

RESEARCH ARTICLE

A symbolic algorithm for computing recursion operators of nonlinear PDEs

D.E. Baldwin^{†*} and W. Hereman[‡]

[†]*Department of Applied Mathematics, University of Colorado, UCB 526, Boulder, CO 80309-0526, USA;* [‡]*Department of Mathematical and Computer Sciences, Colorado School of Mines, Golden, CO 80401-1887, USA*

(Received 31 March 2009)

A recursion operator is an integro-differential operator which maps a generalized symmetry of a nonlinear PDE to a new symmetry. Therefore, the existence of a recursion operator guarantees that the PDE has infinitely many higher-order symmetries, which is a key feature of complete integrability. Completely integrable nonlinear PDEs have a bi-Hamiltonian structure and a Lax pair; they can also be solved with the inverse scattering transform and admit soliton solutions of any order.

A straightforward method for the symbolic computation of polynomial recursion operators of nonlinear PDEs in $(1 + 1)$ dimensions is presented. Based on conserved densities and generalized symmetries, a candidate recursion operator is built from a linear combination of scaling invariant terms with undetermined coefficients. The candidate recursion operator is substituted into its defining equation and the resulting linear system for the undetermined coefficients is solved.

The method is algorithmic and is implemented in *Mathematica*. The resulting symbolic package `PDERecursionOperator.m` can be used to test the complete integrability of polynomial PDEs that can be written as nonlinear evolution equations. With `PDERecursionOperator.m`, recursion operators were obtained for several well-known nonlinear PDEs from mathematical physics and soliton theory.

Keywords: Recursion operator, generalized symmetries, complete integrability, nonlinear PDEs, symbolic software.

AMS Subject Classification: Primary: 37K10, 35Q51, 68W30; Secondary: 37K05, 35Q53, 35Q58,

1. Introduction

Completely integrable nonlinear partial differential equations (PDEs) have a rich mathematical structure and many hidden properties. For example, these PDEs have infinitely many conservation laws and generalized symmetries of increasing order. They have the Painlevé property [3], bi-Hamiltonian (sometimes tri-Hamiltonian) structures [2], Lax pairs [1], Bäcklund and Darboux transformations [44, 54], etc. Completely integrable PDEs can be solved with the Inverse Scattering Transform (IST) [1, 2, 12]. Application of the IST or Hirota's direct method [39, 40] allows one to construct explicit soliton solutions of any order. While there are numerous definitions of complete integrability, Fokas [17] defines an equation as completely integrable if and only if it possesses infinitely many generalized symmetries. A *recursion operator* (also called a formal symmetry or a master symmetry) is a linear integro-differential operator which links such symmetries. The recursion operator

*Corresponding author. Email: recursionOperators@douglasbaldwin.com

is thus a key tool for proving the existence of an infinite hierarchy of generalized symmetries [54] and for computing them sequentially.

The recursion operator story [54, 59] starts with the Korteweg-de Vries (KdV) equation,

$$u_t + 6uu_x + u_{3x} = 0, \quad (1)$$

which is undeniably the most famous completely integrable PDE. The first few generalized symmetries (of infinitely many) for the KdV equation are

$$\begin{aligned} G^{(1)} &= u_x, & G^{(2)} &= 6uu_x + u_{3x}, \\ G^{(3)} &= 30u^2u_x + 20u_xu_{2x} + 10uu_{3x} + u_{5x}. \end{aligned} \quad (2)$$

Note that generalized symmetries depend on the dependent variables of the system as well as the x -derivatives of the dependent variables (in contrast to so-called Lie-point or *geometric* symmetries which only depend on the independent and dependent variables of the system).

The KdV equation is a member of a hierarchy of integrable PDEs, which are higher-order symmetries of the KdV itself. For example, the Lax equation [46], which is the fifth-order member in the hierarchy, is $u_t + G^{(3)} = 0$.

Based on the recursion formula [55] due to Lenard, in 1977 Olver [53] derived an explicit recursion operator for the KdV equation, namely

$$\mathcal{R} = D_x^2 + 4uI + 2u_xD_x^{-1}. \quad (3)$$

In (3), D_x denotes the total derivative with respect to x , D_x^{-1} is its left inverse, and I is the identity operator. Total derivatives act on differential functions [54], i.e. differentiable functions of independent variables, dependent variables, and their derivatives up to an arbitrary but fixed order.

The recursion operator (3) allows one to generate an infinite sequence of local generalized symmetries of the KdV equation. Indeed, starting from “seed” or “root” $G^{(1)}$, repeated application of the recursion operator (3),

$$G^{(j+1)} = \mathcal{R}G^{(j)}, \quad j = 1, 2, \dots, \quad (4)$$

produces the symmetries in (2) and infinitely many more.

Analysis of the form of recursion operators like (3) reveals that they can be split into a (local) differential part, \mathcal{R}_0 , and a (non-local) integral part \mathcal{R}_1 . The differential operator \mathcal{R}_0 involves D_x, D_x^2 , etc., acting on monomials in the dependent variables. Barring strange cases [42], the integral operator \mathcal{R}_1 only involves D_x^{-1} and can be written as the outer product of generalized symmetries and cosymmetries (or conserved covariants) [11, 59]. Furthermore, if \mathcal{R} is a recursion operator, then the Lie derivative [54, 59, 60] of \mathcal{R} with respect to the evolution equation is zero. The latter provides an explicit defining equation for the recursion operator.

For more information on the history of completely integrable systems and recursion operators, see [1, 11, 13, 14, 17, 21–23, 43, 45, 50, 52, 54, 56–59]. Based on studies of formal symmetries and recursion operators, researchers have compiled lists of integrable PDEs [50, 51, 59, 60].

While the computation of the Lie derivative of \mathcal{R} is fairly straightforward, it is computationally intensive and prone to error when done by hand. For example, the computation of the recursion operators of the Kaup-Kupershmidt equation or the Hirota-Satsuma system (see Section 4) may take weeks to complete by

hand and might fill dozens of pages. Using a computer algebra system to carry out the lengthy computations is recommended, yet, it is not without challenge either; systems like *Mathematica* and *Maple* are designed to primarily work with commutative algebraic structures and the computation and application of recursion operators requires non-commutative operations.

There is a variety of methods [48] to construct recursion operators (or master symmetries). As shown in [17, 18, 20, 54], one first finds a bi-Hamiltonian structure (with Hamiltonians Θ_1 and Θ_2) for the given evolution equation and then constructs the recursion operator as $\mathcal{R} = \Theta_1\Theta_2^{-1}$, provided Θ_2 is invertible; for the KdV equation, $\Theta_1 = D_x^3 + 4uD_x + 2u_xI$ and $\Theta_2 = D_x$ form a Hamiltonian pair [47] and $\mathcal{R} = \Theta_1\Theta_2^{-1}$ yields (3). The Hamiltonians are cosymplectic operators, their inverses are symplectic operators [59]. A complicated example of a recursion operator (obtained by composing cosymplectic a symplectic operators of a vector derivative Schrödinger equation) can be found in [61].

A recent approach [49] uses the symbolic method of Gelfand and Dickey [24], and applies to non-local and non-evolutionary equations such as the Benjamin-Ono and Camassa-Holm equations.

At the cost of generality, we advocate a direct approach which applies to polynomial evolution equations. In the spirit of work by Bilge [11], we use the scaling invariance of the given PDE to build a polynomial candidate recursion operator as a linear combination of scaling homogeneous terms with constant undetermined coefficients. The defining equation for the recursion operator is then used to compute the undetermined coefficients.

The goals of our paper are threefold. We present (i) an algorithmic method in a language that appeals to specialists and non-specialists alike, (ii) a symbolic package in *Mathematica* to carry out the tedious computations, (iii) a set of carefully selected examples to illustrate the algorithm and the code.

The theory on which our algorithm is based has been covered extensively in the literature [11, 54, 57–60]. Our paper focuses on *how* things work rather than on *why* they work.

The package `PDERecursionOperator.m` [9] is part of our symbolic software collection for the integrability testing of nonlinear PDEs, including algorithms and *Mathematica* codes for the Painlevé test [6–8] and the computation of conservation laws [4, 5, 15, 16, 29, 32–34, 36], generalized symmetries [26, 30] and recursion operators [10, 26]. As a matter of fact, our package `PDERecursionOperator.m` builds on the code `InvariantsSymmetries.m` [27] for the computation of conserved densities and generalized symmetries for nonlinear PDEs. The code `PDERecursionOperator.m` automatically computes *polynomial* recursion operators for polynomial systems of nonlinear PDEs in $(1+1)$ dimensions, i.e. PDEs in one space variable x and time t . At present, the coefficients in the PDEs cannot *explicitly* depend on x and t . Our code can find recursion operators with coefficients that *explicitly* depend on powers of x and t as long as the maximal degree of these variables is specified. For example, if the maximal degree is set to 1, then the coefficients will be at most linear in both x and t . An example of a recursion operator that explicitly depends on x and t is given in Section 7.2. For extra versatility, the code can be used to test polynomial and rational recursion operators found in the literature, computed by hand, or with other software packages.

Drawing on the analogies with the PDE case, we also developed methods, algorithms, and software to compute conservation laws [31, 33, 36, 38] and generalized symmetries [30] of nonlinear differential-difference equations (DDEs). Although the algorithm is well-established [37], a *Mathematica* package that automatically computes recursion operators of nonlinear DDEs is still under development.

The paper is organized as follows. In Section 2 we briefly discuss our method for computing scaling invariance, conserved densities, and generalized symmetries (which are essential pieces for the computation of recursion operators). Our method for computing and testing recursion operators is discussed in Section 3. In Section 4, we illustrate the subtleties of the method using the KdV equation, the Kaup-Kupershmidt equation, and the Hirota-Satsuma system of coupled KdV (cKdV) equations. The details of computing and testing recursion operators are discussed in Section 5. Section 6 compares our software package to other software packages for computing recursion operators. In Section 7 we give additional examples to demonstrate the capabilities of our software. A discussion of the results and future generalizations are given in Section 8. The use of the software package `PDERecursionOperator.m` [9] is shown in Appendix A.

2. Scaling Invariance, Conservation Laws, and Generalized Symmetries

Consider a polynomial system of evolution equations in $(1 + 1)$ dimensions,

$$\mathbf{u}_t(x, t) = \mathbf{F}(\mathbf{u}(x, t), \mathbf{u}_x(x, t), \mathbf{u}_{2x}(x, t), \dots, \mathbf{u}_{mx}(x, t)), \quad (5)$$

where \mathbf{F} has M components F_1, \dots, F_M , $\mathbf{u}(x, t)$ has M components $u_1(x, t), \dots, u_M(x, t)$ and $\mathbf{u}_{ix} = \partial^i \mathbf{u} / \partial x^i$. Henceforth we write $\mathbf{F}(\mathbf{u})$ although \mathbf{F} (typically) depends on \mathbf{u} and its x -derivatives up to some fixed order m . In the examples, we denote the components of $\mathbf{u}(x, t)$ as u, v, \dots, w . If present, any parameters in the PDEs are assumed to be nonzero and are denoted by Greek letters.

Our algorithms are based on scaling (or dilation) invariance, a feature common to many nonlinear PDEs. If (5) is scaling invariant, then quantities like conserved densities, fluxes, generalized symmetries, and recursion operators are also scaling invariant [54]. Indeed, since their defining equation must be satisfied on solutions of the PDE, these quantities “inherit” the scaling symmetry of the original PDE. Thus, scaling invariance provides an elegant way to construct the form of candidate densities, generalized symmetries, and recursion operators. It suffices to make linear combinations (with constant undetermined coefficients) of scaling-homogeneous terms. Inserting the candidates into their defining equations then leads to a linear system for the undetermined coefficients.

2.1 Scaling Invariance and the Computation of Dilation Symmetries

Many completely integrable nonlinear PDEs are scaling invariant. PDEs that are not scaling invariant can be made so by extending the set of dependent variables with parameters that scale appropriately, see [28, 30] for details.

For example, the KdV equation (1) is invariant under the scaling symmetry

$$(t, x, u) \rightarrow (\lambda^{-3}t, \lambda^{-1}x, \lambda^2u), \quad (6)$$

where λ is an arbitrary parameter. Indeed, upon scaling, a common factor λ^5 can be pulled out. Assigning *weights* (denoted by W) to the variables based on the exponents in λ and setting $W(D_x) = 1$ (or equivalently, $W(x) = W(D_x^{-1}) = -1$) gives $W(u) = 2$ and $W(t) = -3$ (or $W(D_t) = 3$).

