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Exact solitary wave solutions of coupled nonlinear evolution equations using MACSYMA

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A direct series method to find exact travelling wave solutions of nonlinear PDEs is applied to Hirota's system of coupled Korteweg–de Vries equations and to the sine-Gordon equation. The straightforward but lengthy algebraic computations to obtain single and multi-soliton solutions can be carried out with a symbolic manipulation package such as MACSYMA.

1. Introduction

The search for exact solutions of nonlinear PDEs becomes more and more attractive due to the availability of symbolic manipulation programs (MACSYMA, REDUCE, MATHEMATICA, SCRATCHPAD II, and the like) which allow to perform the tedious algebra common to direct methods on a main frame computer or on a PC.

In this paper we generate particular solutions of systems of nonlinear evolution (or wave equations) by a direct series method established by Hereman et al. [1–3]. This method allows to construct single and multi-solitary wave solutions and applies to single equations as well as to coupled systems. The knack of the method is to represent the solutions as infinite series in real exponentials that satisfy the linearized equations. The coefficients of these series must satisfy a highly nonlinear coupled system of recursion relations, which can be solved with any symbolic computer program. The series is then finally summed in closed form and an exact solution of the given system of nonlinear PDEs is obtained.

In section 2 we present the algorithm for the construction of a single solitary wave solution. In section 3 we apply it to a system of coupled Korteweg–de Vries (cKdV) equations [4,5]. In section 4 we show how the method can be generalized to account for N-soliton solutions using the sine-Gordon (SG) equation [6–9] as an example.

2. The algorithm

We outline the algorithm to construct *single* solitary wave solutions to systems of nonlinear evolution or wave equations. Space is lacking to give all the details which may be found in ref. [1–3].

Step 1: Given is a system of two coupled nonlinear PDEs,

$$\begin{aligned} &\mathcal{F}(u, v, u_t, u_x, v_t, v_x, u_{tx}, \ldots, u_{mx}, v_{nx}) = 0, \\ &\mathcal{G}(u, v, u_t, u_x, v_t, v_x, u_{tx}, \ldots, u_{px}, v_{qx}) = 0, \\ &\quad m, n, p, q \in \mathbb{N}, \end{aligned} \tag{1}$$

where $\mathcal{F}$ and $\mathcal{G}$ are supposed to be polynomials in their arguments and where $u_{nx} = \partial^n u/\partial x^n$.

Seeking travelling wave solutions for $u(x, t)$ and $v(x, t)$, we introduce the variable $\xi = x - ct$, where c is the constant velocity. The system (1) then transforms into a coupled system of nonlinear ODEs for $\phi(\xi) \equiv u(x, t)$ and $\psi(\xi) \equiv v(x, t)$. The resulting equations may be integrated with respect to ξ to reduce the order. For simplicity, we ignore integration constants, assuming that the solutions ϕ and ψ and their derivatives vanish at $\xi = \pm\infty$.

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Step 2: We expand ϕ and ψ in a power series

$$\phi = \sum_{n=1}^{\infty} a_n g^n, \quad \psi = \sum_{n=1}^{\infty} b_n g^n, \tag{2}$$

where $g(\xi) = \exp[-K(c)\xi]$ solves the linear part of at least one of the equations in the system. Hence, the wave number K is related to the velocity c by the dispersion law of (one of) the linearized equations. We substitute the expansions (2) into the full nonlinear system (1), rearrange the sums by using Cauchy's rule for multiple series [2] and equate the coefficient of g^n. This leads to a nonlinear system of coupled recursion relations for the coefficients a_n and b_n. Quite often, the relation $K(c)$ and appropriate scales on ϕ and ψ allow to simplify the recursion relations.

