\phi' \tag{6}$$

$$N_q = K_p e^{\pi \tan\phi'} \tag{7}$$

$$N_\gamma = 1.5\left(K_p e^{\pi \tan\phi'} - 1\right)\tan\phi' \tag{8}$$

Eqns.(6) and (7) for N_c and N_q are mathematically rigorous, and due to Prandtl (1921). The N_γ term has proved more challenging and has no direct analytical solution. Martin (2005) has produced exact numerical solutions using the method of characteristics (assuming an associated flow rule), and these results are closely approximated by Eq.(8) from Brinch Hansen (1970). Martin (2005) also shows that Eq.(4) with the bearing capacity factors shown in Eqs.(6-8) is guaranteed to give a result less than or equal to the exact bearing capacity due to the inherent conservatism of Terzaghi's superposition-based approach.

Eqns.(6-8) all include the passive earth pressure coefficient K_p which can be commonly expressed in different several ways, e.g.

$$K_p = \tan^2\left(45° + \frac{\phi'}{2}\right) = \frac{1 + \sin\phi'}{1 - \sin\phi'} = \left(\frac{\cos\phi'}{1 - \sin\phi'}\right)^2 \tag{9}$$

In this work, a less familiar trigonometric identity for K_p is used as given by Eq.(10). This version, which expresses K_p purely in terms of

$\tan\phi'$, has been discussed previously by Griffiths *et al*. (2002), and is convenient for strength reduction calculations based on Methods 1 and 4 in Table 1.

$$K_p = \left(\tan\phi' + \sqrt{1+\tan^2\phi'}\right)^2 \qquad (10)$$

In what follows, and with reference to Eq.(4), each of the three bearing capacity factors given in Eqs. (6-8) will be considered separately, leading to analytical expressions relating the conventional factor of safety based on load increase $\left(FS_q\right)$, to that based on strength reduction $\left(FS_{c',\tan\phi'}\right)$. At the end, an example will be presented with all terms included.

3.1 The N_c term (Eq.6)

In this case it is assumed that $q=0$ and $\gamma=0$, hence

$$\begin{aligned} q_{ult} &= c'N_c \\ &= c'\left(K_p e^{\pi\tan\phi'} - 1\right)\cot\phi' \\ &= c'\left[\left(\tan\phi' + \sqrt{1+\tan^2\phi'}\right)^2 e^{\pi\tan\phi'} - 1\right]\frac{1}{\tan\phi'} \end{aligned} \qquad (11)$$

The strength reduction factor of safety $FS_{c',\tan\phi'}$ will bring the footing to failure, hence

$$q_{all} = \frac{c'}{FS_{c'\tan\phi'}}\left[\left(\frac{\tan\phi'}{FS_{c'\tan\phi'}} + \sqrt{1+\left(\frac{\tan\phi'}{FS_{c'\tan\phi'}}\right)^2}\right)^2 e^{\pi\frac{\tan\phi'}{FS_{c'\tan\phi'}}} - 1\right]\frac{FS_{c'\tan\phi'}}{\tan\phi'} \qquad (12)$$

which after substitution into Eq.(5) gives the function

$$FS_q = \frac{\left(\tan\phi' + \sqrt{1+\tan^2\phi'}\right)^2 e^{\pi\tan\phi'} - 1}{\left(\dfrac{\tan\phi'}{FS_{c'\tan\phi'}} + \sqrt{1+\left(\dfrac{\tan\phi'}{FS_{c'\tan\phi'}}\right)^2}\right)^2 e^{\pi\frac{\tan\phi'}{FS_{c'\tan\phi'}}} - 1} \qquad (13)$$

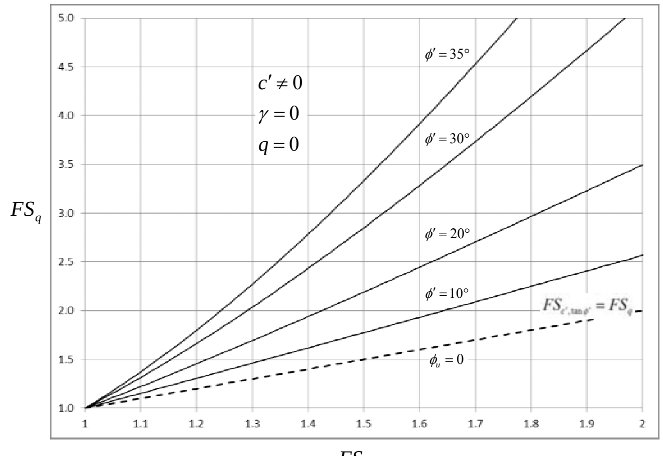

Figure 6. $FS_{c',\tan\phi'}$ vs. FS_q for the N_c term (Eq.13)