The *rank* of a monomial is its total weight; in the KdV equation, all three terms are rank 5. We say that an equation is *uniform in rank* if every term in the equation

has the same rank. Conversely, requiring uniformity in rank in (1) yields

$$W(u) + W(D_t) = 2W(u) + W(D_x) = W(u) + 3W(D_x). \quad (7)$$

Hence, after setting $W(D_x) = 1$, one obtains $W(u) = 2W(D_x) = 2$ and $W(D_t) = 3W(D_x) = 3$. So, scaling symmetries can be computed with linear algebra.

2.2 Computation of Conservation Laws

The first two conservation laws for the KdV equation are

$$D_t(u) + D_x(3u^2 + u_{2x}) = 0, \quad (8)$$

$$D_t(u^2) + D_x(4u^3 - u_x^2 + 2uu_{xx}) = 0, \quad (9)$$

were classically known and correspond to the conservation of mass and momentum (for water waves). Whitham found the third conservation law,

$$D_t(u^3 - \frac{1}{2}u_x^2) + D_x(\frac{9}{2}u^4 - 6uu_x^2 + 3u^2u_{2x} + \frac{1}{2}u_{2x}^2 - u_xu_{3x}) = 0, \quad (10)$$

which corresponds to Boussinesq's moment of instability. For (5), each conservation law has the form

$$D_t\rho(\mathbf{u}(x, t), \mathbf{u}_x(x, t), \dots) + D_xJ(\mathbf{u}(x, t), \mathbf{u}_x(x, t), \dots) = 0, \quad (11)$$

where ρ is the conserved density and J is the associated flux.

Algorithms for computing conserved densities and generalized symmetries are described in [26–28, 30, 33–35]. Our code, `PDERecursionOperator.m` [9], uses these algorithms to compute the densities and generalized symmetries needed to construct the non-local part of the operator. For the benefit of the reader, we present an abbreviated version of these algorithms.

The KdV equation (1) has conserved densities for any even rank. To find the conserved density ρ of rank $R = 6$, we consider all the terms of the form

$$D_x^{R-W(u)i}u^i(x, t), \quad 1 \leq i \leq R/W(u), \quad (12)$$

where D_x is the total derivative with respect to x . Hence, since $W(u) = 2$, we have

$$D_x^4u = u_{4x}, \quad D_x^2u^2 = 2u_x^2 + 2uu_{2x}, \quad D_x^0u^3 = u^3. \quad (13)$$

We then remove divergences and divergence equivalent terms [34], and take a linear combination (with undetermined coefficients) of the remaining terms as the candidate ρ . Terms are divergence equivalent if and only if they differ by a divergence, for instance uu_{2x} and $-u_x^2$ are divergence equivalent because $uu_{2x} - (-u_x^2) = D_x(uu_x)$. Divergences are divergence equivalent to zero, such as $u_{4x} = D_x(u_{3x})$. Thus, the candidate ρ of rank $R = 6$ is

$$\rho = c_1u^3 + c_2u_x^2. \quad (14)$$

To determine the coefficients c_i , we require that (11) holds on the solutions of (5). In other words, we first compute $D_t\rho$ and use (5) to remove \mathbf{u}_t , \mathbf{u}_{tx} , etc.

For the KdV equation,

$$D_t \rho = -(18c_1 u^3 u_x + 3c_1 u^2 u_{3x} + 12c_2 u_x^3 + 12c_2 u u_x u_{2x} + 2c_2 u_x u_{4x}), \quad (15)$$

after u_t, u_{tx} , etc. have been replaced using (1). Then, we require that $D_t \rho$ is a total derivative with respect to x . To do so, for each component u_j of \mathbf{u} , we apply the Euler operator (variational derivative) to $D_t \rho$ and set the result identically equal to zero [28]. The Euler operator for \mathbf{u} is defined as

$$\mathcal{L}_{\mathbf{u}} = \sum_{k=0}^m (-1)^k D_x^k \frac{\partial}{\partial \mathbf{u}_{kx}}, \quad (16)$$

where m is the highest order needed. In our scalar example there is only one component ($\mathbf{u} = u$). Using (15), which is of order $m = 4$,

$$\mathcal{L}_u(D_t \rho) = -18(c_1 + 2c_2)u_x u_{2x} \equiv 0. \quad (17)$$

To find the undetermined coefficients, we consider all monomials in u and its derivatives as independent, giving a linear system for c_i . For the example, $c_1 + 2c_2 = 0$, and taking $c_1 = 1$ and $c_2 = -\frac{1}{2}$ gives

$$\rho = u^3 - \frac{1}{2}u_x^2, \quad (18)$$

which is the conserved density in conservation law (10).

2.3 Computation of Generalized Symmetries

A generalized symmetry, $\mathbf{G}(\mathbf{u})$, leaves the PDE invariant under the replacement $\mathbf{u} \rightarrow \mathbf{u} + \epsilon \mathbf{G}$ within order ϵ [54]. Hence, \mathbf{G} must satisfy the linearized equation

$$D_t \mathbf{G} = \mathbf{F}'(\mathbf{u})[\mathbf{G}], \quad (19)$$

where $\mathbf{F}'(\mathbf{u})[\mathbf{G}]$ is the Fréchet derivative of \mathbf{F} in the direction of \mathbf{G} ,

$$\mathbf{F}'(\mathbf{u})[\mathbf{G}] = \frac{\partial}{\partial \epsilon} \mathbf{F}(\mathbf{u} + \epsilon \mathbf{G})|_{\epsilon=0} = \sum_{i=0}^m (D_x^i \mathbf{G}) \frac{\partial \mathbf{F}}{\partial \mathbf{u}_{ix}}, \quad (20)$$

where m is the order of \mathbf{F} . The KdV equation (1) has generalized symmetries for any odd rank. To find the generalized symmetry of rank $R = 7$, we again consider the terms in (12). This time we do not remove the divergences or divergence equivalent terms. The candidate generalized symmetry is then the linear combination of the monomials generated by (12). For the example, where $W(u) = 2$,

$$D_x^5 u = u_{5x}, \quad D_x^3 u^2 = 6u_x u_{2x} + 2u u_{3x}, \quad D_x u^3 = 3u^2 u_x, \quad (21)$$

so the candidate generalized symmetry of rank $R = 7$ is

$$G = c_1 u^2 u_x + c_2 u_x u_{2x} + c_3 u u_{3x} + c_4 u_{5x}. \quad (22)$$

The undetermined coefficients are then found by computing (19) and using (5) to remove u_t, u_{tx}, u_{txx} , etc. Thus, continuing with the example we have

$$2(2c_1 - 3c_2)u_x^2 u_{2x} + 2(c_1 - 3c_3)uu_{2x}^2 + 2(c_1 - 3c_3)uu_x u_{3x} + (c_2 - 20c_4)u_{3x}^2 + (c_2 + c_3 - 30c_4)u_{2x}u_{4x} + (c_3 - 10c_4)u_x u_{5x} \equiv 0. \quad (23)$$

Again, considering all monomials in u and its derivatives as independent gives a linear system for c_i . From (23),

$$c_1 = 30c_4, \quad c_2 = 20c_4, \quad c_3 = 10c_4. \quad (24)$$

Setting $c_4 = 1$, we find

$$G = 30u^2 u_x + 20u_x u_{2x} + 10uu_{3x} + u_{5x}, \quad (25)$$

which is the fifth-order symmetry $G^{(3)}$ in (2).

3. Algorithm for Computing Recursion Operators

A recursion operator, \mathcal{R} , is a linear integro-differential operator which links generalized symmetries [54],

$$\mathbf{G}^{(j+g)} = \mathcal{R}\mathbf{G}^{(j)}, \quad j = 1, 2, 3, \dots, \quad (26)$$

where g is the gap and $\mathbf{G}^{(j)}$ is the j -th generalized symmetry. In many cases, $g = 1$ because the generalized symmetries differ by a common rank *and*, starting from $\mathbf{G}^{(1)}$, all higher-order symmetries can indeed be consecutively generated with the recursion operator. However, there are exceptions [6] where $g = 2$ or 3 . Examples of the latter are given in Sections 4.2, 4.3, 7.3, and Appendix A. Inspection of the ranks of generalized symmetries usually provides a hint on how to select the gap.

If \mathcal{R} is a recursion operator for (5), then the Lie derivative [35, 54, 59] of \mathcal{R} is zero, which leads to the following defining equation:

$$\frac{\partial \mathcal{R}}{\partial t} + \mathcal{R}'[\mathbf{F}(\mathbf{u})] + \mathcal{R} \circ \mathbf{F}'(\mathbf{u}) - \mathbf{F}'(\mathbf{u}) \circ \mathcal{R} = 0, \quad (27)$$

where \circ denotes a composition of operators, $\mathcal{R}'[\mathbf{F}(\mathbf{u})]$ is the Fréchet derivative of \mathcal{R} in the direction of \mathbf{F} ,

$$\mathcal{R}'[\mathbf{F}(\mathbf{u})] = \sum_{i=0}^m (D_x^i \mathbf{F}(\mathbf{u})) \frac{\partial \mathcal{R}}{\partial \mathbf{u}_{ix}}, \quad (28)$$

and $\mathbf{F}'(\mathbf{u})$ is the Fréchet derivative operator, i.e. a $M \times M$ matrix with entries

$$\mathbf{F}'_{ij}(\mathbf{u}) = \sum_{k=0}^m \left(\frac{\partial F_i}{\partial (u_j)_{kx}} \right) D_x^k, \quad (29)$$

where m is the highest order occurring in the right hand side of (5).

In the scalar case, $\mathbf{F} = F$ and $\mathbf{u} = u$, (29) simplifies into

$$F'(u) = \sum_{k=0}^m \left(\frac{\partial F}{\partial u_{kx}} \right) D_x^k. \tag{30}$$

Rather than solving (27), we will construct a candidate recursion operator and use (27) to determine the unknown coefficients, as shown in the following two steps.

Step 1 Generate the candidate recursion operator.

The rank of the recursion operator is determined by the difference in ranks of the generalized symmetries the recursion operator actually connects,

$$\text{rank } \mathcal{R}_{ij} = \text{rank } \mathbf{G}_i^{(k+g)} - \text{rank } \mathbf{G}_j^{(k)}, \quad i, j = 1, \dots, M, \tag{31}$$

where \mathcal{R} is an $M \times M$ matrix and \mathbf{G} has M components. As before, g is the gap and typically $g = 1$. Yet, there are cases where $g = 2$ or 3 .

The recursion operator naturally splits into two pieces [10],

$$\mathcal{R} = \mathcal{R}_0 + \mathcal{R}_1, \tag{32}$$

where \mathcal{R}_0 is a (local) differential operator and \mathcal{R}_1 is a (non-local) integral operator.

The differential operator \mathcal{R}_0 is a linear combination of terms

$$D_x^{k_0} u_1^{k_1} u_2^{k_2} \dots u_M^{k_M} I, \quad k_0, k_1, \dots \in \mathbb{N}, \tag{33}$$

where the k_i are non-negative integers taken so the monomial has the correct rank and the operator D_x has been propagated to the right. For example,

$$D_x^2 u I = D_x(D_x u I) = D_x(u_x I + u D_x) = u_{2x} I + 2u_x D_x + u D_x^2, \tag{34}$$

which, after multiplying the terms by undetermined coefficients, leads to

$$\mathcal{R}_0 = c_1 u_{2x} I + c_2 u_x D_x + c_3 u D_x^2. \tag{35}$$

We will assume that the integral operator \mathcal{R}_1 is a linear combination of terms

$$G^{(i)} D_x^{-1} \otimes \mathcal{L}_{\mathbf{u}}(\rho^{(j)}), \quad i, j \in \mathbb{N}, \tag{36}$$

of the correct rank [11, 59]. In (36), \otimes is the matrix outer product, and $\mathcal{L}_{\mathbf{u}}(\rho^{(j)})$ is the cosymmetry (Euler operator applied to $\rho^{(j)}$). To standardize \mathcal{R}_1 , propagate D_x to the left. For example, by integration by parts, $D_x^{-1} u_x D_x = u_x I - D_x^{-1} u_{2x} I$.

As shown in [11], the integral operator \mathcal{R}_1 can also be computed as a linear combination of the terms

$$G^{(i)} D_x^{-1} \otimes \psi^{(j)}, \quad i, j \in \mathbb{N}, \tag{37}$$

of the correct rank, where $\psi^{(j)}$ is the covariant (Fréchet derivative of $\rho^{(j)}$). While $G^{(i)} D_x^{-1} \otimes \mathcal{L}_{\mathbf{u}}(\rho^{(j)})$ is strictly non-local, $G^{(i)} D_x^{-1} \otimes \psi^{(j)}$ contains both differential and integral terms. Therefore, it is computationally more efficient to build the candidate recursion operator using $\mathcal{L}_{\mathbf{u}}(\rho^{(j)})$ instead of $\psi^{(j)}$. Finally, the local and non-local operators are added to obtain a candidate recursion operator (32).

