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SOLITARY WAVE SOLUTIONS OF COUPLED NONLINEAR EVOLUTION EQUATIONS USING MACSYMA

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Abstract

A direct series method to find exact travelling wave solutions of nonlinear PDEs is applied to Hirota's system of coupled KdV equations. The straightforward but lengthy algebraic computations are carried out with MACSYMA.

1. Introduction

The search for exact solutions of nonlinear PDEs becomes more and more attractive due to the availability of symbolic manipulation programs (MACSYMA, REDUCE, MATHEMATICA, SCRATCHPAD II, DERIVE and the like) which allow to perform the tedious algebra common to direct methods.

In this paper we generate particular solutions of systems of nonlinear evolution and wave equations by a direct series method established by Hereman *et al* [1-4]. This method allows to construct single and multi-solitary wave solutions and applies to single equations as well as to coupled systems. The knack of the method is to represent the solutions as infinite series in real exponentials that satisfy the linearized equations. The coefficients of these series must satisfy a highly nonlinear coupled system of recursion relations, which can be solved with any symbolic computer program. The series is then finally summed in closed form and an exact solution of the given system of nonlinear PDEs is obtained.

In Section 2 we outline the algorithm and in Section 3 we construct a single solitary wave solution of a system of coupled Korteweg-de Vries (cKdV) equations [5,6].

2. The algorithm

Step 1: Given is a system of two coupled nonlinear PDEs,

$$\begin{aligned} \mathcal{F}(u, v, u_t, u_x, v_t, v_x, u_{tx}, ..., u_{mx}, v_{nx}) &= 0, \\ \mathcal{G}(u, v, u_t, u_x, v_t, v_x, u_{tx}, ..., u_{px}, v_{qx}) &= 0, \end{aligned} \quad (m, n, p, q \in \mathbb{N}), \tag{1}$$

where $\mathcal{F}$ and $\mathcal{G}$ are supposed to be polynomials in their arguments and where $u_{nx} = \frac{\partial^n u}{\partial x^n}$.

Seeking travelling wave solutions $u(x, t)$ and $v(x, t)$, we introduce the variable $\xi = x - ct$, where c is the constant velocity. The system (1) then transforms into a coupled system of nonlinear ODEs for $\phi(\xi) \equiv u(x, t)$ and $\psi(\xi) \equiv v(x, t)$. The resulting equations may be integrated with respect to ξ to reduce the order. For simplicity, we ignore integration constants and assume that the solutions ϕ and ψ and their derivatives vanish at $\xi = \pm\infty$.

Step 2: We expand ϕ and ψ in a power series

$$\phi = \sum_{n=1}^{\infty} a_n \, g^n, \quad \psi = \sum_{n=1}^{\infty} b_n \, g^n, \tag{2}$$

where $g(\xi) = \exp(-K(c)\xi)$ solves the linear part of at least one of the equations in the system. Hence, the wave number K is related to the velocity c by the dispersion law of (one of) the linearized equations. We substitute the expansions (2) into the full nonlinear system (1), rearrange the sums by using Cauchy's rule for multiple series [2] and equate the coefficient of g^n. This leads to a nonlinear system of coupled recursion relations for the coefficients a_n and b_n. Quite often, the relation $K(c)$ and appropriate scales on ϕ and ψ allow to simplify the recursion relations.

Step 3: Assuming that a_n and b_n are polynomials in n, we determine [2] their degrees δ_1 and δ_2. Next, we substitute

$$a_n = \sum_{j=0}^{\delta_1} A_j \, n^j, \quad b_n = \sum_{j=0}^{\delta_2} B_j \, n^j, \tag{3}$$

into the recursion relations. The sums are computed by using the (factored) expressions of the sums of powers of integers [2,7], $S_k = \sum_{i=1}^{n} i^k$. For example, $S_1 = \frac{(n+1)n}{2}$, $S_2 = \frac{n(n+1)(2n+1)}{6}$, etc. The algebraic (nonlinear) equations for the constant coefficients A_j and B_j are obtained by setting to zero the coefficients of the different powers of n. The problem is now completely algebraic and the unknowns A_j and B_j are readily obtained with e.g. MACSYMA.