3.2 The N_q term (Eq.7)

In this case it is assumed that $c'=0$ and $\gamma=0$, hence

$$\begin{aligned} q_{ult} &= qN_q \\ &= qK_p e^{\pi\tan\phi'} \\ &= q\left(\tan\phi' + \sqrt{1+\tan^2\phi'}\right)^2 e^{\pi\tan\phi'} \end{aligned} \qquad (14)$$

The strength reduction factor of safety $FS_{c',\tan\phi'}$ will bring the footing to failure, hence

$$q_{all} = q\left(\frac{\tan\phi'}{FS_{c',\tan\phi'}} + \sqrt{1+\left(\frac{\tan\phi'}{FS_{c',\tan\phi'}}\right)^2}\right)^2 e^{\pi\frac{\tan\phi'}{FS_{c'\tan\phi'}}} \qquad (15)$$

which after substitution into Eq.(5) gives the function

$$FS_q = \frac{\left(\tan\phi' + \sqrt{1+\tan^2\phi'}\right)^2 e^{\pi\tan\phi'}}{\left(\dfrac{\tan\phi'}{FS_{c',\tan\phi'}} + \sqrt{1+\left(\dfrac{\tan\phi'}{FS_{c',\tan\phi'}}\right)^2}\right)^2 e^{\pi\frac{\tan\phi'}{FS_{c'\tan\phi'}}}} \qquad (16)$$

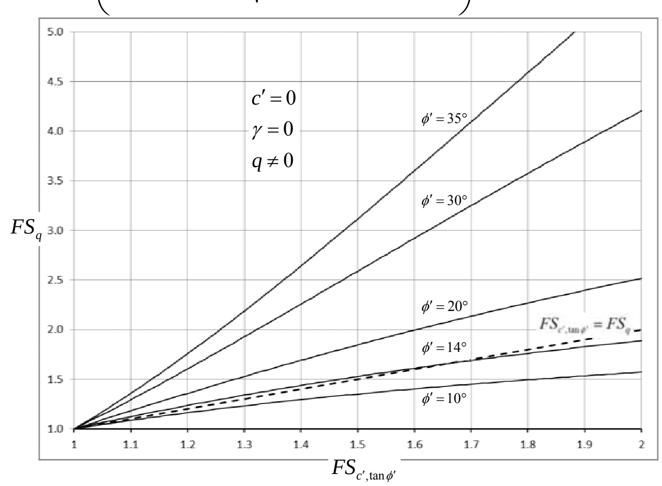

Figure 7. $FS_{c',\tan\phi'}$ vs. FS_q for the N_q term (Eq.16)

3.3 The N_γ term (Eq.8)

In this case it is assumed that $c' = 0$ and $q = 0$, hence

$$q_{ult} = \frac{\gamma B}{2} N_\gamma$$
$$= \frac{\gamma B}{2}\left[1.5\left(K_p e^{\pi \tan\phi'} - 1\right)\tan\phi'\right]$$
$$= \frac{3\gamma B}{4}\left[\left(\tan\phi' + \sqrt{1+\tan^2\phi'}\right)^2 e^{\pi\tan\phi'} - 1\right]\tan\phi'$$

(17)

The strength reduction factor of safety $FS_{c',\tan\phi'}$ will bring the footing to failure, hence

$$q_{all} = \frac{3\gamma B}{4}\left[\left(\frac{\tan\phi'}{FS_{c'\tan\phi'}} + \sqrt{1+\left(\frac{\tan\phi'}{FS_{c'\tan\phi'}}\right)^2}\right)^2 e^{\pi\frac{\tan\phi'}{FS_{c'\tan\phi'}}} - 1\right]\frac{\tan\phi'}{FS_{c'\tan\phi'}}$$

