The Tanh Method: A Tool to Solve Nonlinear Partial Differential Equations with Symbolic Software

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Abstract

The hyperbolic tangent (tanh) method is a powerful technique to symbolically compute traveling waves solutions of one-dimensional nonlinear wave and evolution equations. In particular, the method is well suited for problems where dispersion, convection, and reaction-diffusion phenomena play an important role.
The technique is outlined for the computation of closed-form tanh-solutions for nonlinear partial differential equations and ordinary differential equations. Basic examples illustrate the key steps of the method.
A brief survey is given of symbolic software packages that compute traveling wave solutions in closed form.

Keywords: Nonlinear PDEs, Exact Solutions, Nonlinear Waves, Tanh Method, Symbolic Software.

1. INTRODUCTION

Nonlinear wave phenomena frequently appear in many areas of the natural sciences such as fluid dynamics [1], chemistry (chemical kinetics involving reactions [2]), mathematical biology (population dynamics [3]), solid state physics (lattice vibrations [4]), etc..
Due to the increasing interest in finding exact solutions for those problems, a whole range of analytical solution methods are now available. One of those methods is the tanh method, which was developed some years ago [4]-[7]. It turned out that this method is well suited for problems where dispersive effects, reaction-diffusion phenomena, and convection play an important role. For a wide variety of nonlinear ordinary and partial differential equations (ODEs and PDEs), one can find exact [5, 7] as well as approximate solutions [6] in a straightforward and systematic way.
For completeness, we should mention that this technique is restricted to the search of traveling wave waves. Thus, we essentially deal with one-dimensional shock waves (kink type) and solitary-wave (pulse type) solutions in a moving frame of reference. Based on the tanh method and its generalizations, several symbolic software programs have been developed to find exact traveling wave solutions [8].

2. OUTLINE OF THE TANH METHOD

We briefly discuss the tanh method. We refer to [4]-[7] for more details . The nonlinear wave and evolution equations we want to investigate (for simplicity, in one dimension) are commonly written as

$$u_t = G(u, u_x, u_{xx}, \cdots) \;\text{ or }\; u_{tt} = G(u, u_x, u_{xx}, \cdots). \tag{1}$$

We like to know if these equations admit exact traveling wave solutions and how to compute them.
The first step is to combine the independent variables, x and t, into a new variable, $\xi = k(x - Vt)$, which defines the traveling frame of reference. Here k and V represent the wave number and velocity of the traveling wave. Both are undetermined parameters but we assume that $k > 0$. Accordingly, the dependent variable $u(x, t)$ is replaced by $U(\xi)$. Equations like Eq. (1) are then transformed into

$$-kV\frac{dU}{d\xi} = G(U, k\frac{dU}{d\xi}, k^2\frac{d^2U}{d\xi^2}, \cdots) \tag{2}$$

or

$$k^2V^2\frac{d^2U}{d\xi^2} = G(U, k\frac{dU}{d\xi}, k^2\frac{d^2U}{d\xi^2}, \cdots). \tag{3}$$

Hence, in what follows we deal with ODEs rather than with PDEs. Our main goal is to find exact solutions for those ODEs in tanh form. If that is impossible, one can find approximate solutions [5]. So, we introduce a new independent variable $Y = \tanh \xi$ into the ODE. The coefficients of the ODE in $U(\xi) = F(Y)$ then solely depend on Y, because $\dfrac{d}{d\xi}$ and subsequent derivatives in Eq. (2) and Eq. (3) are now replaced by $(1 - Y^2)\dfrac{d}{dY}$, etc.. Therefore, it makes sense to attempt to find solution(s) as a finite power series in Y,

$$F(Y) = \sum_{n=0}^{N} a_n Y^n, \tag{4}$$

which incorporates solitary-wave and shock-wave profiles (see Section 3). A balancing procedure determines the degree (N) of the power series. The coefficients a_n follow from solving a nonlinear algebraic system. If needed, boundary conditions can be incorporated within this procedure.

3. EXAMPLES

To illustrate the tanh technique and the possibilities it offers, we now investigate some well-known examples of PDEs in detail.