Step 3: Assuming that a_n and b_n are polynomials in n, we determine [2] their degrees δ_1 and δ_2. Next, we substitute

$$a_n = \sum_{j=0}^{\delta_1} A_j n^j, \quad b_n = \sum_{j=0}^{\delta_2} B_j n^j, \qquad n = 1,2,\ldots, \tag{3}$$

into the recursion relations. The sums are computed by using the (factored) expressions of the sums of powers of integers [2,3,10],

$$S_k = \sum_{i=1}^{n} i^k, \quad k = 0, 1, 2, \ldots. \tag{4}$$

For example,

$$S_0 = n, \qquad S_1 = \frac{(n+1)n}{2}, \qquad S_2 = \frac{n(n+1)(2n+1)}{6}, \ldots. \tag{5}$$

The algebraic (nonlinear) equations for the constant coefficients A_j and B_j are obtained by setting to zero the coefficients of the different powers of n. The problem is now completely algebraic and the unknowns A_j and B_j are readily obtained by solving the nonlinear system with e.g. MACSYMA.

Step 4: To find the closed form for ϕ and ψ we substitute eqs. (3) into (2). Hence,

$$\phi = \sum_{n=1}^{\infty} \sum_{j=0}^{\delta_1} A_j n^j g^n \equiv \sum_{j=0}^{\delta_1} A_j F_j(g), \qquad \psi = \sum_{n=1}^{\infty} \sum_{j=0}^{\delta_2} B_j n^j g^n \equiv \sum_{j=0}^{\delta_2} B_j F_j(g), \tag{6}$$

where

$$F_j(g) \equiv \sum_{n=1}^{\infty} n^j g^n, \quad j = 0, 1, 2, \ldots. \tag{7}$$

Using the relation [3]

$$F_{j+1}(g) = g F_j'(g), \quad j = 0, 1, 2, \ldots, \tag{8}$$

one can calculate any $F_j(g)$ starting from

$$F_0(g) = \frac{g}{1-g}. \tag{9}$$

For instance,

$$F_1(g) = \frac{g}{(1-g)^2}, \qquad F_2(g) = \frac{g(1+g)}{(1-g)^3}, \ldots. \tag{10}$$

Finally, we return to the original variables x and t to obtain the desired travelling wave solution of eq. (1).

3. Example 1: the coupled Korteweg–de Vries equations

We consider [4,5],

$$u_t - \alpha(6uu_x + u_{3x}) - 2\beta v v_x = 0, \tag{11a}$$

$$v_t + 3uv_x + v_{3x} = 0, \tag{11b}$$

where $\alpha, \beta \in \mathbb{R}$. These equations describe the interaction of two long waves with different dispersion laws in a nonlinear medium; e.g. shallow water waves in a channel.

One can show that both $u(x, t)$ and $v(x, t)$ travel with the same velocity c, hence we look for $\phi(\xi) \equiv u(x, t)$ and $\psi(\xi) \equiv v(x, t)$ with $\xi = x - ct$.

After one integration with respect to ξ, eq. (11) becomes

$$c\phi + 3\alpha\phi^2 + \alpha\phi_{2\xi} + \beta\psi^2 = 0, \tag{12a}$$

$$-c\psi_\xi + 3\phi\psi_\xi + \psi_{3\xi} = 0, \tag{12b}$$

ignoring integration constants.

Substitution of $g(\xi) = \exp[-K(c)\xi]$ into the linear parts of eq. (12) leads to two dispersion laws, $c = -\alpha K^2$ and $c = K^2$. The solutions ϕ and ψ can only be built up from the same real exponential $g(\xi)$ if $\alpha = -1$, which turns out to be a special case. A detailed study of all the cases reveals that either one of the dispersion laws will lead to the same result. Therefore, we may proceed with $K = \sqrt{c}$, where $c > 0$.

Upon substitution of the conveniently scaled expansions,

$$\phi = \frac{c}{3}\sum_{n=1}^{\infty} a_n g^n, \qquad \psi = \frac{c}{\sqrt{3|\beta|}}\sum_{n=1}^{\infty} b_n g^n, \tag{13}$$

into eq. (12), we get the coupled recursion relations:

$$(1 + \alpha n^2)a_n + \sum_{l=1}^{n-1}(\alpha a_l a_{n-l} + e b_l b_{n-l}) = 0, \tag{14a}$$

$$n(n^2-1)b_n + \sum_{l=1}^{n-1} l b_l a_{n-l} = 0, \tag{14b}$$

both for $n \geq 2$. Furthermore, $(1+\alpha)a_1 = 0$, b_1 is arbitrary, and $e = \pm 1$ if $|\beta| = \pm\beta$. We have to distinguish two cases:

Case 1: $\alpha \neq -1$

From $a_1 = 0$ it follows that $a_{2n-1} = 0$ and $b_{2n} = 0$ ($n = 1, 2, \ldots$). We compute [2] the degrees $\delta_1 = 1$ and $\delta_2 = 0$ of a_{2n} and b_{2n-1}, respectively, and we solve for the coefficients A_1, A_0 and B_0 occurring in $a_{2n} = A_1 n + A_0$; $b_{2n-1} = B_0$, ($n = 1, 2, \ldots$). This calculation can be done by hand or with a symbolic program such as MACSYMA. The result is

$$a_{2n} = 24n(-1)^{n+1}a_0^n, \tag{15a}$$

$$b_{2n-1} = (-1)^{n-1}b_1 a_0^{n-1}, \tag{15b}$$

for $n = 1, 2, \ldots$ and with $a_0 = -eb_1^2/[24(4\alpha+1)] > 0$. Hence, β and $4\alpha+1$ must have opposite signs. Substituting eqs. (15) into (13) and using the formulae for F_0 and F_1, given in eqs. (9) and (10), respectively, we obtain

$$\phi = 8c\sum_{n=1}^{\infty}(-1)^{n+1}n(a_0 g^2)^n = \frac{8ca_0 g^2}{(1+a_0g^2)^2}, \tag{16a}$$

$$\psi = \frac{c}{\sqrt{3|\beta|}}\sum_{n=0}^{\infty}(-1)^n b_1 a_0^n g^{2n+1} = \frac{cb_1 g}{\sqrt{3|\beta|}(1+a_0g^2)}. \tag{16b}$$

Returning to the original variables we finally have

$$u(x,t) = 2c\,\mathrm{sech}^2\left[\sqrt{c}(x-ct)+\delta\right], \tag{17a}$$

$$v(x,t) = \pm c\sqrt{\frac{-2(4\alpha+1)}{\beta}} \times \mathrm{sech}\left[\sqrt{c}(x-ct)+\delta\right], \tag{17b}$$

with $\delta = \frac{1}{2}\ln|1/a_0| = \frac{1}{2}\ln|24(4\alpha+1)/b_1^2|$. Note that the cKdV equations remain indeed invariant for reversing the sign of $v(x,t)$. For $\alpha = -\frac{1}{4}$, clearly $b_n = 0$ ($n = 0, 1, \ldots$); so $v = 0$ and the cKdV equations reduce to the KdV equation. For the special case $\alpha = \frac{1}{2}$ the cKdV equations are known to pass the Painlevé test, the system is completely integrable and an N-soliton solution can be constructed [4,5].

Case 2: $\alpha = -1$

Apparently a_1 is arbitrary and so is b_1. One can calculate all a_n and b_n recursively in the hope to obtain a general closed form, which is not known yet. All attempts failed so far, it seems only possible to obtain a closed form solution provided $b_1^2 = \frac{1}{2}a_1^2$ and for $\beta > 0$ (i.e. $e = 1$). Under these conditions one obtains

$$a_n = 12n(-1)^{n+1}a_0^n, \tag{18a}$$

$$b_n^2 = \tfrac{1}{2}a_n^2 = 72n^2 a_0^{2n}, \tag{18b}$$

for $n = 1, 2, \ldots$, and with $a_0 = a_1/12$. Substitution of eqs. (18) into (13) leads, upon return to the original variables, to

$$u(x, t) = c\ \mathrm{sech}^2\left[\frac{1}{2}\sqrt{c}\,(x - ct) + \delta\right], \tag{19a}$$

$$\begin{aligned} v(x, t) &= \frac{3}{\sqrt{6|\beta|}} u(x, t) \\ &= \frac{3c}{\sqrt{6|\beta|}} \mathrm{sech}^2\left[\frac{1}{2}\sqrt{c}\,(x - ct) + \delta\right], \end{aligned} \tag{19b}$$

with $\delta = \frac{1}{2}\ln|12/a_1|$. Observe that for $v(x, t) = (3/\sqrt{6|\beta|})u(x, t)$ both equations in (11) become identical to the KdV equation, $u_t + 3uu_x + u_{3x} = 0$.