(18)

which after substitution into Eq.(5) gives the function

$$FS_q = \frac{\left[\left(\tan\phi' + \sqrt{1+\tan^2\phi'}\right)^2 e^{\pi\tan\phi'} 11\right] FS_{c'\tan\phi'}}{\left(\frac{\tan\phi'}{FS_{c'\tan\phi'}} + \sqrt{1+\left(\frac{\tan\phi'}{FS_{c'\tan\phi'}}\right)^2}\right)^2 e^{\pi\frac{\tan\phi'}{FS_{c'\tan\phi'}}} - 1}$$

(19)

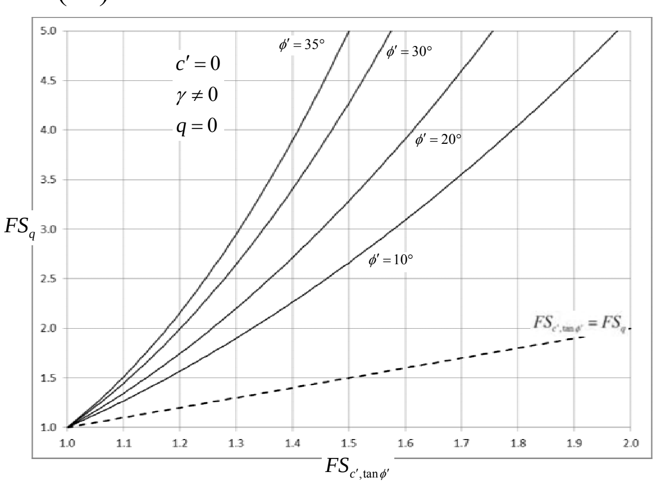

Figure 8. $FS_{c',\tan\phi'}$ vs. FS_q for the N_γ term (Eq.19)

Eqs.(13, 16 and 19) are shown plotted in Figs.6-8 respectively for a range of $\phi'-$values. It may be noted from Fig.6 and Fig.8 for N_c and N_γ respectively, that $FS_q > FS_{c',\tan\phi'}$ for $\phi' > 0$. On the other hand, analysis of Eq.(16) for N_q (see Appendix) indicates that $FS_q < FS_{c',\tan\phi'}$ for all $\phi' < 11.08°$, and can occur for $\phi' > 11.08°$ if $FS_{c',\tan\phi'}$ exceeds some critical value. This effect is demonstrated in Fig. 7, where the line for $\phi' = 14°$ crosses the dashed line $\left(FS_{c',\tan\phi'} = FS_q\right)$ at about $FS_{c',\tan\phi'} = 1.65$. Table 2 summarizes the transition point at which $FS_q < FS_{c',\tan\phi'}$ for a range of ϕ'.

In summary, $FS_q < FS_{c',\tan\phi'}$ is only observed in the N_q term within a reasonable range of $FS_{c',\tan\phi'}$ values, when ϕ' is small (e.g. $\phi' < 15°$). Higher friction angles can also lead to $FS_q < FS_{c',\tan\phi'}$, but only when $FS_{c',\tan\phi'}$ is unrealistically high (e.g. for $\phi' = 25°$, $FS_{c',\tan\phi'} > 7.86$)

Fig. 9 shows plots of $FS_{c',\tan\phi'}$ vs. FS_q for each of the three bearing capacity factors corresponding to $\phi' = 30°$. The N_γ and N_q lines display, respectively, the greatest and least differences between the factors of safety. It may be noted that in a bearing capacity analysis of a soil with $\phi' = 30°$, a strength reduction factor of safety of $FS_{c',\tan\phi'} = 1.5$ corresponds approximately to a load increase factor of safety in the range $2.5 < FS_q < 3.5$.