3.1 The Korteweg-de Vries (KdV) equation

The KdV equation is one of the most famous nonlinear PDEs. It was derived in fluid mechanics to describe shallow water waves in a rectangular channel:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + b\frac{\partial^3 u}{\partial x^3} \;=\; 0. \tag{5}$$

The positive parameter b refers to a dispersive effect.
Following the above procedure we transform Eq. (5) into

$$-kV\frac{dU}{d\xi} + kU\frac{dU}{d\xi} + bk^3\frac{d^3U}{d\xi^3} \;=\; 0. \tag{6}$$

Next, we introduce $Y = \tanh(\xi)$ and replace Eq. (6) by

$$\begin{aligned} &- kV(1-Y^2)\frac{dF(Y)}{dY} + kF(Y)(1-Y^2)\frac{dF(Y)}{dY} \\ &+ bk^3(1-Y^2)\frac{d}{dY}\Big\{(1-Y^2)\frac{d}{dY}\Big[(1-Y^2)\frac{dF(Y)}{dY}\Big]\Big\} \\ &= \;0. \end{aligned} \tag{7}$$

After substitution of Eq. (4) into Eq. (7), one readily verifies that the highest powers of Y appear as Y^{2N+1} in the second term and Y^{N+3} in the last term of Eq. (7). Balancing these then leads to $2N + 1 = N + 3$ or $N = 2$. A possible solution can thus be found by using Eq. (4) truncated at $N = 2$. Substitution of that polynomial into Eq. (7) leads to

$$F(Y) = a_0 - 12bk^2Y^2 \quad \text{and} \quad V = a_0 - 8bk^2. \tag{8}$$

If we require that the solution vanishes for $\xi \to \infty$ $(Y \to 1)$, we get

$$F(Y) = 12bk^2(1 - Y^2) \quad \text{with} \quad V = 4bk^2, \tag{9}$$

or, in the original variables,

$$u(x,t) = 12bk^2 \mathrm{sech}^2 k(x - Vt), \tag{10}$$

the well-known solitary wave in bell-shape form.

3.2 The Burgers equation

The Burgers equation,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - a\frac{\partial^2 u}{\partial x^2} \;=\; 0, \tag{11}$$

is one of the most famous nonlinear diffusion equations. The positive parameter a refers to a dissipative effect. As in the previous example, we first transform Eq. (11) into

$$-kV\frac{dU}{d\xi} + kU\frac{dU}{d\xi} - ak^2\frac{d^2U}{d\xi^2} \;=\; 0. \tag{12}$$

Next, we introduce $Y = \tanh(\xi)$, which yields

$$\begin{aligned} &- kV(1-Y^2)\frac{dF(Y)}{dY} + \; kF(Y)(1-Y^2)\frac{dF(Y)}{dY} \\ &- ak^2(1-Y^2)\frac{d}{dY}\Big[(1-Y^2)\frac{dF(Y)}{dY}\Big] \\ &= \;0. \end{aligned} \tag{13}$$

After substitution of Eq. (4) into Eq. (13), we balance the highest powers of Y. They arise as Y^{2N+1} in the second term and Y^{N+2} in the last term of Eq. (13). Hence, $2N + 1 = N + 2$. So, in this example $N = 1$. A candidate solution can thus be found using Eq. (4) with $N = 1$. Upon substitution into Eq. (13) we get

$$F(Y) = a_0 - 2akY \quad \text{with} \quad V = a_0, \tag{14}$$

where k is an arbitrary parameter (as was the case for the KdV equation).
Requiring that $U(\xi)$ (and its derivatives) vanish for $\xi \to \infty$, or equivalently, $Y \to 1$, we get

$$F(Y) = 2ak(1 - Y) \qquad \text{with} \qquad V = 2ak, \tag{15}$$

or

$$u(x,t) = 2ak[1 - \tanh k(x - Vt)], \tag{16}$$

the well-known shock-wave solution for the Burgers equation.

3.3 The Fisher equation

The Fisher equation [3],

$$\frac{\partial u}{\partial t} \;=\; \frac{\partial^2 u}{\partial x^2} + u(1 - u), \tag{17}$$

serves as a basic model for population dynamics or chemical kinetics. The solution shows exponential growth counteracted by nonlinear damping (where the damping rate is proportional to u). Moreover, linear diffusion is added.
To find a solution we first transform Eq. (17) into

$$-kV\frac{dU(\xi)}{d\xi} = k^2\frac{d^2U(\xi)}{d\xi^2} + U(\xi)(1-U(\xi)). \quad (18)$$