4. Example 2: the sine-Gordon equation

The generalization of our method to construct N-soliton solutions is straightforward but the calculations become rather cumbersome and are therefore very suitable for symbolic manipulators such as MACSYMA. In this section we will just treat an illuminating example. Details about the method and another example, namely the ubiquitous KdV equation for which we constructed the N-soliton solution, may be found in ref. [2,3].

The SG in light-cone coordinates,

$$u_{xt} = \sin u, \tag{20}$$

describes e.g. the propagation of crystal dislocations, superconductivity in a Josephson junction, and ultra short optical pulse propagation in a resonant medium [6–8]. In mathematics, the SG is long known in the differential geometry of surfaces of constant negative Gaussian curvature [7,8].

To apply our series method we must remove the transcendental nonlinearity in eq. (20) and transform it into a coupled system with strictly polynomial terms [2]:

$$\Phi_{xt} - \Phi - \Phi\Psi = 0, \tag{21a}$$

$$2\Psi + \Psi^2 + \Phi_t^2 = 0, \tag{21b}$$

where $\Phi = u_x$ and $\Psi = \cos u - 1$. For the construction of the single solitary wave solution, one proceeds as in the previous example. In summary, upon substitution of the scaled expansions,

$$\begin{aligned} \Phi(x - ct) &= \phi(\xi) = \frac{1}{\sqrt{-c}} \sum_{n=1}^{\infty} a_n g^n(\xi), \\ \Psi(x - ct) &= \psi(\xi) = \sum_{n=1}^{\infty} b_n g^n(\xi), \end{aligned} \tag{22}$$

with $g(\xi) = \exp[-K(c)\xi]$ into eqs. (21), one obtains the coupled recursion relations,

$$(n^2 - 1)a_n - \sum_{l=1}^{n-1} a_l b_{n-l} = 0, \tag{23a}$$

$$2b_n + \sum_{l=1}^{n-1} \left(b_l b_{n-l} + l(n-l)a_l a_{n-1}\right) = 0, \tag{23b}$$

for $n \geq 2$, and where a_1 is arbitrary and $b_1 = 0$. We also used the dispersion law $K^2 = -c$ for $c < 0$, obtained from the linear part of eq. (21a), to simplify eqs. (23).

The solution of eqs. (23) is

$$a_{2n} = 0, \quad b_{2n} = 8(-1)^n n a_0^{2n}, \qquad n = 1, 2, \ldots, \tag{24a}$$

$$a_{2n+1} = 4(-1)^n a_0^{2n+1}, \quad b_{2n+1} = 0, \quad n = 0, 1, \ldots, \tag{24b}$$

with $a_0 = \frac{1}{4}a_1 > 0$.

Substituting eqs. (24) into (22) and using the formulae for F_0 and F_1, we obtain

$$\begin{aligned} \phi &= \frac{4}{\sqrt{-c}} \sum_{n=0}^{\infty} (-1)^n (a_0 g)^{2n+1} \\ &= \frac{4}{\sqrt{-c}} \frac{a_0 g}{1 + (a_0 g)^2}, \end{aligned} \tag{25a}$$

$$\psi = -8 \sum_{n=1}^{\infty} (-1)^{n+1} n (a_0 g)^{2n} = \frac{-8(a_0 g)^2}{\left[1 + (a_0 g)^2\right]^2}. \tag{25b}$$

Returning to the original variables yields

$$\begin{aligned} &\cos[u(x, t)] - 1 \\ &\quad = 1 - 2\ \mathrm{sech}^2\left[\frac{1}{\sqrt{-c}}(x - ct) + \delta\right], \end{aligned} \tag{26a}$$

$$u(x,t) = \pm \frac{2}{\sqrt{-c}} \int \operatorname{sech}\left[\frac{1}{\sqrt{-c}}(x-ct)+\delta\right] \mathrm{d}x$$
$$= \pm 4 \arctan\left\{\exp\left[\frac{1}{\sqrt{-c}}(x-ct)+\delta\right]\right\}, \tag{26b}$$

with $\delta = \ln|4/a_1|$. This is the well-known kink-type solution [6–9] of the SG equation.