Table 2. Ranges of $FS_{c',\tan\phi'}$ for which $FS_q < FS_{c',\tan\phi'}$ from the N_q Eq.(16)

ϕ' (degrees)	**$FS_{c',\tan\phi'}$**
<11.08	All values
12	>1.18
14	>1.65
20	>4.02
25	>7.86

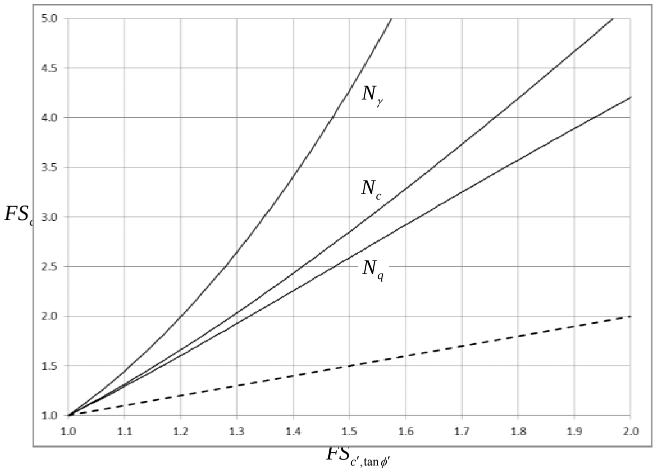

Figure 9. $FS_{c',\tan\phi'}$ vs. FS_q for ϕ'=30°

3.4 Example problem including all terms (Eq.4)

In order to include all the terms of the bearing capacity equation, an example from Salencon and Matar (1982) is now considered where, with reference to Fig. 5, $\phi' = 30°$, $c' = 16$ kPa, $\gamma = 18$ kN/m^3, $B = 4$m and $q = 18$ kPa.

Based on Eqs.(4 and 6-8), the bearing capacity factors are $N_c = 30.1$, $N_q = 18.4$ and $N_\gamma = 15.1$, and the bearing capacity is given by

$$\begin{aligned} q_{ult} &= 16 \times 30.1 + 18 \times 18.4 + \frac{18 \times 4}{2} \times 15.1 \\ &= 1356 \text{ kPa} \end{aligned} \tag{20}$$

3.4.1 Load increase

If a typical load-based factor of safety against bearing failure of $FS_q = 3$ is used, this leads to an allowable bearing pressure of

$$q_{all} = \frac{1356}{3} = 452 \text{ kPa} \tag{21}$$

3.4.2 Strength reduction

The value of $FS_{c',\tan\phi'}$ that would be needed to reduce the bearing capacity given in Eq.(20) from $q_{ult} = 1356$ kPa to $q_{ult} = q_{all} = 452$ kPa is given by the nonlinear equation

$$\begin{aligned} 452 = 16 &\left[\left(\frac{\tan 30°}{FS_{c'\tan\phi'}} + \sqrt{1 + \left(\frac{\tan 30°}{FS_{c'\tan\phi'}} \right)^2} \right)^2 e^{\pi \frac{\tan 30°}{FS_{c'\tan\phi'}}} - 1 \right] \frac{1}{\tan 30°} + \\ 18 &\left(\frac{\tan 30°}{FS_{c',\tan\phi'}} + \sqrt{1 + \left(\frac{\tan 30°}{FS_{c',\tan\phi'}} \right)^2} \right)^2 e^{\pi \frac{\tan 30°}{FS_{c'\tan\phi'}}} + \\ \frac{3 \times 18 \times 4}{4} &\left[\left(\frac{\tan 30°}{FS_{c'\tan\phi'}} + \sqrt{1 + \left(\frac{\tan 30°}{FS_{c'\tan\phi'}} \right)^2} \right)^2 e^{\pi \frac{\tan 30°}{FS_{c'\tan\phi'}}} - 1 \right] \frac{\tan 30°}{FS_{c'\tan\phi'}} \end{aligned}$$

(22)

which after solution gives

$$FS_{c',\tan\phi'} = 1.60 \tag{23}$$

This example gives $FS_q \approx 2FS_{c',\tan\phi'}$, which is to be expected for bearing capacity on a soil with. $\phi' = 30°$ It is clear from the examples considered, that geotechnical design is more sensitive to strength reduction than load increase, i.e. an allowable bearing capacity based on $FS_{c',\tan\phi'} = 1.5$ (say) will be considerably lower (more conservative) than one based on $FS_q = 1.5$.

4 CONCLUDING REMARKS

The paper has examined differences between the factor of safety defined by strength reduction $\left(FS_{c',\tan\phi'}\right)$ and that defined by gravity increase $\left(FS_\gamma\right)$ in slope stability or load increase $\left(FS_q\right)$ in bearing capacity.