Step 2 Determine the unknown coefficients.

To determine the unknown coefficients in the recursion operator, we substitute the candidate into the defining equation (27). After normalizing the form of the terms (propagating the D_x through the expression toward the right), we group the terms in like powers of \mathbf{u} , \mathbf{u}_x , \mathbf{u}_{xx} , \dots , I , D_x , D_x^2 , \dots , and D_x^{-1} . Requiring that these terms vanish identically gives a linear system for the c_i . Solving this linear system and substituting the coefficients into the candidate operator gives the recursion operator for (5). If $c_i = 0$ for all i , then either the gap g is incorrect or (5) does not have a recursion operator.

While the gap (g) is usually 1, 2 or 3, it is not obvious which value to take for g . In Sections 4.2 and 4.3 we give a couple of examples where $g = 2$. Starting from $\mathbf{G}^{(1)}$, the recursion operator then generates the higher-order symmetries $\mathbf{G}^{(3)}$, $\mathbf{G}^{(5)}$, \dots . Analogously, starting from $\mathbf{G}^{(2)}$ the recursion operator produces $\mathbf{G}^{(4)}$, $\mathbf{G}^{(6)}$, and so on. In Section 7.3 we show an example where $g = 3$. Further details on how to select the gap are given in Section 5.

4. Examples

4.1 The Korteweg-de Vries Equation

To illustrate the method, we use the KdV equation (1). Reversing the sign of t ,

$$u_t = F(u) = 6uu_x + u_{3x} \tag{38}$$

for scalar $u(x, t)$. From (2), the difference in ranks of the generalized symmetries is

$$\text{rank } G^{(3)} - \text{rank } G^{(2)} = \text{rank } G^{(2)} - \text{rank } G^{(1)} = 2. \tag{39}$$

Therefore, we will assume $g = 1$, and build a recursion operator with rank $\mathcal{R} = 2$. Thus, the local operator has the terms D_x^2 and $D_x^0 u I = u I$ of rank 2. So, the candidate differential operator is

$$\mathcal{R}_0 = c_1 D_x^2 + c_2 u I. \tag{40}$$

Using $\rho^{(1)} = u$ and $G^{(1)} = u_x$, the non-local operator is

$$\mathcal{R}_1 = c_3 G^{(1)} D_x^{-1} \mathcal{L}_u(\rho^{(1)}) = c_3 u_x D_x^{-1} \mathcal{L}_u(u) = c_3 u_x D_x^{-1}, \tag{41}$$

where we used \mathcal{L}_u given in (16). Thus, the candidate recursion operator is

$$\mathcal{R} = \mathcal{R}_0 + \mathcal{R}_1 = c_1 D_x^2 + c_2 u I + c_3 u_x D_x^{-1}. \tag{42}$$

Note that each term in (42) indeed has rank 2.

Now, we separately compute the pieces needed to evaluate (27). Using (30), we readily compute

$$F'(u) = D_x^3 + 6u D_x + 6u_x I. \tag{43}$$

Since the candidate recursion operator (42) is t -independent, we have $\partial \mathcal{R} / \partial t = 0$. Next, using (28) and (42) we compute

$$\mathcal{R}'[F(u)] = (6c_2 uu_x + c_2 u_{3x}) I + (6c_3 u_x^2 + 6c_3 uu_{2x} + c_3 u_{4x}) D_x^{-1}. \tag{44}$$

Using (42) and (43), we compute

$$\begin{aligned} \mathcal{R} \circ F'(u) &= c_1 D_x^5 + (6c_1 + c_2) u D_x^3 + (18c_1 + c_3) u_x D_x^2 \\ &\quad + 6(c_2 u^2 + 3c_1 u_{2x}) D_x + 6(c_2 u u_x + c_3 u u_x + c_1 u_{3x}) I, \end{aligned} \tag{45}$$

and

$$\begin{aligned} F'(u) \circ \mathcal{R} &= c_1 D_x^5 + (6c_1 + c_2) u D_x^3 + (6c_1 + 3c_2 + c_3) u_x D_x^2 \\ &\quad + 3(2c_2 u^2 + c_2 u_{2x} + c_3 u_{2x}) D_x + (12c_2 u u_x + 6c_3 u u_x \\ &\quad + c_2 u_{3x} + 3c_3 u_{3x}) I + (6c_3 u_x^2 + 6c_3 u u_{2x} + c_3 u_{4x}) D_x^{-1}. \end{aligned} \tag{46}$$

Substituting (44), (45) and (46) into (27) and grouping like terms, we find

$$(4c_1 - c_2) u_x D_x^2 + (6c_1 - c_2 - c_3) u_{2x} D_x + (2c_1 - c_3) u_{3x} I \equiv 0. \tag{47}$$

So, $2c_1 = c_3$ and $c_2 = 2c_3$. Taking $c_3 = 2$, gives

$$\mathcal{R} = D_x^2 + 4uI + 2u_x D_x^{-1}, \tag{48}$$

which is indeed the recursion operator (3) of the KdV equation [53].

4.2 The Kaup-Kupershmidt Equation

Consider the Kaup-Kupershmidt (KK) equation [28, 59],

$$u_t = F(u) = 20u^2 u_x + 25u_x u_{2x} + 10u u_{3x} + u_{5x}. \tag{49}$$

To find the dilation symmetry for (49), we require that all the terms in (49) have the same rank:

$$\begin{aligned} W(u) + W(D_t) &= 3W(u) + W(D_x) = 2W(u) + 3W(D_x) \\ &= 2W(u) + 3W(D_x) = W(u) + 5W(D_x). \end{aligned} \tag{50}$$

If we set $W(D_x) = 1$, then $W(u) = 2$, $W(D_t) = 5$ and the rank of (49) is 7.

Using `InvariantsSymmetries.m` [27], we compute the conserved densities

$$\rho^{(1)} = u, \quad \rho^{(2)} = -8u^3 + 3u_x^2, \tag{51}$$

and the generalized symmetries

$$G^{(1)} = u_x, \quad G^{(2)} = F(u) = 20u^2 u_x + 25u_x u_{2x} + 10u u_{3x} + u_{5x} \tag{52}$$

of (49). We do not show $G^{(3)}$ through $G^{(6)}$ explicitly due to length. From the weights above, the ranks of the first six generalized symmetries are

$$\begin{aligned} \text{rank } G^{(1)} &= 3, & \text{rank } G^{(2)} &= 7, & \text{rank } G^{(3)} &= 9, \\ \text{rank } G^{(4)} &= 13, & \text{rank } G^{(5)} &= 15, & \text{rank } G^{(6)} &= 19. \end{aligned} \tag{53}$$

We assume that $\text{rank } \mathcal{R} = 6$ and $g = 2$, since $\text{rank } G^{(2)} - \text{rank } G^{(1)} \neq \text{rank } G^{(3)} - \text{rank } G^{(2)}$ but $\text{rank } G^{(3)} - \text{rank } G^{(1)} = \text{rank } G^{(4)} - \text{rank } G^{(2)} = 6$. Thus, taking all terms of the form $D_x^i u^j$ ($i, j \in \mathbb{N}$) such that $\text{rank } (D_x^i u^j) = 6$ gives

$$\begin{aligned} \mathcal{R}_0 = & c_1 D_x^6 + c_2 u D_x^4 + c_3 u_x D_x^3 + (c_4 u^2 + c_5 u_{2x}) D_x^2 \\ & + (c_6 u u_x + c_7 u_{3x}) D_x + (c_8 u^3 + c_9 u_x^2 + c_{10} u u_{2x} + c_{11} u_{4x}) I. \end{aligned} \quad (54)$$

Using the densities and generalized symmetries above, we compute

$$G^{(1)} D_x^{-1} \mathcal{L}_u(\rho^{(2)}) = u_x D_x^{-1} \mathcal{L}_u(-8u^3 + 3u_x^2) = -6u_x D_x^{-1} (4u^2 + u_{2x}) \quad (55)$$

and

$$G^{(2)} D_x^{-1} \mathcal{L}_u(\rho^{(1)}) = F(u) D_x^{-1} \mathcal{L}_u(u) = F(u) D_x^{-1}. \quad (56)$$

Thus, the candidate non-local operator is

$$\mathcal{R}_1 = c_{12} u_x D_x^{-1} (4u^2 + u_{2x}) I + c_{13} (20u^2 u_x + 25u_x u_{2x} + 10u u_{3x} + u_{5x}) D_x^{-1}. \quad (57)$$

Substituting $\mathcal{R} = \mathcal{R}_0 + \mathcal{R}_1$ into (27) gives 49 linear equations for c_i . Solving yields

$$\begin{aligned} c_1 = -\frac{1}{2} c_{12}, \quad c_2 = -6c_{12}, \quad c_3 = c_5 = -18c_{12}, \quad c_4 = -16c_{12}, \\ c_6 = -\frac{69}{2} c_{12}, \quad c_7 = -\frac{49}{2} c_{12}, \quad c_8 = -\frac{35}{2} c_{12}, \quad c_9 = -\frac{13}{2} c_{12}, \\ c_{10} = -60c_{12}, \quad c_{11} = -41c_{12}, \quad c_{13} = -c_{12}, \end{aligned} \quad (58)$$

where c_{12} is arbitrary. Setting $c_{12} = -2$, we obtain

$$\begin{aligned} \mathcal{R} = & D_x^6 + 12u D_x^4 + 36u_x D_x^3 + (36u^2 + 49u_{2x}) D_x^2 \\ & + 5(24u u_x + 7u_{3x}) D_x + (32u^3 + 69u_x^2 + 82u u_{2x} + 13u_{4x}) I \\ & + 2u_x D_x^{-1} (4u^2 + u_{2x}) I + 2F(u) D_x^{-1}, \end{aligned} \quad (59)$$

which was computed in [59] as the composition of the cosymplectic and symplectic operators of (49).

Since $g = 2$ the symmetries are not generated sequentially via (4). Actually, $G^{(1)}$ and $G^{(2)}$ in (52) are the ‘‘seeds’’ (or roots) and one must use (26). Indeed, from $G^{(1)}$ one obtains $G^{(3)} = \mathcal{R}G^{(1)}$, $G^{(5)} = \mathcal{R}G^{(3)}$, and so on. From $G^{(2)}$, upon repeated application of \mathcal{R} , one obtains $G^{(4)}$, $G^{(6)}$, etc. Thus, using the recursion operator one can generate an infinity of generalized symmetries, confirming that (49) is completely integrable.

4.3 The Hirota-Satsuma System

Consider the system of coupled KdV equations due to Hirota and Satsuma [1],

$$\begin{aligned} u_t = F_1(\mathbf{u}) &= 3u u_x - 2v v_x + \frac{1}{2} u_{3x}, \\ v_t = F_2(\mathbf{u}) &= -3u v_x - v_{3x}, \end{aligned} \quad (60)$$

which model shallow water waves. Solving the equations for the weights,

$$\begin{cases} W(u) + W(D_t) = 2W(u) + 1 = W(u) + 3 = 2W(v) + 1, \\ W(v) + W(D_t) = W(u) + W(v) + 1 = W(v) + 3, \end{cases} \quad (61)$$

yields $W(u) = W(v) = 2$ and $W(D_t) = 3$.