Let us now turn to the construction of the N-soliton solution of eq. (20), for which various techniques are available [7,8]. Neither one of these methods is very transparent in explaining how N-solitons are built up from N real exponentials. In the spirit of our earlier work [2], we focus on this aspect of soliton generation and show how multi-solitons evolve from the mixing of real exponential solutions of the underlying linear equation.

As a generalization of eqs. (22) we substitute

$$\Phi^{(1)} = \sum_{i=1}^{N} c_i g_i(x,t) = \sum_{i=1}^{N} c_i a_i \exp(K_i x - \Omega_i t), \tag{27}$$

into the linear part of eq. (21a), implying that

$$\Omega_i = -\frac{1}{K_i}, \quad i = 1, 2, \ldots, N. \tag{28}$$

The amplitudes a_i could be absorbed into the exponentials to account for the arbitrary phase factors; the constants $c_i(K_i)$ will be chosen later. Observe that the starting term in the expansion of Ψ, say $\Psi^{(2)}$, must be of the form

$$\Psi^{(2)} = \sum_{i=1}^{N}\sum_{j=1}^{N} d_{ij} g_i g_j = \sum_{i=1}^{N}\sum_{j=1}^{N} d_{ij} a_i a_j \times \exp\left[(K_i+K_j)x - (\Omega_i+\Omega_j)t\right], \tag{29}$$

so that $-2\Psi^{(2)}$ balances with

$$\Phi_t^{2(1)} = \sum_{i=1}^{N}\sum_{j=1}^{N} c_i c_j \Omega_i \Omega_j g_i g_j = \sum_{i=1}^{N}\sum_{j=1}^{N} c_i c_j a_i a_j \Omega_i \Omega_j \times \exp\left[(K_i+K_j)x - (\Omega_i+\Omega_j)t\right] \tag{30}$$

in eq. (21b). Thus, using the dispersion law (28), we find

$$d_{ij} = -\tfrac{1}{2} c_i c_j \Omega_i \Omega_j = -\frac{1}{2}\frac{c_i c_j}{K_i K_j} = \frac{1}{2} c_i c_j \frac{\Omega_i + \Omega_j}{K_i + K_j}. \tag{31}$$

Note that any term in the expansion of Φ (Ψ, respectively) will only have an odd (even, respectively) number of g's, which is in agreement with eq. (25a) and (25b), respectively.

The analogue of the first term in eq. (23a), i.e. $(n^2-1)a_n$, will result from the action of the linear operator

$$L\bullet = \frac{\partial^2 \bullet}{\partial x \partial t} - 1\bullet \tag{32}$$

on the $(2n+1)$-th term in the expansion of Φ, namely,

$$\Phi^{(2n+1)} = \underbrace{\sum_{i=1}^{N}\sum_{j=1}^{N}\cdots\sum_{s=1}^{N}}_{2n+1 \text{ summations}} c_{ij\ldots s} g_i g_j \cdots g_s,$$
$$n = 0, 1, \ldots. \tag{33}$$

The analogue of the second term in eq. (23a) will, in its most symmetric form, look like

$$\frac{1}{2}\sum_{l=0}^{n-1} \Phi^{(2l+1)}\underbrace{(K_i, K_j, \ldots, K_o)}_{2l+1 \text{ arguments}} \times \Psi^{(2n-2l)}\underbrace{(K_p, K_q, \ldots, K_s)}_{2(n-l) \text{ arguments}} + \Psi^{(2n-2l)}\underbrace{(K_i, K_j, \ldots, K_l)}_{2(n-l) \text{ arguments}} \times \Phi^{(2l+1)}\underbrace{(K_m, K_n, \ldots, K_s)}_{2l+1 \text{ arguments}}. \tag{34}$$