Analysis of a slope stability example showed that in all cases considered in which $\phi' > 0$, $FS_\gamma > FS_{c',\tan\phi'}$. In part, this is due to the appearance of unit weight in both the numerator and denominator of the factor of safety equation, but more significantly, for slopes inclined at β to the horizontal, it was shown that as $FS_\gamma \to \infty$, $\phi' \to \beta$, i.e. a slope in which $\phi' > \beta$ can never be brought to failure by gravity increase.

Analysis of the bearing capacity equation led to the development of closed form expressions directly relating FS_q to $FS_{c',\tan\phi'}$ for each of the three bearing capacity factors. As before, it was observed that when $\phi' > 0$, $FS_q > FS_{c',\tan\phi'}$ for nearly all cases, except for low values of ϕ' in the

N_q-term. A bearing capacity example involving a soil with $\phi'=30°$ and all terms of the bearing capacity equation present, led to $FS_q \approx 2FS_{c',\tan\phi'}$. The difference would be greater with higher friction angles, however the result suggests that for a typical friction angle, a target factor of safety based on strength reduction of around $FS_{c',\tan\phi'} \approx 1.5$ is not significantly different to a target factor of safety based on load increase of $FS_q \approx 3.0$. In general, the results further demonstrate the greater sensitivity of geotechnical design outcomes to strength factoring over load factoring.

5 NOTATION

c'	Effective cohesion
c_u	Undrained shear strength
B	Footing width
D	Depth ratio
FS	Conventional factor of safety
$FS_{c',\tan f'}$	FS-c' and $\tan\phi'$ strength reduction
FS_γ	FS-unit weight γ increase
FS_q	FS-footing load q increase
$FS_{c'}$	FS-c' strength reduction
$FS_{\tan f'}$	FS-tanϕ' strength reduction
FS_{k_h}	FS-acceleration factor k_h increase
FS_{r_u}	FS-pp coefft r_u increase
H	Slope height
K_p	Passive earth pressure coefficient
N_c, N_q, N_γ	Bearing capacity factors
q	Surcharge
q_{all}	Allowable bearing pressure
q_{ult}	Bearing capacity
β	Slope angle
γ	Unit weight
γ_{sat}	Saturated unit weight
ϕ'	Effective friction angle
ϕ_u	Total undrained friction angle (= 0)

6 APPENDIX

From Figure 3 it is clear that $FS_q \le FS_{c',\tan\phi'}$ in certain ranges. For example, when $\phi'=14°$ cross-over occurs at about $FS_{c',\tan\phi'}=1.65$ while when $\phi'=10°$ there appears to be no cross-over.

To examine this limiting condition more closely (Martin 2015), let $FS_q = FS_{c',\tan\phi'} = F$ and solve for $\tan\phi'$. Let $\tan\phi' = t$ for algebraic simplicity, and write Eqn.(16) as

$$V = \frac{\left(t+\sqrt{1+t^2}\right)^2 e^{\pi t}}{\left(\frac{t}{F}+\sqrt{1+\left(\frac{t}{F}\right)^2}\right)^2 e^{\frac{\pi t}{F}}} - F = 0$$

Clearly $V=0$ when $F=1$ for all t, so examine the value of t when $V=0$ and $F\approx 1$.

Let $F=1+\varepsilon$ where ε is small. Then

$$\frac{t}{F}=\frac{t}{1+\varepsilon}\approx t(1-\varepsilon)=t-t\varepsilon$$

$$1+\left(\frac{t}{F}\right)^2 = 1+\frac{t^2}{(1+\varepsilon)^2}\approx 1+t^2(1-2\varepsilon)$$

$$\begin{aligned}\sqrt{1+\left(\frac{t}{F}\right)^2} &= \sqrt{1+t^2-2t^2\varepsilon} = \sqrt{1+t^2}\left(1-\frac{2t^2\varepsilon}{1+t^2}\right)^{1/2}\\ &\approx \sqrt{1+t^2}\left(1-\frac{t^2\varepsilon}{1+t^2}\right)\\ &= \sqrt{1+t^2}-\frac{t^2\varepsilon}{\sqrt{1+t^2}}\end{aligned}$$

$$\begin{aligned}\left(\frac{t}{F}+\sqrt{1+\left(\frac{t}{F}\right)^2}\right)^2 &= \left(t-t\varepsilon+\sqrt{1+t^2}-\frac{t^2\varepsilon}{\sqrt{1+t^2}}\right)^2\\ &= \left(T-t\varepsilon-\frac{t^2\varepsilon}{\sqrt{1+t^2}}\right)^2\\ &= \left(T-\frac{Tt\varepsilon}{\sqrt{1+t^2}}\right)^2\\ &= T^2\left(1-\frac{t\varepsilon}{\sqrt{1+t^2}}\right)^2\end{aligned}$$

where $T=t+\sqrt{1+t^2}$.