We further assume that $U(\xi)$ and its first derivative tend to 0 as $\xi \to \infty$.
In terms of $Y = \tanh\xi$ we get

$$\begin{aligned} & kV(1-Y^2)\frac{dF(Y)}{dY} + F(Y)(1-F(Y)) \\ & + k^2(1-Y^2)\Big[\frac{d}{dY}\Big[(1-Y^2)\frac{dF(Y)}{dY}\Big]\Big] \\ & = 0. \end{aligned} \quad (19)$$

Balancing the most nonlinear term with the highest derivative term results in $2N = N + 2$, so that $N = 2$. Taking into account the boundary conditions, it turns out that

$$F(Y) = d_0(1-Y)^2 \quad (20)$$

The velocity is than fixed:

$$V = \frac{16k^2+1}{4k}. \quad (21)$$

After some algebra, the following values for the parameters are obtained:

$$k = \frac{1}{2\sqrt{6}},\ d_0 = \frac{1}{4} \quad \text{with} \quad V = \frac{5}{\sqrt{6}}, \quad (22)$$

where only positive k and V are considered. The final solution of the Fisher equation then reads

$$u(x,t) = \frac{1}{4}\left\{1 - \tanh\frac{1}{2\sqrt{6}}\left[x - \frac{5}{\sqrt{6}}t\right]\right\}^2, \quad (23)$$

which represents a shock-wave moving to the right. Note that the wave number k is fixed in this example.

From these elementary examples it should be obvious that various generalizations of tanh method can designed and implemented.
For instance, instead of working with the tanh function one could work with the sech function. In that case, the computations involve a square root of $(1 - \tanh^2)$ as well. The methodology remains the same, yet, the mathematical steps are more cumbersome.
Further generalizations include the search for polynomial solutions in terms of the Jacobi s_n and c_n functions.
Instead of single PDEs, one can apply the method to nonlinear systems of couple PDEs. Although the algebra involved becomes quite tedious, they are treated likewise.
Finally, the tanh method can be extended to solve certain semi-discrete equations, where the space variable is discretized while time is kept continuous. In which case one uses a phase-shift property of tanh to link neighboring tanh functions, like $\tanh(k(n \pm 1 - Vt)$, to $\tanh(k(n - Vt)$, without phase shift.

4. THE ALGORITHM

To develop an automated package for the computation of exact solutions of single nonlinear PDEs (or coupled system of PDEs) based on the tanh function method, the following steps must be implemented.

Step 1: Transform the PDE into a nonlinear ODE.

Step 2: Determine the degree of the polynomial solution in Y. Equate every two possible highest exponents in the equation to get a linear system for N. Solve that system rejecting any solution N which is not a positive integer.

Step 3: Derive the nonlinear algebraic system for the coefficients $a_n (n = 0, 1, 2, \ldots, N)$. Solve the nonlinear system (this is the most difficult step). The following assumptions apply:

(i) All parameters appearing in the problem are considered strictly positive. If some parameters are required to be zero, one has to recalculate N again since the degree of the polynomial solution might have changed.

(ii) The coefficient of the highest power term must be nonzero (to be consistent with Step 2).

(iii) The wave number is assumed to be positive.

Step 4: Substitute the solutions for the coefficients and parameters into the original equation. Reverse Step 1 to obtain the explicit solution(s) of the problem in its original variables. Test the final solution(s) for correctness.

5. CONCLUSION

The tanh method provides a straightforward algorithm to compute particular kink and pulse type solutions for a large class of nonlinear PDEs. The success of the method lies in the fact that one circumvents integration to get closed-form solutions at the cost of (sometimes tedious) algebra.
Using well-known PDEs, it was shown that the tanh technique is a powerful and straightforward solution method to find closed-form, analytical expressions for traveling waves of nonlinear wave and evolution equations. The method is ease to use and leads to physically relevant solitary wave (pulse shaped) and shock-wave (kink shaped) profiles. Numerous single as well as coupled PDEs and ODEs have been solved with the method and its generalizations (see e.g. [5] and [8]). Based on the

method, powerful symbolic software packages have been developed to search for exact solutions of large classes of nonlinear PDEs. We mention e.g. **PDESpecialSolutions.m**, written in *Mathematica* which allows one to fully automaticaly compute traveling wave solutions as polynomials in tanh, sech, combinations thereof, cn or sn functions [8]. In addition, the package **DDESpecialSolutions.m** computes tanh solutions for nonlinear differential-difference equations (DDEs) [9].
Finally, the knowledge of exact traveling waves obtained with the tanh method may be useful to numerical analysts in the design of numerical solvers for nonlinear PDEs and DDEs.

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