The analogue of eq. (23b) reads

$$\Psi^{(2n)} = \underbrace{\sum_{i=1}^{N}\sum_{j=1}^{N}\cdots\sum_{r=1}^{N}}_{2n \text{ summations}} d_{ij\ldots r} g_i g_j \cdots g_r$$
$$= -\frac{1}{2}\sum_{l=1}^{n-1} \Phi^{(2l)} \underbrace{(K_i, K_j, \ldots, K_n)}_{2l \text{ arguments}}$$
$$\times \Psi^{(2n-2l)} \underbrace{(K_o, K_p, \ldots, K_r)}_{2(n-l) \text{ arguments}}$$
$$+ \sum_{l=0}^{n-1} \Phi_t^{(2l+1)} \underbrace{(K_i, K_j, \ldots, K_o)}_{2l+1 \text{ arguments}}$$
$$\times \Phi_t^{(2n-2l-1)} \underbrace{(K_p, K_q, \ldots, K_s)}_{2(n-l)-1 \text{ arguments}},$$
$$n = 1, 2, \ldots, \tag{35}$$

and allows to subsequently determine the coefficients $d_{ij\ldots r}$.

To make this less obscure, let us give an example. $\Phi^{(1)}$ and $\Psi^{(2)}$ being computed in eq. (27), and eq. (29) with (31), we equate

$$L\Phi^{(3)} = \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} \left[-(\Omega_i + \Omega_j + \Omega_k) \times (K_i + K_j + K_k) - 1 \right] c_{ijk} g_i g_j g_k$$
$$= \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} (K_i + K_j)(K_j + K_k) \times (K_k + K_i) \frac{c_{ijk}}{K_i K_j K_k} g_i g_j g_k, \tag{36}$$

to

$$\tfrac{1}{2}\left\{ \Phi^{(1)}(K_i)\Psi^{(2)}(K_j, K_k) + \Psi^{(2)}(K_i, K_j)\Phi^{(1)}(K_k) \right\}$$
$$= -\frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} (K_i + K_j)(K_j + K_k) \times (K_k + K_i) \frac{c_i c_j c_k}{K_i K_j K_k} g_i g_j g_k, \tag{37}$$

thus

$$c_{ijk} = \frac{-1}{4(K_i + K_j)(K_j + K_k)}, \tag{38}$$

provided we set $c_i = 1$. After some lengthy calculations, carried out with MACSYMA, we obtain

$$c_{ij\ldots s} = \frac{(-1)^n}{4^n (K_i + K_j)(K_j + K_k)\cdots(K_r + K_s)},$$
$$n = 0, 1, \ldots,$$
$$d_{ij\ldots r} = \frac{-2(-1)^n(\Omega_i + \Omega_j + \cdots + \Omega_r)}{4^n (K_i + K_j)(K_j + K_k)\cdots(K_q + K_r)},$$
$$n = 1, 2, \ldots. \tag{38}$$

The final objective is then to write

$$\Phi = \sum_{n=0}^{\infty} \Phi^{(2n+1)}$$
$$= \sum_{i=1}^{N} g_i + \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\left(\frac{-1}{4}\right) \times \frac{g_i g_j g_k}{(K_i + K_j)(K_j + K_k)} + \cdots$$
$$+ \sum_{i=1}^{N}\sum_{j=1}^{N}\cdots\sum_{s=1}^{N}\left(\frac{-1}{4}\right)^n \times \frac{g_i g_j \cdots g_s}{(K_i + K_j)(K_j + K_k)\cdots(K_r + K_s)} + \cdots, \tag{39}$$

and a similar expression for Ψ, in their closed forms. Various authors [2,7,9] have shown that this can be done by introducing the $N \times N$ identity matrix I and the $N \times N$ matrix B with elements

$$B_{ij} = \frac{1}{2}\frac{\sqrt{a_i a_j}}{(K_i + K_j)} \times \exp\left\{ \tfrac{1}{2}\left[(K_i + K_j)x - (\Omega_i + \Omega_j)t \right] \right\}. \tag{40}$$

