Nonlinear Wave Equations: Analytic and Computational Techniques

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Preface

The field of nonlinear waves is an active mathematical research area with a long tradition and storied history. Starting with the observation of John Scott Russell in 1834 of the great "wave of translation," now called a *soliton*, nonlinear dispersive wave equations have proven over the years to be fundamental for modeling nonlinear wave phenomena in such diverse fields as fluid and gas dynamics, nonlinear optics, low-temperature physics, biology, and more. The discipline can be succinctly described as a study of waves usually resulting from a precise balance between nonlinearity and dispersion. While dispersion attenuates in physical problems, it does so weakly and typically over long spatial and temporal scales. Coupled with an intricate process of wave mixing due to nonlinearity, the analysis and computation of nonlinear dispersive waves becomes challenging and complicated. Furthermore, computing solitons and describing their elastic collisions requires mathematical methods that are quite distinct from those used to analyze and solve linear partial differential equations.

Major progress in the study of soliton equations came with the discovery of the "Inverse Scattering Transform" (IST) by Gardner, Greene, Kruskal, and Miura in 1967. As the name suggests, the IST combines techniques from inverse problems and scattering theory to solve a special class of nonlinear dispersive equations, called completely integrable systems, of which the Korteweg-de Vries (KdV) and nonlinear Schrödinger (NLS) equations are prototypical examples. The IST method can be viewed as a nonlinear analog of the Fourier transform. Not only does the IST allow one to solve the initial-value problem for nonlinear integrable equations for fairly general initial conditions, it has also provided unprecedented insight into how nonlinear and dispersive effects influence one another. Research in integrable systems has developed into a rich discipline at the crossroads between analysis, geometry, algebra, and mathematical physics. Recently discovered connections with the study of Riemann–Hilbert problems and Riemann surfaces are examples of such intersections.

Unfortunately, not every nonlinear dispersive system can be solved with the IST. Notably, the problem of modeling free surface irrotational waves in incompressible and inviscid fluids is beyond the scope of the IST. As a "parent" model for several of the most important problems in integrable systems though, one might question how phenomena seen in integrable systems manifest themselves in this more complicated, yet physically more realistic setting. Likewise, within the last several years, techniques from integrable systems have found their way into research of the free surface problem. Despite such efforts to solve a problem with a nearly 200 year history, the benefits and limitations of these novel lines of attack are still unclear.

With the purpose of bringing together researchers working on these different problems, a special session called "Nonlinear Waves and Integrable Systems" was organized at the AMS Western Sectional Meeting in Boulder, Colorado, in the spring of 2013. The organizers were fortunate in having such a diverse group of speakers agree to present their research and commit to write a research paper for this volume which attempts to capture the scope of the special session. While by no means exhaustive, this volume addresses and explains many of the major analytic and computational techniques used across several sub-disciplines of the study of nonlinear wave equations.

The volume begins with an article by Ben Herbst, Garrett Nieddu, and A. David Trubatch who return to questions from the genesis of the field of integrable systems. The authors study whether discretizations of continuous integrable systems inherit their dynamic properties, in this case "recurrence," from the integrability of the continuous model.

Next, a series of papers on techniques related to the IST is presented. Francesco Demontis, Cornelis van der Mee, and Federica Vitale study the scattering data of a defocusing Zakharov–Shabat system with non-zero boundary conditions. This problem is of particular relevance to the study of dispersive shock-waves, a subject of much recent interest.

The next two papers address questions related to the Novikov–Veselov (NV) equation, a two-dimensional generalization of the KdV equation. The first paper, co-authored by Ryan Croke, Jennifer L. Mueller, Michael Music, Peter Perry, Samuli Siltanen, and Andreas Stahel, is devoted to the use of the IST to study the NV equation. The authors illustrate the use of powerful techniques from inverse problems and scattering theory to determine for what types of initial conditions the NV equation is well posed. In the second paper, the stability of particular solutions to the NV equation is studied by Ryan Croke, Jennifer L. Mueller, and Andreas Stahel.

One last paper concerned with integrable systems is written by Gregory Lyng who studies the semi-classical limit of the focusing NLS equation showing the role of Riemann–Hilbert problems in investigating integrable systems.

The subsequent series of papers addresses questions on the free surface problem in fluids. The first paper by Jon Wilkening looks at a classic question on the correspondence of phenomena found in the KdV equation and the free surface problem from which the KdV equation is derived.

In the next paper, Katie Oliveras and Bernard Deconinck investigate modulation instabilities in the free surface problem. Modulational instabilities were first found in the NLS equation, which is derived from the free surface problem. Using a formulation of the free surface problem first proposed by Ablowitz, Fokas and Musslimani (AFM) – an approach strongly motivated by methods from integrable systems – the authors extend the results from the NLS equation to a more physically realistic model.

In the same spirit, Katie Oliveras and Vishal Vasan show how to use the AFM formulation to determine the height of free surface waves from pressure measurements along the "sea-floor". This work has practical importance for oceanographers, and represents a significant improvement over existing methodologies.

Finally, Jon Wilkening and Vishal Vasan investigate several competing numerical schemes for simulating the free surface problem. The merits and drawbacks of the AFM approach for numerical solutions are examined in detail, and superior alternatives to this approach are provided and explained.

The editors wish to thank the authors who contributed to this volume. Without their diligent efforts this volume would not exist. Likewise, we wish to express our gratitude to the anonymous reviewers who helped to make this volume a strong contribution to the nonlinear waves and broader mathematical community. Finally, we would like to thank all of the participants in the special session who helped to motivate the creation of this volume.

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