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LIE POINT SYMMETRIES OF CLASSICAL FIELD THEORIES

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Abstract

We report on an investigation of Lie point symmetries of two systems of nonlinear PDEs, both representing classical field theories. The program SYMMGRP.MAX is used to compute the determining equations, which amount to several hundreds. With the use of relativistic notation, these are completely solved and therefore the coefficients of the vector field are obtained in closed form. Our investigation confirms that the symmetries previously reported in the literature are correct and also complete.

Introduction

We present results of an investigation of Lie point symmetries of two systems of nonlinear PDEs, both arising in the context of classical field theories. The first system describes an electromagnetic field coupled with a complex scalar field. The second system models an electromagnetic field coupled with a two-dimensional complex spinor.

We use relativistic notation throughout, except that summations over repeated indices are not implicit. Let x^μ (μ = 1, 2, 3, 4) denote the usual real space-time coordinates, with $x^4 = t$. Indices are raised and lowered with the help of the Minkowski metric (-1, -1, -1, +1). Summations over greek indices range from 1 to 4. Furthermore let A^ν denote four real functions, and let ϕ, ψ_1 and ψ_2 denote three complex functions of the x^μ. The real constants e, λ_c and M are such that $e \neq 0$, $\lambda_c > 0$ and $M \geq 0$. We use σ^μ for a set of four 2×2 hermitian matrices, so that σ^4 is the identity and σ^1, σ^2 and σ^3 are the Pauli matrices. Also ψ denotes a two-component object with components ψ_1 and ψ_2. A star stands for complex conjugation, and a dagger for hermitian conjugation.

Field Equations

In the above notations, the first set of equations we will investigate can be written as

$$\sum_\mu \partial_\mu \left\{ \partial^\mu A^\nu - \partial^\nu A^\mu \right\} + ie \left\{ \phi \partial^\nu \phi^* - \phi^* \partial^\nu \phi \right\} + 2e^2 \phi^* \phi A^\nu = 0 ,$$

$$\sum_\mu \left\{ \partial_\mu + ieA_\mu \right\} \left\{ \partial^\mu + ieA^\mu \right\} \phi + M^2 \phi + \lambda_c \left(\phi \phi^* \right) \phi = 0 . \tag{1}$$

This is a system of six nonlinear partial differential equations for the six dependent variables A^μ (μ = 1, 2, 3, 4), Re(ϕ) and Im(ϕ). There are, of course, four independent variables x^μ (μ = 1, 2, 3, 4). The second set of equations we will consider

$$\sum_\mu \partial_\mu \left\{ \partial^\mu A^\nu - \partial^\nu A^\mu \right\} - e \psi^\dagger \sigma^\nu \psi = 0 , \qquad \sum_\mu \sigma^\mu \left\{ \partial_\mu + ieA_\mu \right\} \psi = 0 , \tag{2}$$

consists of eight equations for eight dependent variables A^μ, Re(ψ_1), Im(ψ_1), Re(ψ_2) and Im(ψ_2).

In field-theoretical language, A^μ usually represents the electromagnetic potential, ϕ is a scalar field of mass M, and ψ is a two-dimensional (massless) spinor. More about the physical significance of these variables can be found in Ref. [1].

Determining Equations

We first concentrate on Eqs. (1), the coupled electromagnetic and scalar fields. We look for the Lie point transformations involving the independent and dependent variables which will leave the solution set of (1) invariant. Locally, such transformations are generated by a vector field of the form [2]

$$\alpha = \sum_\mu H^\mu \frac{\partial}{\partial x^\mu} + \sum_\mu \Phi^\mu \frac{\partial}{\partial A^\mu} + \sum_{a=5,6} \Phi^a \frac{\partial}{\partial \phi^a} , \tag{3}$$

where H^μ, Φ^μ and Φ^a are functions of x^ν, A^ν and ϕ^b, and where $\phi^5 = \mathrm{Re}(\phi)$ and $\phi^6 = \mathrm{Im}(\phi)$. The vector field α must be such that its second prolongation, acting on Eqs. (1), vanishes on the solution set of (1). This requirement yields the so-called determining equations for the coefficients H^μ, Φ^μ and Φ^a. Knowledge of the closed form of these coefficients then allows to find the Lie point transformations upon integration of a system of first order equations.