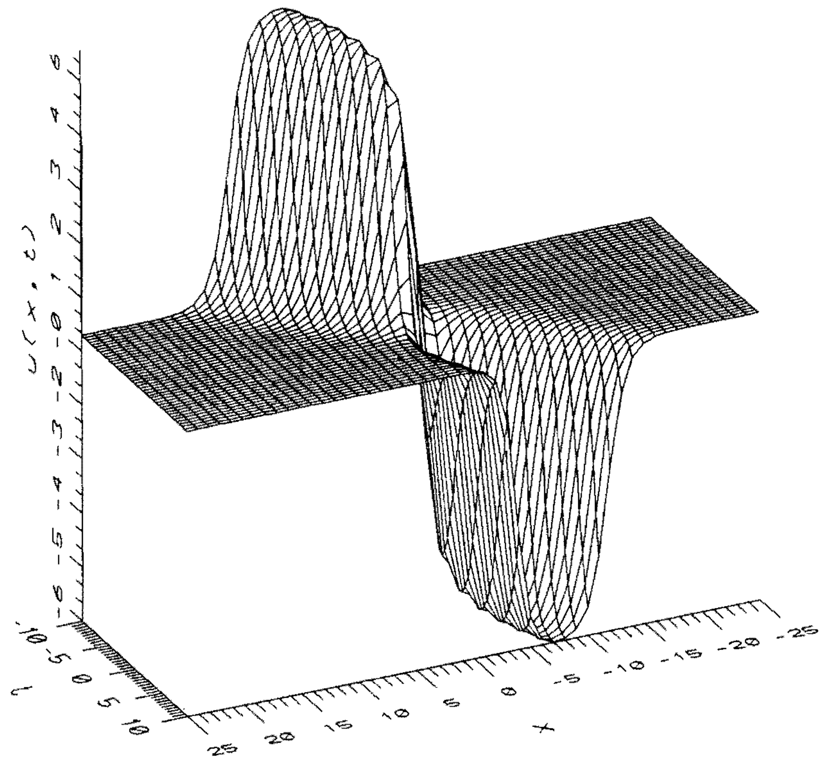

Fig. 1. The two-soliton solution of the sine-Gordon equation.

The N-soliton solution to the SG equation (21) is then found to be

$$\Phi(x, t) = 4[\mathrm{Tr}(\arctan B)]_x, \tag{41a}$$

$$\Psi(x, t) = -2\{\ln[\det(I + B^2)]\}_{xt}, \tag{41b}$$

where Tr stands for trace. Finally, with $\Phi = u_x$ we have

$$u(x, t) = \pm 4\ \mathrm{Tr}(\arctan B) = \pm\left(\frac{2}{\mathrm{i}}\right) \mathrm{Tr}\left\{\ln\left[\frac{I + \mathrm{i}B}{I - \mathrm{i}B}\right]\right\}, \tag{42}$$

which satisfies eq. (20). This solution is in agreement with the one obtained in e.g. ref. [9].

It is left as an exercise to the reader to verify that for $N = 1$ eq. (42) simplifies to eq. (26b). The two-soliton solution,

$$u(x, t) = 4 \arctan\left\{\left(\frac{K_1 + K_2}{K_1 - K_2}\right)\left[\{\exp[K_1 x - \Omega_1 t + \delta_1] - \exp[K_2 x - \Omega_2 t + \delta_2]\}\{1 + \exp[(K_1 + K_2)x - (\Omega_1 + \Omega_2)t + \delta_1 + \delta_2]\}^{-1}\right]\right\}, \tag{43}$$

is found by setting $N = 2$ in eq. (42). This solution is shown in fig. 1 or $K_1 = 1$, $K_2 = \sqrt{2}$, thus $\Omega_1 = 1$, $\Omega_2 = -\frac{1}{2}\sqrt{2}$, and with $\delta_1 = \delta_2 = 0$. A further discussion of other kink and anti-kink solutions of the SG equation may be found in ref. [6–8].

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