The first few conserved densities and generalized symmetries computed with `InvariantsSymmetries.m` [27] are

$$\begin{aligned} \rho^{(1)} &= u, & \rho^{(2)} &= 3u^2 - 2v^2, \\ \mathbf{G}^{(1)} &= \begin{pmatrix} u_x \\ v_x \end{pmatrix}, & \mathbf{G}^{(2)} &= \begin{pmatrix} F_1(\mathbf{u}) \\ F_2(\mathbf{u}) \end{pmatrix} = \begin{pmatrix} 3uu_x - 2vv_x + \frac{1}{2}u_{3x} \\ -3uv_x - v_{3x} \end{pmatrix}. \end{aligned} \quad (62)$$

$G^{(3)}$ and $G^{(4)}$ are not shown explicitly due to length. Based on the above weights,

$$\text{rank } \rho^{(1)} = 2, \quad \text{rank } \rho^{(2)} = 4, \quad (63)$$

and

$$\begin{aligned} \text{rank } \mathbf{G}^{(1)} &= \begin{pmatrix} 3 \\ 3 \end{pmatrix}, & \text{rank } \mathbf{G}^{(2)} &= \begin{pmatrix} 5 \\ 5 \end{pmatrix}, \\ \text{rank } \mathbf{G}^{(3)} &= \begin{pmatrix} 7 \\ 7 \end{pmatrix}, & \text{rank } \mathbf{G}^{(4)} &= \begin{pmatrix} 9 \\ 9 \end{pmatrix}. \end{aligned} \quad (64)$$

We first set $g = 1$, so that $\text{rank } \mathcal{R}_{ij} = 2, i, j = 1, 2$. If indeed the generalized symmetries were linked consecutively, then

$$\mathcal{R}_0 = \begin{pmatrix} c_1 D_x^2 + c_2 uI + c_3 vI & c_4 D_x^2 + c_5 uI + c_6 vI \\ c_7 D_x^2 + c_8 uI + c_9 vI & c_{10} D_x^2 + c_{11} uI + c_{12} vI \end{pmatrix}. \quad (65)$$

Using (62), we have

$$\mathcal{R}_1 = c_{13} \mathbf{G}^{(1)} D_x^{-1} \otimes \mathcal{L}_{\mathbf{u}}(\rho^{(1)}) = c_{13} \begin{pmatrix} u_x \\ v_x \end{pmatrix} D_x^{-1} \otimes (\mathcal{L}_u(\rho^{(1)}) \mathcal{L}_v(\rho^{(1)})) \quad (66)$$

$$= c_{13} \begin{pmatrix} u_x \\ v_x \end{pmatrix} D_x^{-1} \otimes (I \ 0) = c_{13} \begin{pmatrix} u_x D_x^{-1} & 0 \\ v_x D_x^{-1} & 0 \end{pmatrix}. \quad (67)$$

Substituting $\mathcal{R} = \mathcal{R}_0 + \mathcal{R}_1$ into (27), we find $c_1 = \dots = c_{13} = 0$. Thus, the choice $g = 1$ appears to be *incorrect*. Noting that the ranks of the symmetries in (64) differ by 2, we repeat the process with $g = 2$, so that $\text{rank } \mathcal{R}_{ij} = 4, i, j = 1, 2$. Thus,

$$\mathcal{R} = \begin{pmatrix} (\mathcal{R}_0)_{11} & (\mathcal{R}_0)_{12} \\ (\mathcal{R}_0)_{21} & (\mathcal{R}_0)_{22} \end{pmatrix} + c_{41} \mathbf{G}^{(1)} D_x^{-1} \otimes \mathcal{L}_{\mathbf{u}}(\rho^{(2)}) + c_{42} \mathbf{G}^{(2)} D_x^{-1} \otimes \mathcal{L}_{\mathbf{u}}(\rho^{(1)}), \quad (68)$$

where $(\mathcal{R}_0)_{ij}, i, j = 1, 2$, are linear combinations (with different undetermined coefficients) of $\{D_x^4, uD_x^2, vD_x^2, u_x D_x, v_x D_x, u^2, uv, v^2, u_{2x}, v_{2x}\}$. For instance,

$$\begin{aligned} (\mathcal{R}_0)_{12} &= c_{11} D_x^4 + (c_{12} u + c_{13} v) D_x^2 + (c_{14} u_x + c_{15} v_x) D_x \\ &\quad + (c_{16} u^2 + c_{17} uv + c_{18} v^2 + c_{19} u_{2x} + c_{20} v_{2x}) I. \end{aligned} \quad (69)$$

Using (62), the first term of \mathcal{R}_1 in (68) is

$$\begin{aligned} \mathcal{R}_1^{(1)} &= c_{41} \mathbf{G}^{(1)} D_x^{-1} \otimes \mathcal{L}_{\mathbf{u}}(\rho^{(2)}) = c_{41} \begin{pmatrix} u_x \\ v_x \end{pmatrix} D_x^{-1} \otimes (6uI - 4vI) \\ &= c_{41} \begin{pmatrix} 3u_x D_x^{-1} uI & -2u_x D_x^{-1} vI \\ 3v_x D_x^{-1} uI & -2v_x D_x^{-1} vI \end{pmatrix}. \end{aligned}$$

The second term of \mathcal{R}_1 in (68) is

$$\begin{aligned} \mathcal{R}_1^{(2)} &= c_{42} \mathbf{G}^{(2)} D_x^{-1} \otimes \mathcal{L}_{\mathbf{u}}(\rho^{(1)}) = c_{42} \begin{pmatrix} F_1(\mathbf{u}) \\ F_2(\mathbf{u}) \end{pmatrix} D_x^{-1} \otimes (I \ 0) \\ &= c_{42} \begin{pmatrix} F_1(\mathbf{u}) D_x^{-1} & 0 \\ F_2(\mathbf{u}) D_x^{-1} & 0 \end{pmatrix}. \end{aligned}$$

Substituting the form of $\mathcal{R} = \mathcal{R}_0 + \mathcal{R}_1 = \mathcal{R}_0 + \mathcal{R}_1^{(1)} + \mathcal{R}_1^{(2)}$ into (27), the linear system for c_i has a non-trivial solution. Solving the linear system, we finally obtain

$$\mathcal{R} = \begin{pmatrix} (\mathcal{R})_{11} & (\mathcal{R})_{12} \\ (\mathcal{R})_{21} & (\mathcal{R})_{22} \end{pmatrix}, \tag{70}$$

with

$$\begin{aligned} (\mathcal{R})_{11} &= D_x^4 + 8uD_x^2 + 12u_x D_x + 8 \left(2u^2 + u_{2x} - \frac{2}{3}v^2 \right) I \\ &\quad + 4u_x D_x^{-1} uI + 2(6uu_x + u_{3x} - 4vv_x) D_x^{-1}, \\ (\mathcal{R})_{12} &= -\frac{20}{3}vD_x^2 - \frac{16}{3}v_x D_x - \frac{4}{3}(4uv + v_{2x}) I - \frac{8}{3}u_x D_x^{-1} vI, \\ (\mathcal{R})_{21} &= -10v_x D_x - 12v_{2x} I + 4v_x D_x^{-1} uI - 4(3uv_x + v_{3x}) D_x^{-1}, \\ (\mathcal{R})_{22} &= -4D_x^4 - 16uD_x^2 - 8u_x D_x - \frac{16}{3}v^2 I - \frac{8}{3}v_x D_x^{-1} vI. \end{aligned}$$

The above recursion operator was computed in [59] as the composition of the cosymplectic and symplectic operators of (60).

In agreement with $g = 2$, there are two seeds. Using (26) and starting from $\mathbf{G}^{(1)}$ in (62), the recursion operator (70) generates the infinite sequence of generalized symmetries with odd labels. Starting from $\mathbf{G}^{(2)}$, the recursion operator (70) generates the infinite sequence of generalized symmetries with even labels. The existence of a recursion operator confirms that (60) is completely integrable.

5. Key Algorithms

In this section, we present details of the algorithm. To illustrate the key algorithms in Sections 5.2 and 5.3, we will use the dispersiveless long wave system [1],

$$\begin{aligned} u_t &= F_1(\mathbf{u}) = uv_x + u_x v, \\ v_t &= F_2(\mathbf{u}) = u_x + vv_x, \end{aligned} \tag{71}$$

which is used in applications involving shallow water waves.

5.1 Integro-Differential Operators

Recursion operators are non-commutative by nature and certain rules must be used to simplify expressions involving integro-differential operators. While the multiplication of differential and integral operators is completely described by

$$D_x^i D_x^j = D_x^{i+j}, \quad i, j \in \mathbb{Z}, \tag{72}$$

the propagation of a differential operator through an expression is trickier. To propagate the differential operator to the right, we use Leibniz' rule

$$D_x^n Q = \sum_{k=0}^n \binom{n}{k} Q^{(k)} D_x^{n-k}, \quad n \in \mathbb{N}, \tag{73}$$

where Q is an expression and $Q^{(k)}$ is the k -th derivative with respect to x of Q . Unlike the finite series for a differential operator, Leibniz' rule for an inverse differential operator is

$$D_x^{-1} Q = Q D_x^{-1} - Q' D_x^{-2} + Q'' D_x^{-3} - \dots = \sum_{k=0}^{\infty} (-1)^k Q^{(k)} D_x^{-k-1}. \tag{74}$$

Therefore, rather than dealing with an infinite series, we only use Leibniz' rule for the inverse differential operator when there is a differential operator to the right of the inverse operator. In such cases we use

$$D_x^{-1} Q D_x^n = Q D_x^{n-1} - D_x^{-1} Q' D_x^{n-1}. \tag{75}$$

Repeated application of (75) yields

$$D_x^{-1} Q D_x^n = \sum_{k=0}^{n-1} (-1)^k Q^{(k)} D_x^{n-k-1} + (-1)^n D_x^{-1} Q^{(n)} I. \tag{76}$$

By using these identities, all the terms are either of the form $\tilde{P} D_x^n$ or $\tilde{P} D_x^{-1} \tilde{Q} I$, where \tilde{P} and \tilde{Q} are polynomials in \mathbf{u} and its x derivatives.

5.2 Algorithm for Building the Candidate Recursion Operator

Step 1 Find the dilation symmetry.

The dilation symmetry is found by requiring that each equation in (5) is uniform in rank, i.e. every monomial in that equation has the same rank. If (5) is not uniform in rank we use a trick. In that case, we multiply those terms that are not uniform in rank by auxiliary parameters (α, β, \dots) with weights. Once the computations are finished we set the auxiliary parameters equal to one.

Since the linear system for the weights is always underdetermined, we set $W(D_x) = 1$ and this (typically) fixes the values for the remaining weights.

For the example under consideration, (71), we have the linear system

$$\begin{aligned} W(u) + W(D_t) &= W(u) + W(v) + 1 = W(u) + W(v) + 1, \\ W(v) + W(D_t) &= W(u) + 1 = 2W(v) + 1. \end{aligned} \tag{77}$$

Thus, $W(v) = \frac{1}{2}W(u)$, $W(D_t) = \frac{1}{2}W(u) + 1$, provided $W(D_x) = 1$. If we select $W(u) = 2$, then the scaling symmetry for (71) becomes

$$(t, x, u, v) \rightarrow (\lambda^{-2}t, \lambda^{-1}x, \lambda^2u, \lambda v). \tag{78}$$

In the code `PDERecursionOperator.m`, the user can set the values of weights with `WeightRules -> {weight[u] -> 2}`.

Step 2 Determine the rank of the recursion operator.

Since the gap g cannot be determined *a priori*, we assume $g = 1$. Should this choice not lead to a result, one could set $g = 2$ or 3 . In the code, the user can set the `Gap` to any positive integer value (see Appendix A).

To determine the rank of the recursion operator, we compute the first $g + 1$ generalized symmetries and then use

$$\text{rank } \mathcal{R}_{ij} = \text{rank } \mathbf{G}_i^{(k+g)} - \text{rank } \mathbf{G}_j^{(k)}, \quad i, j = 1, 2, \tag{79}$$

to determine the rank of \mathcal{R} . Hence, the rank of the recursion operator \mathcal{R} can be represented (in matrix form) as follows

$$\text{rank } \mathcal{R} = \begin{pmatrix} \text{rank } \mathcal{R}_{11} & \text{rank } \mathcal{R}_{12} \\ \text{rank } \mathcal{R}_{21} & \text{rank } \mathcal{R}_{22} \end{pmatrix}. \tag{80}$$

In exceptional cases, the rank of the recursion operator might be lower (or higher) than computed by (79). In the code, the user has some additional control over the rank of the recursion operator. For example, in an attempt to find a simpler recursion operator, the rank of the recursion operator can be shifted down by one by setting `RankShift -> -1`. Similarly, to increase the rank of the recursion operator one can set `RankShift -> 1` (see Appendix A).

For (71), the first two generalized symmetries and their ranks are

$$\mathbf{G}^{(1)} = \begin{pmatrix} u_x \\ v_x \end{pmatrix}, \quad \text{rank } \mathbf{G}^{(1)} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \tag{81}$$

$$\mathbf{G}^{(2)} = \begin{pmatrix} uv_x + vu_x \\ u_x + vv_x \end{pmatrix}, \quad \text{rank } \mathbf{G}^{(2)} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}. \tag{82}$$

Then, using (79) and (80),

$$\text{rank } \mathcal{R} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}. \tag{83}$$

Step 3 Generate the (local) differential operator \mathcal{R}_0 .

Given the rank of the recursion operator, we take a linear combination of

$$D_x^{k_0} u_1^{k_1} u_2^{k_2} \dots u_M^{k_M} \alpha^{k_{M+1}} \beta^{k_{M+2}} \dots, \quad k_0, k_1, \dots \in \mathbb{N}, \tag{84}$$

where the k_i are taken so the monomial has the correct rank, the operator D_x has been propagated to the right, and α, β, \dots are the weighted parameters from Step 1 (if present).

For (71), the (local) differential operator is

$$\mathcal{R}_0 = \begin{pmatrix} c_1 D_x + c_2 v I & c_3 D_x^2 + c_4 u I + c_5 v D_x + c_6 v^2 I + c_7 v_x I \\ c_8 I & c_9 D_x + c_{10} v I \end{pmatrix}. \quad (85)$$

Step 4 Generate the (non-local) integral operator \mathcal{R}_1 .