Step 4: To find the closed form for ϕ and ψ we insert (3) in (2). Hence,

$$\phi = \sum_{n=1}^{\infty} \sum_{j=0}^{\delta_1} A_j \, n^j \, g^n \equiv \sum_{j=0}^{\delta_1} A_j \, F_j(g), \quad \psi = \sum_{n=1}^{\infty} \sum_{j=0}^{\delta_2} B_j \, n^j \, g^n \equiv \sum_{j=0}^{\delta_2} B_j \, F_j(g), \tag{4}$$

with $F_j(g) \equiv \sum_{n=1}^{\infty} n^j g^n$, $(j = 0, 1, 2, ...)$. Using the relation [3]: $F_{j+1}(g) = gF_j'(g)$, $(j = 0, 1, 2, ...)$, one can derive any $F_j(g)$ starting from $F_0(g) = \frac{g}{1-g}$. For example, $F_1(g) = \frac{g}{(1-g)^2}$, $F_2(g) = \frac{g(1+g)}{(1-g)^3}$, etc. Finally, we return to the original variables x and t and obtain the desired travelling wave solution of (1).

3. Example: The coupled Korteweg-de Vries (cKdV) equations

We consider [5,6],

$$u_t - \alpha(6uu_x + u_{3x}) - 2\beta vv_x = 0, \tag{7a}$$

$$v_t + 3uv_x + v_{3x} = 0, \tag{7b}$$

where $\alpha, \beta \in \mathbb{R}$. These equations describe the interaction of two long waves with different dispersion laws in a nonlinear medium.

One can prove that both $u(x,t)$ and $v(x,t)$ travel with the same velocity c, hence $\phi(\xi) \equiv u(x,t)$ and $\psi(\xi) \equiv v(x,t)$ with $\xi = x - ct$. After one integration with respect to ξ, (7) becomes

$$c\phi + 3\alpha\phi^2 + \alpha\phi_{2\xi} + \beta\psi^2 = 0, \tag{8a}$$

$$-c\psi_\xi + 3\phi\psi_\xi + \psi_{3\xi} = 0, \tag{8b}$$

ignoring integration constants.

Substitution of $g = \exp(-K(c)\xi)$ into the linear parts of (8) leads to two dispersion laws, $c = -\alpha K^2$ and $c = K^2$. The solutions ϕ and ψ can only be built up from the same real exponential if $\alpha = -1$, which turns out to be a special case. A detailed study of all the cases reveals that either one of the dispersion laws will lead to the same result. Therefore, we may proceed with $K = \sqrt{c}, \ c > 0$.

Upon substitution of the conveniently scaled expansions,

$$\phi = \frac{c}{3} \sum_{n=1}^{\infty} a_n \, g^n, \quad \psi = \frac{c}{\sqrt{3|\beta|}} \sum_{n=1}^{\infty} b_n \, g^n, \tag{9}$$

into (8), we get the coupled recursion relations:

$$(1 + \alpha \, n^2) \, a_n + \sum_{l=1}^{n-1} (\alpha \, a_l \, a_{n-l} + e \, b_l \, b_{n-l}) = 0, \tag{10a}$$

$$n \, (n^2 - 1) \, b_n + \sum_{l=1}^{n-1} l \, b_l \, a_{n-l} = 0, \tag{10b}$$

for $n \geq 2$ and with $(1 + \alpha)a_1 = 0$, $\ b_1$ arbitrary, and where $e = \pm 1$ if $|\beta| = \pm\beta$. We have to distinguish two cases:

CASE 1: $\alpha \neq -1$

Since $a_1 = 0$ it follows that $a_{2n-1} = 0$, and $b_{2n} = 0$, $(n = 1, 2, ...)$. We compute [2] the degrees $\delta_1 = 1$ and $\delta_2 = 0$ of a_{2n} and b_{2n-1}, respectively, and we solve for the coefficients A_1, A_0 and B_0

occurring in $a_{2n} = A_1 n + A_0$; $b_{2n-1} = B_0$, $(n = 1, 2, ...)$. This calculation can be done by hand or with a symbolic program such as MACSYMA. The result is

$$a_{2n} = 24\, n\, (-1)^{n+1}\, {a_0}^n, \tag{11a}$$
$$b_{2n-1} = (-1)^{n-1}\, b_1\, {a_0}^{n-1}, \tag{11b}$$