Determining equations can be obtained entirely algorithmically. A number of symbolic packages exist to find them. We have used the Macsyma program SYMMGRP.MAX [3]. Running the program and eliminating simple dependencies, we find that

$$H^\mu = H^\mu(x^\lambda) , \quad \Phi^\mu = \Phi^\mu(x^\lambda, A^\nu) , \quad \Phi^a = \Phi^a(x^\lambda, A^\nu, \phi^b) . \tag{4}$$

The program then yielded 210 (not all independent) determining equations for the coefficients of the vector field. It turns out that the overdetermined system can be simplified remarkably by writing it in relativistic notation. After elimination of some redundant equations and minor rearrangements, we find that the determining equations are as follows:

$$\frac{\partial^2 \Phi^\mu}{\partial A^\kappa \partial A^\lambda} = 0 ; \qquad \frac{\partial \Phi^a}{\partial A^\mu} = 0 ; \qquad \frac{\partial^2 \Phi^a}{\partial \phi^b \partial \phi^c} = 0 ; \tag{5a}$$

$$\frac{\partial \Phi^\mu}{\partial A_\nu} + \frac{\partial H^\nu}{\partial x_\mu} = 0 = \frac{\partial \Phi^\mu}{\partial A_\nu} - \frac{\partial H^\mu}{\partial x_\nu} , \mu \neq \nu ; \qquad \frac{\partial H^\mu}{\partial x^\mu} - \frac{\partial H^\nu}{\partial x^\nu} = 0 = \frac{\partial \Phi^\mu}{\partial A^\mu} - \frac{\partial \Phi^\nu}{\partial A^\nu} ; \tag{5b}$$

$$2 \frac{\partial^2 \Phi^\mu}{\partial A_\lambda \partial x_\nu} - \frac{\partial^2 \Phi^\nu}{\partial A_\lambda \partial x_\mu} + \frac{\partial^2 H^\nu}{\partial x_\lambda \partial x_\mu} = 0 , \qquad \begin{array}{l} \lambda \neq \mu \\ \nu \neq \mu \end{array} ; \tag{5c}$$

$$\frac{\partial^2 \Phi^\kappa}{\partial x^\kappa \partial A^\kappa} + \frac{\partial^2 \Phi^\lambda}{\partial x^\lambda \partial A^\kappa} + \frac{\partial^2 \Phi^\mu}{\partial x^\mu \partial A^\kappa} - \frac{\partial^2 H^\nu}{\partial x^\nu \partial x^\kappa} = 0 , \qquad \kappa, \lambda, \mu, \nu \neq ; \tag{5d}$$

$$2 \frac{\partial^2 \Phi^\nu}{\partial x_\mu \partial A^\nu} - \frac{\partial^2 \Phi^\mu}{\partial x_\nu \partial A^\nu} - \frac{\partial^2 H^\mu}{\partial x_\kappa \partial x^\kappa} - \frac{\partial^2 H^\mu}{\partial x_\lambda \partial x^\lambda} - \frac{\partial^2 H^\mu}{\partial x_\mu \partial x^\mu} = 0 , \quad \kappa, \lambda, \mu, \nu \neq ; \tag{5e}$$

$$\Phi^a = \phi^a \frac{\partial \Phi^b}{\partial \phi^b} - \phi^b \frac{\partial \Phi^b}{\partial \phi^a} - \phi^a \left(\frac{\partial \Phi^4}{\partial A^4} + \frac{\partial H^4}{\partial x^4} \right) , \qquad a \neq b ; \tag{5f}$$

$$\Phi^a = \phi^b \frac{\partial \Phi^a}{\partial \phi^b} - \phi^a \frac{\partial \Phi^b}{\partial \phi^b} + \phi^a \left(\frac{\partial \Phi^4}{\partial A^4} + \frac{\partial H^4}{\partial x^4} - 2 \frac{\partial H^1}{\partial x^1} \right) , \qquad a \neq b ; \tag{5g}$$

$$2 e \varepsilon_a A^\mu \left(\frac{\partial \Phi^a}{\partial \phi^b} + \frac{\partial \Phi^b}{\partial \phi^a} \right) - 2 \frac{\partial^2 \Phi^a}{\partial x_\mu \partial \phi^a} + \sum_\lambda \frac{\partial^2 H^\mu}{\partial x_\lambda \partial x^\lambda} = 0 , \qquad a \neq b ; \tag{5h}$$