Since the integral operator involves the outer product of generalized symmetries and cosymmetries, we compute the conserved densities up to

$$\max_{i,j} \{ \text{rank } \mathcal{R}_{ij} - \text{rank } (\mathbf{G}^{(1)})_i + W(u_j) + W(D_x) \}, \quad i, j = 1, \dots, M. \quad (86)$$

We add $W(u_j)$ in (86) because the Euler operator \mathcal{L}_{u_j} decreases the weight of the conserved density by the weight of u_j . In most cases, we take a linear combination of the terms

$$G^{(i)} D_x^{-1} \otimes \mathcal{L}_{\mathbf{u}}(\rho^{(j)}), \quad i, j \in \mathbb{N}, \quad (87)$$

of the correct rank as the candidate non-local operator. However, there are cases in which we must take a linear combination of the monomials in *each term* of type (87) with different coefficients.

For (71), we only need the cosymmetry of density $\rho^{(1)} = v$,

$$\mathcal{L}_{\mathbf{u}}(\rho^{(1)}) = (0 \ I). \quad (88)$$

Hence,

$$G^{(1)} D_x^{-1} \otimes \mathcal{L}_{\mathbf{u}}(\rho^{(1)}) = \begin{pmatrix} 0 & u_x D_x^{-1} \\ 0 & v_x D_x^{-1} \end{pmatrix}. \quad (89)$$

Thus, the (non-local) integral operator is

$$\mathcal{R}_1 = \begin{pmatrix} 0 & c_{11} u_x D_x^{-1} \\ 0 & c_{12} v_x D_x^{-1} \end{pmatrix}. \quad (90)$$

Step 5 Add the local and the non-local operators to form \mathcal{R} .

The candidate recursion operator is

$$\mathcal{R} = \mathcal{R}_0 + \mathcal{R}_1. \quad (91)$$

So, the candidate recursion operator for (71) is

$$\mathcal{R} = \begin{pmatrix} c_1 D_x + c_2 v I & c_3 D_x^2 + c_4 u I + c_5 v D_x + c_6 v^2 I + c_7 v_x I + c_{11} u_x D_x^{-1} \\ c_8 I & c_9 D_x + c_{10} v I + c_{12} v_x D_x^{-1} \end{pmatrix}. \quad (92)$$

5.3 Algorithm for Determining the Unknown Coefficients

Step 1 Compute the terms in the defining equation (27).

Step 1.1 Compute $\mathcal{R}_t = \frac{\partial \mathcal{R}}{\partial t}$.

The computation of \mathcal{R}_t is easy. Since the candidate recursion operator is t -independent one has $\mathcal{R}_t = \mathbf{0}$.

Step 1.2 Compute $\mathcal{R}'[\mathbf{F}(\mathbf{u})]$.

The Fréchet derivative of \mathcal{R} in the direction of $\mathbf{F}(\mathbf{u})$ is given in (28). Unlike the Fréchet derivative (20) of $\mathbf{F}(\mathbf{u})$ in the direction of \mathbf{G} (used in the computation of generalized symmetries), \mathcal{R} and $\mathbf{F}(\mathbf{u})$ in (28) are operators.

Applied to the example (71) with 2 components,

$$\mathcal{R}'[\mathbf{F}(\mathbf{u})] = \begin{pmatrix} (\mathcal{R}'[\mathbf{F}(\mathbf{u})])_{11} & (\mathcal{R}'[\mathbf{F}(\mathbf{u})])_{12} \\ (\mathcal{R}'[\mathbf{F}(\mathbf{u})])_{21} & (\mathcal{R}'[\mathbf{F}(\mathbf{u})])_{22} \end{pmatrix}, \quad (93)$$

where

$$(\mathcal{R}'[\mathbf{F}(\mathbf{u})])_{ij} = \sum_{k=0}^m \left(D_x^k \mathbf{F}(\mathbf{u}) \right) \frac{\partial (\mathcal{R})_{ij}}{\partial \mathbf{u}_{kx}}, \quad i, j = 1, 2. \quad (94)$$

Explicitly,

$$(\mathcal{R}'[\mathbf{F}(\mathbf{u})])_{11} = c_2 (u_x + vv_x) I, \quad (95)$$

$$(\mathcal{R}'[\mathbf{F}(\mathbf{u})])_{21} = 0, \quad (96)$$

$$\begin{aligned} (\mathcal{R}'[\mathbf{F}(\mathbf{u})])_{12} = & +c_5 (u_x + vv_x) D_x + \left(c_4 uv_x + (2c_6 + c_4)vu_x + 2c_6v^2v_x \right. \\ & \left. + c_7vv_{2x} + c_7v_x^2 + c_7u_{2x} \right) I + c_{11}(2u_xv_x + uv_{2x} + vu_{2x})D_x^{-1}, \end{aligned} \quad (97)$$

$$(\mathcal{R}'[\mathbf{F}(\mathbf{u})])_{22} = c_{10} (u_x + vv_x) I + c_{12} (u_{2x} + vv_{2x} + v_x^2) D_x^{-1}. \quad (98)$$

Step 1.3 Compute $\mathbf{F}'(\mathbf{u})$.

Use formula (29) to compute $\mathbf{F}'(\mathbf{u})$. Continuing with example (71),

$$\mathbf{F}'(\mathbf{u}) = \begin{pmatrix} vD_x + v_x I & uD_x + u_x I \\ D_x & vD_x + v_x I \end{pmatrix}. \quad (99)$$

Step 1.4 Compose \mathcal{R} and $\mathbf{F}'(\mathbf{u})$.

The composition of the $M \times M$ matrices \mathcal{R} and $\mathbf{F}'(\mathbf{u})$ is an order preserving inner product of the two matrices. For example (71),

$$\mathcal{R} \circ \mathbf{F}'(\mathbf{u}) = \begin{pmatrix} (\mathcal{R} \circ \mathbf{F}'(\mathbf{u}))_{11} & (\mathcal{R} \circ \mathbf{F}'(\mathbf{u}))_{12} \\ (\mathcal{R} \circ \mathbf{F}'(\mathbf{u}))_{21} & (\mathcal{R} \circ \mathbf{F}'(\mathbf{u}))_{22} \end{pmatrix}, \quad (100)$$

with

$$\begin{aligned} (\mathcal{R} \circ \mathbf{F}'(\mathbf{u}))_{11} = & c_3 D_x^3 + (c_1 + c_5)vD_x^2 + (2c_1v_x + c_2v^2 + c_4u + c_6v^2 + c_7v_x)D_x \\ & + (c_1v_{2x} + c_2vv_x + c_{11}u_x) I, \end{aligned} \quad (101)$$

$$\begin{aligned} (\mathcal{R} \circ \mathbf{F}'(\mathbf{u}))_{12} = & c_3vD_x^3 + (c_1u + 3c_3v_x + c_5v^2) D_x^2 + (2c_1u_x + c_2uv + 3c_3v_{2x} \\ & + c_4uv + 2c_5vv_x + c_6v^3 + c_7vv_x)D_x + (c_1u_{2x} + c_2vu_x + c_3v_{3x} \\ & + c_4uv_x + c_5vv_{2x} + c_6v^2v_x + c_7v_x^2 + c_{11}vu_x) I, \end{aligned} \quad (102)$$

$$(\mathcal{R} \circ \mathbf{F}'(\mathbf{u}))_{21} = c_9 D_x^2 + (c_8 + c_{10})vD_x + (c_8 + c_{12})v_x I, \quad (103)$$

$$\begin{aligned} (\mathcal{R} \circ \mathbf{F}'(\mathbf{u}))_{22} = & c_9vD_x^2 + (c_8u + 2c_9v_x + c_{10}v^2) D_x + (c_8u_x + c_9v_{2x} \\ & + c_{10}vv_x + c_{12}vv_x) I. \end{aligned} \quad (104)$$

Similarly,

$$\mathbf{F}'(\mathbf{u}) \circ \mathcal{R} = \begin{pmatrix} (\mathbf{F}'(\mathbf{u}) \circ \mathcal{R})_{11} & (\mathbf{F}'(\mathbf{u}) \circ \mathcal{R})_{12} \\ (\mathbf{F}'(\mathbf{u}) \circ \mathcal{R})_{21} & (\mathbf{F}'(\mathbf{u}) \circ \mathcal{R})_{22} \end{pmatrix}, \quad (105)$$

with

$$(\mathbf{F}'(\mathbf{u}) \circ \mathcal{R})_{11} = c_1 v D_x^2 + (c_1 v_x + c_2 v^2 + c_8 u) D_x + (2c_2 v v_x + c_8 u_x) I, \quad (106)$$

$$\begin{aligned} (\mathbf{F}'(\mathbf{u}) \circ \mathcal{R})_{12} = & c_3 v D_x^3 + (c_3 v_x + c_5 v^2 + c_9 u) D_x^2 + (c_4 u v + 2c_5 v v_x + c_6 v^3 \\ & + c_7 v v_x + c_9 u_x + c_{10} u v) D_x + (c_4 u v_x + c_4 v u_x + 3c_6 v^2 v_x \\ & + c_7 v_x^2 + c_7 v v_{2x} + c_{10} u v_x + c_{10} v u_x + c_{11} v u_x + c_{12} u v_x) I \\ & + (c_{11} u_x v_x + c_{11} v u_{2x} + c_{12} u v_{2x} + c_{12} u_x v_x) D_x^{-1}, \end{aligned} \quad (107)$$

$$(\mathbf{F}'(\mathbf{u}) \circ \mathcal{R})_{21} = c_1 D_x^2 + (c_2 + c_8) v D_x + (c_2 + c_8) v_x I, \quad (108)$$

$$\begin{aligned} (\mathbf{F}'(\mathbf{u}) \circ \mathcal{R})_{22} = & c_3 D_x^3 + (c_5 + c_9) v D_x^2 + (c_4 u + c_5 v_x + c_6 v^2 + c_7 v_x \\ & + c_9 v_x + c_{10} v^2) D_x + (c_4 u_x + 2c_6 v v_x + c_7 v_{2x} + 2c_{10} v v_x \\ & + c_{11} u_x + c_{12} v v_x) I + (c_{11} u_{2x} + c_{12} v v_{2x} + c_{12} v_x^2) D_x^{-1}. \end{aligned} \quad (109)$$

Step 1.5 Sum the terms in the defining equation.

For (71), summing the terms in the defining equation (27), we find

$$\begin{aligned} c_3 D_x^3 + c_5 v D_x^2 + (c_1 + c_7) v_x D_x + (c_4 - c_8) u D_x + c_6 v^2 D_x \\ + (c_2 - c_8 + c_{11}) u_x I + c_1 v_{2x} I \equiv 0, \end{aligned} \quad (110)$$

$$\begin{aligned} (c_1 - c_9) u D_x^2 + 2c_3 v_x D_x^2 + (c_2 - c_{10}) u v D_x + (c_5 - c_9 + 2c_1) u_x D_x \\ + c_5 v v_x D_x + 3c_3 v_{2x} D_x - (c_1 - c_7) u_{2x} I + (c_4 - c_{10} - c_{12}) u v_x I \\ + (c_2 + 2c_6 - c_{10}) v u_x I + c_5 v v_{2x} I + c_7 v_x^2 I + c_3 v_{3x} I \\ + (c_{11} - c_{12}) u v_{2x} D_x^{-1} + (c_{11} - c_{12}) u_x v_x D_x^{-1} \equiv 0, \end{aligned} \quad (111)$$

$$(c_1 - c_9) D_x^2 + (c_2 - c_{10}) v D_x + (c_2 - c_{12}) v_x I \equiv 0, \quad (112)$$

$$\begin{aligned} c_3 D_x^3 + c_5 v D_x^2 + (c_4 - c_8) u D_x + (c_5 + c_7 - c_9) v_x D_x \\ + c_6 v^2 D_x + (c_4 - c_8 - c_{10} + c_{11}) u_x I + 2c_6 v v_x I \\ + (c_7 - c_9) v_{2x} I + (c_{11} - c_{12}) u_{2x} D_x^{-1} \equiv 0. \end{aligned} \quad (113)$$

Step 2 Extract the linear system for the undetermined coefficients.