for $n = 1, 2, ...$ and with $a_0 = -e{b_1}^2/24(4\alpha + 1) > 0$. Hence, β and $4\alpha + 1$ must have opposite signs. Substituting (11) in (9) and using the formulae for F_0 and F_1 following (4), we obtain

$$\phi = 8\, c \sum_{n=1}^{\infty} (-1)^{n+1}\, n\, (a_0\, g^2)^n = \frac{8\, c\, a_0\, g^2}{(1 + a_0\, g^2)^2}, \tag{12a}$$
$$\psi = \frac{c}{\sqrt{3|\beta|)}} \sum_{n=0}^{\infty} (-1)^n\, b_1\, {a_0}^n\, g^{2n+1} = \frac{c\, b_1\, g}{\sqrt{3|\beta|)}(1 + a_0\, g^2)}. \tag{12b}$$

Returning to the original variables gives

$$u(x, t) = 2\, c\, \mathrm{sech}^2[\sqrt{c}(x - ct) + \delta], \tag{13a}$$
$$v(x, t) = \pm c \sqrt{\frac{-2(4\alpha + 1)}{\beta}}\, \mathrm{sech}[\sqrt{c}(x - ct) + \delta], \tag{13b}$$

with $\delta = \frac{1}{2} \ln |24(4\alpha + 1)/{b_1}^2|$. Note that the cKdV equations remain indeed invariant for reversing the sign of v. For $\alpha = \frac{-1}{4}$, clearly $b_n = 0$, $(n = 0, 1, ...)$; so $v = 0$ and the cKdV equations reduce to the KdV with $\alpha = 6$. For the special case $\alpha = \frac{1}{2}$ the cKdV equations are known to pass the Painlevé test, the system is completely integrable and an N-soliton solution can be constructed [5,6].

CASE 2: $\alpha = -1$

Apparently a_1 is arbitrary and so is b_1. Recursively, one can calculate all a_n and b_n in the hope to obtain a general closed form, which is not known yet. All attempts failed so far, it seems only possible to obtain a closed form solution provided ${b_1}^2 = {a_1}^2/2$ and $\beta > 0$ (i.e. $e = 1$), in which case we obtain

$$a_n = 12\, n\, (-1)^{n+1}\, {a_0}^n, \tag{14a}$$
$${b_n}^2 = \frac{{a_n}^2}{2} = 72\, n^2\, {a_0}^{2n}, \tag{14b}$$

for $n = 1, 2, ...$, and with $a_0 = a_1/12$. Substitution of (14) into (9) leads, upon return to the original variables, to

$$u(x, t) = c\, \mathrm{sech}^2[\frac{1}{2}\sqrt{c}(x - ct) + \delta], \tag{15a}$$
$$v(x, t) = \frac{3}{\sqrt{6|\beta|}}\, u(x, t) = \frac{3\, c}{\sqrt{6|\beta|}}\, \mathrm{sech}^2[\frac{1}{2}\sqrt{c}(x - ct) + \delta], \tag{15b}$$

with $\delta = \frac{1}{2} \ln |12/a_1|$. Observe that for $v(x, t) = \frac{3}{\sqrt{6\beta}}\, u(x, t)$ both equations in (7) become identical to the KdV equation with $\alpha = 3$.

References

[1] Hereman W, Korpel A and Banerjee P P, Wave Motion **7**, 283-90 (1985)

[2] Hereman W, Banerjee P P, Korpel A, Assanto G, Van Immerzeele A and Meerpoel A, J. Phys. A: Math. Gen. **19**, 607-28 (1986)

[3] Hereman W, *The construction of implicit and explicit solitary wave solutions of nonlinear partial differential equations*, Proc. Math. Soc. Egypt **57** (1988) to appear. Also: Proc. Conf. Appl. Math., Cairo, Egypt, January 3-6, 1987, pp. 291-312 (1987)

[4] Hereman W and Takaoka M, *Solitary wave solutions of nonlinear evolution and wave equations using a direct method and MACSYMA*, submitted to J. Phys. A: Math. Gen. (1990)

[5] Hirota R and Satsuma J, Phys. Lett. A **85**, 407-10 (1981)

[6] Satsuma J and Hirota R, J. Phys. Soc. Jpn. **51**, 3390-7 (1982)

[7] Spiegel M R, *Mathematical Handbook of Formulas and Tables, Shaum's Outline Series*, McGraw-Hill, New York (1980)