Also $e^{\frac{\pi t}{F}} \approx e^{\pi t(1-\varepsilon)} = e^{\pi t} e^{-\pi t\varepsilon}$ hence

$$V \approx \frac{T^2 e^{\pi t}}{T^2\left(1-\varepsilon t/\sqrt{1+t^2}\right)^2 e^{\pi t} e^{-\pi t\varepsilon}} - (1+\varepsilon)$$

$$\approx e^{\pi t\varepsilon}\left(1+2\varepsilon t/\sqrt{1+t^2}\right) - 1 - \varepsilon$$

$$\approx 1 + 2\varepsilon t/\sqrt{1+t^2} + \pi t\varepsilon - 1 - \varepsilon$$

$$= \varepsilon\left(\frac{2t}{\sqrt{1+t^2}} + \pi t - 1\right)$$

For $V = 0$ when ε is small, it must be the case that

$$\left(\frac{2t}{\sqrt{1+t^2}} + \pi t - 1\right) = 0$$

Solving for t gives $t = \tan\phi' = 0.1959$ and $\phi' = 11.08°$ hence $FS_q < FS_{c',\tan\phi'}$ in the N_q term provided $\phi' < 11.08°$

7 REFERENCES

Atkinson, J. (2007). *The Mechanics of Soil and Foundations* (2nd Edition). Taylor & Francis, Abingdon, Oxford

Brinch Hansen, J. (1970). A revised and extended formula for bearing capacity. *The Danish Geotechnical Institute*, Bulletin No. 28.

Dawson, E.M., Motamed, F., Nesarajah, S. and Roth, W. Geotechnical stability analysis by strength reduction. In *Slope Stabiilty 2000*, (Eds. D.V. Griffiths *et al.*), GSP 101, pp.99-113, Pub. ASCE, Reston, USA.

Duncan, J.M. and Wright, S.G. (2005) *Soil Strength and Slope Stability*. John Wiley & Sons Inc., Hoboken, New Jersey

Griffiths, D.V., Fenton, G.A. and Tveten, D.E. (2002). Probabilistic geotechnical analysis: How difficult does it need to be? Proc. Int. Conf. on, *Probabilistics in Geotechnics: Technical and Economic Risk Estimation*, (eds. R. Poettler *et al.*), Pub. VGE, Essen, Germany, pp.3-20.

Martin, C.M. (2005). Exact bearing capacity calculations using the method of characteristics. Proc. *11th Int. Conf. of IACMAG*, Turin, Vol. 4, pp 441-450.

Martin, P.A. (2015). *Private communication*. Department of Applied Mathematics and Statistics, Colorado School of Mines.

Pantelidis, L. and Griffiths, D.V. (2014). Integrating Eurocode 7 (load and resistance factor design) with different factoring strategies in slope design. *Can Geotech J*, vol.51, No.2, pp.208-216.

Prandtl, L. (1921). Über die Eindringungsfestigkeit (Härte) plastischer Baustoffe und die Festigkeit von Schneiden. Zeit. angew. *Math. Mech*., Vol. 1, pp 15-20.

Salençon J. and Matar M. (1982). Bearing capacity of circular shallow foundations. In *Foundation Engineering* (ed. G. Pilot), 159-168, Presses de l'ENPC, Paris (France).

Salgado, R. (2008). *The Engineering of Foundations*. McGraw-Hill, New York.

Terzaghi, K. , Peck, R.B. and Mesri, G. (1996). *Soil Mechanics in Engineering Practice*. 3rd edition, John Wiley & Sons, Chichester.

ACKNOWLEDGEMENT

The author wishes to thank Dr. A. Wayllace, Dr. M. Cerrolaza and Mr. J.C Viviescas for their help with Spanish translation.