Group the terms in like powers of $\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, \dots, I, D_x, D_x^2, \dots$ and D_x^{-1} . Then, grouping like terms and setting their coefficients equal to zero yields a linear system

for the undetermined coefficients. For (71), we obtain

$$\begin{aligned}
 c_1 = 0, \quad c_3 = 0, \quad c_5 = 0, \quad c_6 = 0, \quad c_7 = 0, \quad c_4 - c_8 = 0, \\
 c_1 - c_9 = 0, \quad 2c_1 + c_5 - c_9 = 0, \quad c_5 + c_7 - c_9 = 0, \quad c_2 - c_{10} = 0, \\
 c_2 + 2c_6 - c_{10} = 0, \quad c_2 - c_{10} = 0, \quad c_4 - c_8 - c_{10} + c_{11} = 0, \quad c_2 - c_8 + c_{11} = 0, \\
 c_4 - c_{10} - c_{12} = 0, \quad c_{11} - c_{12} = 0, \quad c_2 - c_{12} = 0, \quad c_{11} - c_{12} = 0.
 \end{aligned}
 \tag{114}$$

Step 3 Solve the linear system and build the recursion operator.

Solve the linear system and substitute the constants into the candidate recursion operator. For (71), we find

$$c_1 = c_3 = c_5 = c_6 = c_7 = c_9 = 0, \quad 2c_2 = c_4 = 2c_{10} = 2c_{11} = 2c_{12} = c_8,
 \tag{115}$$

so taking $c_8 = 1$ gives

$$\mathcal{R} = \begin{pmatrix} \frac{1}{2}vI & uI + \frac{1}{2}u_x D_x^{-1} \\ I & \frac{1}{2}vI + \frac{1}{2}v_x D_x^{-1} \end{pmatrix}.
 \tag{116}$$

In [59] this recursion operator was obtained as the composition of the cosymplectic and symplectic operators of (71).

Starting from $G^{(1)}$, repeated application of (116) generates an infinite number of generalized symmetries of (71), establishing its completely integrable.

6. Other Software Packages

There has been little work on using computer algebra methods to find and test recursion operators. In 1987, Fuchssteiner et al. [23] wrote *PASCAL*, *Maple*, and *Macsyma* codes for testing recursion operators. While these packages could verify if a recursion operator is correct, they were unable to either generate the form of the operator or test a candidate recursion operator with undetermined coefficients. Bilge [11] did substantial work on finding recursion operators interactively with *REDUCE*.

Sanders and Wang [56, 59] wrote *Maple* and *Form* codes to aid in the computation of recursion operators. Their software was used to compute the symplectic, cosymplectic, as well as recursion operators of the 39 PDEs listed in [59] and [60].

Recently, Meshkov [48] implemented a package in *Maple* for investigating complete integrability from the geometric perspective. If the zero curvature representation of the system is known, then his software package can compute the recursion operator. To our knowledge, our package `PDERecursionOperator.m` [9] is the only fully automated software package for computing and testing polynomial recursion operators of polynomial evolution equations.

7. Additional Examples

7.1 The Nonlinear Schrödinger Equation

For convenience, we write the standard nonlinear Schrödinger equation (NLS),

$$iu_t + u_{xx} + 2|u|^2u = 0,
 \tag{117}$$

as the system of two real equations,

$$\begin{aligned} u_t &= -(u_{xx} + 2u^2v), \\ v_t &= v_{xx} + 2uv^2, \end{aligned} \tag{118}$$

where $v = \bar{u}$ and i has been absorbed in t .

To determine the weights, we assume $W(u) = W(v)$ so that (118) has dilation symmetry $(t, x, u, v) \rightarrow (\lambda^{-2}t, \lambda^{-1}x, \lambda u, \lambda v)$. Hence, $W(u) = W(v) = 1, W(D_t) = 2,$ and $W(D_x) = 1,$ as usual. The first densities and symmetries are

$$\rho^{(1)} = uv, \quad \rho^{(2)} = u_xv, \tag{119}$$

$$\mathbf{G}^{(1)} = \begin{pmatrix} u \\ -v \end{pmatrix}, \quad \mathbf{G}^{(2)} = \begin{pmatrix} u_x \\ v_x \end{pmatrix}. \tag{120}$$

Thus, rank $\mathcal{R}_{ij} = 1, i, j = 1, 2,$ and the candidate local operator is

$$\mathcal{R}_0 = \begin{pmatrix} c_1D_x + (c_2u + c_3v)I & c_4D_x + (c_5u + c_6v)I \\ c_7D_x + (c_8u + c_9v)I & c_{10}D_x + (c_{11}u + c_{12}v)I \end{pmatrix}. \tag{121}$$

The candidate non-local operator is

$$\mathcal{R}_1 = \mathbf{G}^{(1)}D_x^{-1} \otimes \mathcal{L}_u(\rho^{(1)}) = \begin{pmatrix} -c_{13}uD_x^{-1}vI & -c_{14}uD_x^{-1}uI \\ c_{15}vD_x^{-1}vI & c_{16}vD_x^{-1}uI \end{pmatrix}. \tag{122}$$

Substituting $\mathcal{R} = \mathcal{R}_0 + \mathcal{R}_1$ into (27), solving for the undetermined coefficients, and setting $c_{16} = -2,$ we find

$$\mathcal{R} = \begin{pmatrix} D_x + 2uD_x^{-1}vI & 2uD_x^{-1}uI \\ -2vD_x^{-1}vI & -D_x - 2vD_x^{-1}uI \end{pmatrix}. \tag{123}$$

Starting from “seed” $G^{(1)},$ the generalized symmetries can be constructed sequentially using (4). This establishes the complete integrability of (118).

In [59], Wang split (117) into an alternate system of two real equations by setting $u = v + iw.$ Using the cosymplectic and symplectic operators of that system, she obtained a recursion operator which is equivalent to (123).

7.2 The Burgers’ Equation

Consider the Burgers’ equation [54],

$$u_t = uu_x + u_{xx}, \tag{124}$$

which has the dilation symmetry $(t, x, u) \rightarrow (\lambda^{-2}t, \lambda^{-1}x, \lambda u),$ or $W(D_t) = 2,$ $W(u) = 1,$ with $W(D_x) = 1.$ For (124),

$$\rho^{(1)} = u, \quad G^{(1)} = u_x, \quad G^{(2)} = uu_x + u_{xx}. \tag{125}$$

Assuming that $g = 1,$ the candidate recursion operator of rank $\mathcal{R} = 1$ is

$$\mathcal{R} = c_1D_x + c_2uI + c_3G^{(1)}D_x^{-1}\mathcal{L}_u(\rho^{(1)}) = c_1D_x + c_2uI + c_3u_xD_x^{-1}. \tag{126}$$

Using the defining equation (27), we determine that $c_1 = 2c_3$ and $c_2 = c_3$. Taking $c_3 = \frac{1}{2}$, gives the recursion operator reported in [53],

$$\mathcal{R} = D_x + \frac{1}{2} (uI + u_x D_x^{-1}) = D_x + \frac{1}{2} D_x (u D_x^{-1}). \quad (127)$$

As expected, starting from $G^{(1)}$, one computes $G^{(2)} = \mathcal{R}G^{(1)}$, $G^{(3)} = \mathcal{R}G^{(2)} = \mathcal{R}^2 G^{(1)}$, etc., confirming that (124) is completely integrable.

The Burgers' equation also has the recursion operator [54],

$$\tilde{\mathcal{R}} = t\mathcal{R} + \frac{1}{2} (xI + D_x^{-1}) = tD_x + \frac{1}{2} (tu + x)I + \frac{1}{2} (tu_x + 1) D_x^{-1}, \quad (128)$$

which explicitly depends on x and t . Using $W(t) = -2$, $W(x) = W(D_x^{-1}) = -1$ and $W(u) = 2$, one can readily verify that each term in (128) has rank -1 .

To find recursion operators like (128), which depend explicitly on x and t , we can again use scaling symmetries to build $\tilde{\mathcal{R}}$. However, one must select the maximum degree for x and t . For instance, for degree 1 the coefficients in the recursion operator will at most depend on x and t (but not on xt , x^2 , or t^2 which are quadratic). To control the highest exponent in x and t , in the code the user can set `MaxExplicitDependency` to any non-negative integer value (see Appendix A).

With `MaxExplicitDependency -> 1`, the candidate local operator then is

$$\tilde{\mathcal{R}}_0 = c_1 t D_x + (c_2 x + c_3 t u) I. \quad (129)$$

The first symmetries that explicitly depend on x and t (of degree 1) are

$$\tilde{G}^{(1)} = 1 + tu_x, \quad \tilde{G}^{(2)} = \frac{1}{2} (u + xu_x) + tuu_x + tu_{xx}. \quad (130)$$

Thus, the candidate non-local operator is

$$\tilde{\mathcal{R}}_1 = c_4 \tilde{G}^{(1)} D_x^{-1} \mathcal{L}_u(\rho^{(1)}) = c_4 (tu_x + 1) D_x^{-1}. \quad (131)$$

Requiring that $\tilde{\mathcal{R}} = \tilde{\mathcal{R}}_0 + \tilde{\mathcal{R}}_1$ satisfies the defining equation (27), next solving for the constants c_1 through c_4 , and finally setting $c_4 = \frac{1}{2}$, yields the recursion operator (128). Using (128), one can construct an additional infinite sequence of generalized symmetries. Furthermore, $\tilde{G}^{(2)} = \tilde{\mathcal{R}}G^{(1)}$, $\tilde{G}^{(3)} = \tilde{\mathcal{R}}G^{(2)} = \tilde{\mathcal{R}}\mathcal{R}G^{(1)}$, etc. Connections between \mathcal{R} and $\tilde{\mathcal{R}}$ and their symmetries are discussed in [54] and [59].

7.3 The Drinfel'd-Sokolov-Wilson Equation

Consider the Drinfel'd-Sokolov-Wilson system [1, 41],

$$\begin{aligned} u_t &= 3vv_x, \\ v_t &= 2uv_x + u_xv + 2v_{3x}, \end{aligned} \quad (132)$$

which has static soliton solutions that interact with moving solitons without deformation. The scaling symmetry for (132) is $(t, x, u, v) \rightarrow (\lambda^{-3}t, \lambda^{-1}x, \lambda^2u, \lambda^2v)$. Expressed in weights, $W(D_t) = 3$, $W(D_x) = 1$, and $W(u) = W(v) = 2$. The first few conserved densities and generalized symmetries are

$$\rho^{(1)} = u, \quad \rho^{(2)} = v^2, \quad \rho^{(3)} = \frac{4}{27}u^3 - \frac{2}{3}uv^2 - \frac{1}{9}u_x^2 + v_x^2, \quad (133)$$

$$\mathbf{G}^{(1)} = \begin{pmatrix} u_x \\ v_x \end{pmatrix}, \quad \mathbf{G}^{(2)} = \begin{pmatrix} 3vv_x \\ u_xv + 2uv_x + 2v_{3x} \end{pmatrix}, \quad (134)$$

and

$$\mathbf{G}^{(3)} = \begin{pmatrix} -10u^2u_x + 15v^2u_x + 30uvv_x - 25u_xu_{2x} + 45v_xv_{2x} - 10uu_{3x} + 30vv_{3x} - 2u_{5x} \\ 10u^2v_x + 15v^2v_x + 10uvu_x + 45u_xv_{2x} + 35v_xu_{2x} + 30uv_{3x} + 10vu_{3x} + 18v_{5x} \end{pmatrix}. \quad (135)$$

Despite the fact that

$$\text{rank } \mathbf{G}^{(1)} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}, \quad \text{rank } \mathbf{G}^{(2)} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}, \quad (136)$$

we can not take $g=1$ or 2 . Surprisingly, for (132) we must set $g=3$ and rank $\mathcal{R}_{ij} = 6, i, j = 1, 2$. So, the candidate local operator has elements involving D_x^6 . For example,

$$\begin{aligned} (\mathcal{R}_0)_{11} = & c_1D_x^6 + (c_2u + c_5v) D_x^4 + (c_8u_x + c_{10}v_x) D_x^3 + (c_3u^2 + c_6v^2 + c_{12}u_{2x} \\ & + c_{13}v_{2x} + c_{18}uv) D_x^2 + (c_{14}u_{3x} + c_{15}v_{3x} + c_{20}uu_x + c_{21}uv_x + c_{25}vu_x + c_{26}vv_x) D_x \\ & + (c_4u^3 + c_7v^3 + c_{27}vu_{2x} + c_{28}vv_{2x} + c_{29}u_xv_x + c_9u_x^2 + c_{11}v_x^2 + c_{16}u_{4x} \\ & + c_{17}v_{4x} + c_{19}uv^2 + c_{22}uu_{2x} + c_{23}uv_{2x} + c_{24}u^2v) I. \end{aligned} \quad (137)$$

The candidate non-local operator is

$$\mathcal{R}_1 = \sum_{i=1}^4 \mathbf{G}^{(i)} D_x^{-1} \otimes \mathcal{L}_u(\rho^{(5-i)}) = \begin{pmatrix} (\mathcal{R}_1)_{11} & (\mathcal{R}_1)_{12} \\ (\mathcal{R}_1)_{21} & (\mathcal{R}_1)_{22} \end{pmatrix}, \quad (138)$$

where

$$\begin{aligned} (\mathcal{R}_1)_{11} = & - \left(\frac{1}{9}c_{117}u_{5x} + \frac{25}{18}c_{118}u_xu_{2x} + \frac{5}{9}c_{119}uu_{3x} + \frac{5}{9}c_{120}u^2u_x \right. \\ & \left. - \frac{5}{6}c_{121}v^2u_x - \frac{5}{3}c_{122}vv_{3x} - \frac{5}{2}c_{123}v_xv_{2x} \right) D_x^{-1} - \frac{2}{3}c_{124}u_xD_x^{-1}v^2 \\ & + \frac{2}{9}c_{125}u_xD_x^{-1}u_{2x} + \frac{4}{9}c_{126}u_xD_x^{-1}u^2 + \frac{5}{3}c_{127}uvv_xD_x^{-1}, \end{aligned} \quad (139)$$

$$(\mathcal{R}_1)_{12} = -2c_{128}u_xD_x^{-1}v_{2x} - \frac{4}{3}c_{129}u_xD_x^{-1}uv + 3c_{130}vv_xD_x^{-1}v, \quad (140)$$

$$\begin{aligned} (\mathcal{R}_1)_{21} = & \left(c_{131}v_{5x} + \frac{5}{9}c_{132}u^2v_x + \frac{5}{9}c_{133}vu_{3x} + \frac{5}{6}c_{134}v^2v_x \right. \\ & \left. + \frac{5}{3}c_{135}uv_{3x} + \frac{35}{18}c_{136}v_xu_{2x} + \frac{5}{2}c_{137}u_xv_{2x} \right) D_x^{-1} - \frac{2}{3}c_{138}v_xD_x^{-1}v^2 \\ & + \frac{2}{9}c_{139}v_xD_x^{-1}u_{2x} + \frac{4}{9}c_{140}v_xD_x^{-1}u^2 + \frac{5}{9}c_{141}uvv_xD_x^{-1}, \end{aligned} \quad (141)$$

$$\begin{aligned} (\mathcal{R}_1)_{22} = & -2c_{142}v_xD_x^{-1}v_{2x} + 2c_{143}v_{3x}D_x^{-1}v + 2c_{144}uv_xD_x^{-1}v \\ & - \frac{4}{3}c_{145}v_xD_x^{-1}uv + c_{146}vv_xD_x^{-1}v. \end{aligned} \quad (142)$$

The terms in (27) fill 160 pages and grouping like terms results in a system of 508 linear equations for c_i . Solving these linear equations and setting $c_{146} = -9$ gives the recursion operator

$$\mathcal{R} = \begin{pmatrix} (\mathcal{R})_{11} & (\mathcal{R})_{12} \\ (\mathcal{R})_{21} & (\mathcal{R})_{22} \end{pmatrix} \quad (143)$$

with

$$\begin{aligned} (\mathcal{R})_{11} = & D_x^6 + 6uD_x^4 + 18u_xD_x^3 + \left(9u^2 - 21v^2 + \frac{49}{2}u_{2x}\right)D_x^2 + \left(30uu_x - 75vv_x \right. \\ & \left. + \frac{35}{2}u_{3x}\right)D_x + \left(4u^3 - 12uv^2 + \frac{41}{2}uu_{2x} + \frac{13}{2}u_{4x} + \frac{69}{4}u_x^2 - \frac{111}{2}vv_{2x} \right. \\ & \left. - \frac{141}{4}v_x^2\right)I + \left(5u^2u_x + 5uu_{3x} - 15uvv_x - 15vv_{3x} - \frac{15}{2}v^2u_x + \frac{25}{2}u_xu_{2x} \right. \\ & \left. - \frac{45}{2}v_xv_{2x} + u_{5x}\right)D_x^{-1} + \frac{1}{2}u_xD_x^{-1}u_{2x}I - \frac{3}{2}u_xD_x^{-1}v^2I + u_xD_x^{-1}u^2I, \quad (144) \end{aligned}$$

$$\begin{aligned} (\mathcal{R})_{12} = & -42vD_x^4 - 51v_xD_x^3 - \left(48uv + \frac{63}{2}v_{2x}\right)D_x^2 - \left(33uv_x + 60vu_x + \frac{21}{2}v_{3x}\right)D_x \\ & - \left(18v^3 + 15u_xv_x + 6u^2v + \frac{15}{2}uv_{2x} + \frac{39}{2}vu_{2x} + \frac{3}{2}v_{4x}\right)I \\ & - 27vv_xD_x^{-1}vI - 3u_xD_x^{-1}uvI - \frac{9}{2}u_xD_x^{-1}v_{2x}I, \quad (145) \end{aligned}$$

$$\begin{aligned} (\mathcal{R})_{21} = & -14vD_x^4 - 67v_xD_x^3 - \left(16uv + \frac{243}{2}v_{2x}\right)D_x^2 - \left(18vu_x + 53uv_x \right. \\ & \left. + \frac{219}{2}v_{3x}\right)D_x - \left(46u_xv_x + 2u^2v + 6v^3 + \frac{99}{2}uv_{2x} + \frac{99}{2}v_{4x} + \frac{27}{2}vu_{2x}\right)I \\ & - \left(15uv_{3x} + 5u^2v_x + 5uvu_x + 5vu_{3x} + 9v_{5x} + \frac{15}{2}v^2v_x + \frac{35}{2}v_xu_{2x} \right. \\ & \left. + \frac{45}{2}u_xv_{2x}\right)D_x^{-1} + \frac{1}{2}v_xD_x^{-1}u_{2x}I - \frac{3}{2}v_xD_x^{-1}v^2I + v_xD_x^{-1}u^2I, \quad (146) \end{aligned}$$

and

$$\begin{aligned} (\mathcal{R})_{22} = & -27D_x^6 - 54uD_x^4 - 108u_xD_x^3 - \left(27u^2 + 33v^2 + \frac{243}{2}u_{2x}\right)D_x^2 \\ & - \left(54uu_x + 105vv_x + \frac{135}{2}u_{3x}\right)D_x - \left(24uv^2 + \frac{27}{2}uu_{2x} + \frac{27}{4}u_x^2 \right. \\ & \left. + \frac{147}{2}vv_{2x} + \frac{27}{2}u_{4x} + \frac{201}{4}v_x^2\right)I - 9\left(2uv_x + 2v_{3x} + vu_x\right)D_x^{-1}vI \\ & - 3v_xD_x^{-1}uvI - \frac{9}{2}v_xD_x^{-1}v_{2x}I \quad (147) \end{aligned}$$

This recursion operator can also be computed [59] by composing the cosymplectic and symplectic operators of (132). Since $g = 3$ the symmetries are not generated via (4). Instead, there are three seeds, $G^{(1)}$, $G^{(2)}$, and $G^{(3)}$ given in (134) and (135). Using (26), from $G^{(1)}$ one obtains $G^{(4)} = \mathcal{R}G^{(1)}$, $G^{(7)} = \mathcal{R}G^{(4)} = \mathcal{R}^2G^{(1)}$, and so

on. From $G^{(2)}$, upon repeated application of \mathcal{R} , one gets $G^{(5)}, G^{(8)}$, etc., whereas $G^{(3)}$ generates $G^{(6)}, G^{(9)}$, etc. Thus, the recursion operator generates a threefold infinity of generalized symmetries, confirming that (132) is completely integrable.

This example illustrates the importance of computer algebra software in the study of integrability, in particular, for the computation of recursion operators. The length of the computations makes it impossible to compute the recursion operators for all but the simplest systems by hand.

8. Conclusions and Future Work

To our knowledge, no one has ever attempted to fully automate an algorithm for finding or testing recursion operators. The commutative nature of computer algebra systems makes it a non-trivial task to efficiently implement the non-commutative rules needed for working with integro-differential operators.

Based on our recursion operator algorithm and its implementation in `PDERecursionOperator.m`, a large class of nonlinear PDEs can be tested for complete integrability in a straightforward manner. Currently our code computes polynomial recursion operators for polynomial PDEs (with constant coefficients) which can be written in evolution form, $\mathbf{u}_t = \mathbf{F}(\mathbf{u}, \mathbf{u}_x, \dots, \mathbf{u}_{mx})$.

With the tools developed for finding and testing recursion operators, it would be possible to extend the algorithm to find master symmetries as well as cosymplectic, symplectic and conjugate recursion operators. A symplectic operator maps (generalized) symmetries into cosymmetries, while a cosymplectic operator maps cosymmetries into generalized symmetries. Hence, the recursion operator for a system is the composition of the cosymplectic operator and the symplectic operator of the system. A conjugate recursion operator maps conserved densities of lower order to conserved densities of higher order. The master symmetry can be used to generate an infinite hierarchy of time-dependent generalized symmetries. It would be worthwhile to add to our *Mathematica* code an automated test of the ‘‘hereditary’’ condition [23] for recursion operators.

Acknowledgements

This material is based upon work supported by the National Science Foundation (NSF) under Grant Nos. CCF-0830783 and CCR-9901929. This work was also partially supported by a NDSEG Fellowship awarded to DB. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of NSF.

WH thanks Jan Sanders (Free University Amsterdam, The Netherlands) and Jing Ping Wang (University of Kent, Canterbury, UK) for many valuable discussions.

Appendix A. Using the Software Package `PDERecursionOperator.m`

The package `PDERecursionOperator.m` has been tested with *Mathematica* 4.0 through 7.0 using more than 30 PDEs. The Backus-Naur form of the main function (`RecursionOperator`) is

```

⟨Main Function⟩ → RecursionOperator[⟨Equations⟩, ⟨Functions⟩,
                                   ⟨Variables⟩, ⟨Parameters⟩, ⟨Options⟩]
⟨Options⟩ → Verbose → ⟨Bool⟩ | WeightsVerbose → ⟨Bool⟩ |
            Gap → ⟨Positive Integer⟩ |
            MaxExplicitDependency → ⟨Nonnegative Integer⟩ |
            RankShift → ⟨Integer⟩ |
    
```

```

WeightRules → ⟨List of Rules⟩ |
WeightedParameters → ⟨List of Weighted Parameters⟩ |
UnknownCoefficients → ⟨Symbol⟩
⟨Bool⟩ → True | False
⟨List of Rules⟩ → {weight[u] → ⟨Integer⟩, weight[v] → ⟨Integer⟩, ...}

```

When using a PC, place the packages PDERecursionOperator.m and InvariantsSymmetries.m in a directory, for example, myDirectory on drive C. Start a *Mathematica* notebook session and execute the commands:

```

In[1] := SetDirectory["C:\\myDirectory"]; (* Specify directory *)

In[2] := Get["PDERecursionOperator.m"] (* Read in the package *)

In[3] := RecursionOperator[ (* Burgers' equation *)
  D[u[x,t],t]==u[x,t]*D[u[x,t],x]+D[u[x,t],{x,2}],
  u[x,t], {x,t}]
Out[3] =

```

$$\{\{2C_3D_x + C_3uI + C_3u_xD_x^{-1}\}\}$$

We can find a recursion operator for Burgers' equation which explicitly depends on x and t (linearly) by using the option MaxExplicitDependency:

```

In[4] := RecursionOperator[ (* Burgers' equation *)
  D[u[x,t],t]==u[x,t]*D[u[x,t],x]+D[u[x,t],{x,2}],
  u[x,t], {x,t}, MaxExplicitDependency -> 1]
Out[4] =

```

$$\{\{2C_5tD_x + C_5(x + tu)I + C_5(1 + tu_x)D_x^{-1}\}\}$$

```

In[5] := RecursionOperator[ (* Potential mKdV equation *)
  D[u[x,t],t]==1/3*D[u[x,t],x]^3+D[u[x,t],{x,3}],
  u[x,t], {x,t},
  WeightRules -> {weight[u] -> 1}, Gap -> 2]

```

$$\{\{3C_{19}D_x^2 + (C_1 + 2C_{19}u_x^2)I - 2C_{19}u_xD_x^{-1}u_{2x}I\}\}$$

In this example, we must use the WeightRules option to fix the weights and the Gap option to set $g = 2$.

```

In[6] := RecursionOperator[ (* Diffusion system *)
  { D[u[x,t],t]==D[u[x,t],{x,2}]+v[x,t]^2,
    D[v[x,t],t]==D[v[x,t],{x,2}] },
  { u[x,t],v[x,t] }, {x,t},
  WeightRules -> {weight[u] -> weight[v]},
  RankShift -> -1 ]

```

$$\{\{C_5D_x, C_2D_x + C_5vD_x^{-1}\}, \{0, C_5D_x\}\}$$

In this system of equations, we again use the option WeightRules to fix the weights. We also use the option RankShift to force RecursionOperator to search for re-

cursion operator of a lower than expected rank.

The option `Verbose` prints out a trace of the calculations, while the option `WeightsVerbose` prints out a trace of the calculation of the scaling symmetry. If one or more parameters have a weight, these weights can be specified using `WeightedParameters`. The undetermined constants can be set to any variable using the option `UnknownCoefficients` (the default is C_i).

References

- [1] M.J. Ablowitz and P.A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, London Math. Soc. Lect. Note Series vol. 149, Cambridge University Press, Cambridge (1991).
- [2] M.J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform*, SIAM Studies in Applied Mathematics, SIAM, Philadelphia, PA (1981).
- [3] M.J. Ablowitz, A. Ramani, and H. Segur, *A connection between nonlinear evolution equations and ordinary differential equations of P-type. I*, J. Math. Phys. 21 (1980), pp. 715–721.
- [4] P.J. Adams, *Symbolic computation of conserved densities and fluxes for systems of partial differential equations with transcendental nonlinearities*, M.S. thesis, Colorado School of Mines, Golden, Colorado (2003).
- [5] P.J. Adams and W. Hereman, *TransPDEDensityFlux.m: A Mathematica package for the symbolic computation of conserved densities and fluxes for systems of partial differential equations with transcendental nonlinearities* (2002).
- [6] D. Baldwin, *Symbolic algorithms and software for the Painlevé test and recursion operators for nonlinear partial differential equations*, M.S. thesis, Colorado School of Mines, Golden, Colorado (2004).
- [7] D. Baldwin and W. Hereman, *PainleveTest.m: A Mathematica package for the Painlevé test of systems of ODEs and PDEs* (2003).
- [8] ———, Symbolic software for the Painlevé test of nonlinear ordinary and partial differential equations, J. Nonl. Math. Phys. 13 (2006) 90–110.
- [9] ———, *PDERecursionOperator.m: A Mathematica package for the symbolic computation of recursion operators for nonlinear partial differential equations*, 2003–2009; software available at <http://inside.mines.edu/~wherman/software/PDERecursionOperator>.
- [10] D. Baldwin, W. Hereman, and J. Sayers, *Symbolic algorithms for the Painlevé test, special solutions, and recursion operators for nonlinear PDEs*, in *Group Theory and Numerical Analysis*, P. Winternitz, D. Gomez-Ullate, A. Iserles, D. Levi, P.J. Olver, R. Quispel, and P. Tempesta, eds., CRM Proc. & Lect. Ser. vol. 39, AMS, Providence, RI, 2005, pp. 17–32.
- [11] A.H. Bilge, *On the equivalence of linearization and formal symmetries as integrability tests for evolution equations*, J. Phys. A: Math. Gen. 26 (1993), pp. 7511–7519.
- [12] F. Calogero and A. Degasperis, *Spectral Transform and Solitons I*, North Holland, Amsterdam (1982).
- [13] R.K. Dodd, J.C. Eilbeck, J.D. Gibbon, and M.C. Morris, *Solitons and Nonlinear Wave Equations*, Academic Press, London (1982).
- [14] I. Dorfman, *Dirac Structures and Integrability of Nonlinear Evolution Equations*, Wiley, Chichester, UK (1993).
- [15] H. Eklund, *Symbolic computation of conserved densities and fluxes for nonlinear systems of differential-difference equations*, M.S. thesis, Colorado School of Mines, Golden, Colorado (2003).
- [16] H. Eklund and W. Hereman, *DDEDensityFlux.m: A Mathematica package for the symbolic computation of conserved densities and fluxes for nonlinear systems of differential-difference equations* (2003).
- [17] A.S. Fokas, *Symmetries and integrability*, Stud. Appl. Math. 77 (1987), pp. 253–299.
- [18] A.S. Fokas and B. Fuchssteiner, *On the structure of symplectic operators and hereditary symmetries*, II Nuovo Cim. 28 (1980), pp. 229–303.
- [19] B. Fuchssteiner, *Application of hereditary symmetries to nonlinear evolution equations*, Nonl. Anal., Theory, Methods & Appls. 3 (1979), pp. 849–862.
- [20] B. Fuchssteiner and A.S. Fokas, *Symplectic structures, their Bäcklund transformations and hereditary symmetries*, Physica D 4 (1981), pp. 47–66.
- [21] B. Fuchssteiner, S. Ivanov, and W. Wiwianka, *Algorithmic determination of infinite-dimensional symmetry groups for integrable systems in 1+1 dimensions*, Math. Comp. Model. 25 (1997), pp. 91–100.
- [22] B. Fuchssteiner and W. Oevel, *The bi-Hamiltonian structure of some nonlinear fifth- and seventh-order differential equations and recursion formulas for their symmetries and conserved covariants*, J. Math. Phys. 23 (1982), pp. 358–363.
- [23] B. Fuchssteiner, W. Oevel, and W. Wiwianka, *Computer-algebra methods for investigation of hereditary operators of higher order soliton equations*, Comp. Phys. Comm. 44 (1987), pp. 47–55.
- [24] I.M. Gelfand and L.A. Dickey, *Asymptotic properties of the resolvent of Sturm-Liouville equations, and the algebra of Korteweg-de Vries equations*, Russ. Math. Surv. 30 (1975), pp. 77–113.
- [25] Ü. Göktaş, *Symbolic computation of conserved densities for systems of evolution equations*, M.S. thesis, Colorado School of Mines, Golden, Colorado (1996).
- [26] ———, *Algorithmic computation of symmetries, invariants and recursion operators for systems of nonlinear evolution and differential-difference equations*, Ph.D. diss., Colorado School of Mines, Golden, Colorado (1998).
- [27] Ü. Göktaş and W. Hereman, *Mathematica package InvariantSymmetries.m: Symbolic computation of conserved densities and generalized symmetries of nonlinear PDEs and differential-difference*

- equations, 1997; software available at <http://library.wolfram.com/infocenter/MathSource/570> and <http://inside.mines.edu/~whereman/software/invarsym>.
- [28] ———, *Symbolic computation of conserved densities for systems of nonlinear evolution equations*, J. Symb. Comp. 24 (1997), pp. 591–621.
- [29] ———, *Computation of conservation laws for nonlinear lattices*, Physica D 132 (1998), pp. 425–436.
- [30] ———, *Algorithmic computation of higher-order symmetries for nonlinear evolution and lattice equations*, Adv. Comp. Math. 11 (1999), pp. 55–80.
- [31] Ü. Göktaş, W. Hereman, and G. Erdmann, *Computation of conserved densities for systems of nonlinear differential-difference equations*, Phys. Lett. A 236 (1997), pp. 30–38.
- [32] W. Hereman, *Symbolic computation of conservation laws of nonlinear partial differential equations in multi-dimensions*, Int. J. Quant. Chem. 106 (2006), pp. 278–299.
- [33] W. Hereman, P.A. Adams, H.L. Eklund, M.S. Hickman, and B.M. Herbst, *Direct methods and symbolic software for conservation laws of nonlinear equations*, in *Advances in Nonlinear Waves and Symbolic Computation*, Z. Yan, ed., Nova Science Publishers, New York (2009), pp. 19–79.
- [34] W. Hereman, M. Colagrosso, R. Sayers, A. Ringler, B. Deconinck, M. Nivala, and M. Hickman, *Continuous and discrete homotopy operators and the computation of conservation laws*, in *Differential Equations with Symbolic Computation*, D. Wang and Z. Zheng, eds., Birkhäuser Verlag, 2005, pp. 249–285.
- [35] W. Hereman and Ü. Göktaş, *Integrability tests for nonlinear evolution equations*, in *Computer Algebra Systems: A Practical Guide*, M. Wester, ed., Wiley, New York, 1999, pp. 211–232.
- [36] W. Hereman, Ü. Göktaş, M. Colagrosso, and A. Miller, *Algorithmic integrability tests for nonlinear differential and lattice equations*, Comp. Phys. Comm. 115 (1998), pp. 428–446.
- [37] W. Hereman, J.A. Sanders, J. Sayers, and J.P. Wang, *Symbolic computation of polynomial conserved densities, generalized symmetries, and recursion operators for nonlinear differential-difference equations*, in *Group Theory and Numerical Analysis*, P. Winternitz, D. Gomez-Ullate, A. Iserles, D. Levi, P.J. Olver, R. Quispel, and P. Tempesta, eds., CRM Proc. & Lect. Ser. vol. 39, AMS, Providence, RI, 2005, pp. 267–282.
- [38] M. Hickman and W. Hereman, *Computation of densities and fluxes of nonlinear differential-difference equations*, Proc. Roy. Soc. Lond. A 459 (2003), pp. 2705–2729.
- [39] R. Hirota, *Direct methods in soliton theory*, in *Solitons*, R. Bullough and P. Caudrey, eds., Topics in Current Physics vol. 17, Springer-Verlag, New York, 1980, Ch. 5, pp. 157–176.
- [40] R. Hirota, *The Direct Method in Soliton Theory* [English translation], A. Nagai, J. Nimmo, and C. Gilson, transl. & eds., Cambridge University Press, Cambridge, UK (2004).
- [41] R. Hirota, B. Grammaticos, and A. Ramani, *Soliton structure of the Drinfel'd-Sokolov-Wilson equation*, J. Math. Phys. 27 (1986), pp. 1499–1505.
- [42] A. Karasu (Kalkanli), A. Karasu, and S.Yu. Sakovich, *A strange recursion operator for a new integrable system of coupled Korteweg-de Vries equations*, Acta Appl. Math. 83 (2004), pp. 85–94.
- [43] B.G. Konopelchenko, *Nonlinear Integrable Equations*, Lect. Notes in Phys. vol. 270, Springer Verlag, New York, PA (1981).
- [44] M. Lakshmanan and P. Kaliappan, *Lie transformations, nonlinear evolution equations, and Painlevé forms*, J. Math. Phys. 24 (1983), pp. 795–806.
- [45] G.L. Lamb, *Elements of soliton Theory*, Wiley, New York (1980).
- [46] P. Lax, *Integrals of nonlinear equations of evolution and solitary waves*, Comm. Pure Appl. Math. 21 (1968), pp. 467–490.
- [47] F. Magri, *A simple model of the integrable Hamiltonian equation*, J. Math. Phys. 19 (1978), pp. 1156–1162.
- [48] A.G. Meshkov, *Computer package for investigation of the complete integrability*, in *Proc. of Institute of Maths. of NAS of Ukraine* vol. 30, 2000, pp. 35–46.
- [49] A.V. Mikhailov and V.S. Novikov, *Perturbative symmetry approach*, J. Phys. A 35 (2002), pp. 4775–4790.
- [50] A.V. Mikhailov, A.B. Shabat, and V.V. Sokolov, *The symmetry approach to classification of integrable equations*, in *What Is Integrability?*, V.E. Zakharov, ed., Springer Verlag, Berlin Heidelberg, 1991, pp. 115–184.
- [51] A.V. Mikhailov, A.B. Shabat, and R.I. Yamilov, *The symmetry approach to the classification of non-linear equations. Complete lists of integrable systems*, Usp. Mat. Nauk. 42 (1987), pp. 3–53.
- [52] A.C. Newell, *Solitons in Mathematics and Physics*, CBMS-NSF Regional Conference Series in Applied Mathematics vol. 48, SIAM, Philadelphia, PA (1985).
- [53] P.J. Olver, *Evolution equations possessing infinitely many symmetries*, J. Math. Phys. 18 (1977), pp. 1212–1215.
- [54] ———, *Applications of Lie groups to Differential Equations*, Graduate Texts in Mathematics vol. 107, 2nd ed., Springer-Verlag, New York (1993).
- [55] J. Praught and R.G. Smirnov, *Andrew Lenard: A mystery unraveled*, Symmetry Integrability & Geometry: Meth. Appl. (SIGMA) 1 (2005), Paper 006, pp. 1–7.
- [56] J. Sanders and J.P. Wang, *Combining Maple and Form to decide on integrability questions*, Comp. Phys. Comm. 115 (1998), pp. 447–459.
- [57] ———, *Integrable systems and their recursion operators*, Nonl. Anal. 47 (2001), pp. 5213–5240.
- [58] ———, *On recursion operators*, Physica D 149 (2001), pp. 1–10.
- [59] J.P. Wang, *Symmetries and conservation laws of evolution equations*, Ph.D. diss., Thomas Stieltjes Institute for Mathematics, Amsterdam (1998).
- [60] ———, *A list of (1 + 1) dimensional integrable equations and their properties*, J. Nonl. Math. Phys. 9 (2002), pp. 213–233.
- [61] R. Willox, W. Hereman, and F. Verheest, *Complete integrability of a modified vector derivative nonlinear Schrödinger equation*, Physica Scripta 52 (1995), pp. 21–26.