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$$A^\mu \left(\frac{\partial \Phi^a}{\partial \phi^a} - \frac{\partial \Phi^b}{\partial \phi^b} \right) + \frac{\varepsilon_a}{e} \frac{\partial^2 \Phi^a}{\partial x_\mu \partial \phi^b} - \Phi^\mu - 2A^\mu \frac{\partial H^4}{\partial x^4} + \sum_\lambda A^\lambda \frac{\partial H^\mu}{\partial x^\lambda} = 0 , \qquad a \neq b ; \qquad (5i)$$

$$\sum_\lambda \left(\frac{\partial^2 \Phi_\mu}{\partial x_\lambda \partial x^\lambda} - \frac{\partial^2 \Phi^\lambda}{\partial x^\mu \partial x^\lambda} \right) + 2e \left(\phi^5 \frac{\partial \Phi^6}{\partial x^\mu} - \phi^6 \frac{\partial \Phi^5}{\partial x^\mu} \right) + 4e^2 A_\mu \left\{ \phi^6 \Phi^6 + \phi^5 \Phi^5 \right\}$$
$$+ 2e^2 \left\{ (\phi^5)^2 + (\phi^6)^2 \right\} \left(\Phi_\mu - \sum_\lambda A^\lambda \frac{\partial \Phi_\mu}{\partial A^\lambda} + 2 A_\mu \frac{\partial H^4}{\partial x^4} \right) = 0 ; \qquad (5j)$$

$$-\sum_\lambda \left(\frac{\partial^2 \Phi^a}{\partial x_\lambda \partial x^\lambda} + \varepsilon_b e \left[2A^\lambda \frac{\partial \Phi^b}{\partial x^\lambda} + \phi^b \frac{\partial \Phi^\lambda}{\partial x^\lambda} \right] \right) + M^2 \left(\phi^5 \frac{\partial \Phi^a}{\partial \phi^5} + \phi^6 \frac{\partial \Phi^a}{\partial \phi^6} - \Phi^a - 2\phi^a \frac{\partial H^4}{\partial x^4} \right)$$
$$+ \lambda_c \left(\left[(\phi^5)^2 + (\phi^6)^2 \right] \left[\phi^6 \frac{\partial \Phi^a}{\partial \phi^6} + \phi^5 \frac{\partial \Phi^a}{\partial \phi^5} - 2\phi^a \frac{\partial H^4}{\partial x^4} - \Phi^a \right] - 2\phi^a \left[\phi^5 \Phi^5 + \phi^6 \Phi^6 \right] \right) \qquad (5k)$$
$$+ e^2 \sum_\lambda \left(A_\lambda A^\lambda \left[2\phi^a \frac{\partial H^4}{\partial x^4} + \Phi^a - \frac{\partial \Phi^a}{\partial \phi^5} \phi^5 - \frac{\partial \Phi^a}{\partial \phi^6} \phi^6 \right] + 2\phi^a A_\lambda \Phi^\lambda \right) = 0 , \qquad a \neq b .$$

Here κ, λ, μ and ν run from 1 to 4; a, b and c run from 5 to 6; and $\varepsilon_5 = 1 = -\varepsilon_6$.

Solution and Lie Point Symmetries

To solve the determining equations (5a) - (5k) requires work of some length, so we only outline the main steps. One can show that, owing to (4), the most general solution of (5a) and (5b) is

$$\Phi^\mu = \sum_\nu f^\mu{}_\nu (x^\lambda) A^\nu + f(x^\lambda) A^\mu + f^\mu (x^\lambda) , \qquad (6)$$

$$\Phi^a = \sum_{b=5,6} g^a{}_b (x^\lambda) \phi^b + g^a (x^\lambda) , \qquad \frac{\partial H^\mu}{\partial x^\nu} = f^\mu{}_\nu (x^\lambda) + \delta^\mu{}_\nu F(x^\lambda) .$$

Here we have $f^\mu{}_\nu = 0$ if $\mu = \nu$, $f^{\mu\nu} = -f^{\nu\mu}$, and $\delta^\mu{}_\nu$ is the Kronecker delta. From (5c) - (5e), we find

$$F = -f + C , \qquad \frac{\partial f^{\mu\lambda}}{\partial x^\nu} = \delta^\lambda{}_\nu \frac{\partial f}{\partial x_\mu} , \qquad \begin{matrix} \mu \neq \nu \\ \mu \neq \lambda \end{matrix} . \qquad (7)$$

In (7), C is a constant. We can eventually deduce that $f^\mu{}_\nu$ only depends on x^μ and x^ν, and is linear in these variables. We thus write

$$-f^2{}_1 = c_5 - 2c_{13} y + 2c_{14} x = f^1{}_2 , \qquad f^1{}_3 = c_6 - 2c_{13} z + 2c_{15} x = -f^3{}_1 ,$$

$$f^4{}_1 = c_8 + 2c_{13} t - 2c_{16} x = f^1{}_4 , \qquad f^2{}_3 = c_7 - 2c_{14} z + 2c_{15} y = -f^3{}_2 , \qquad (8)$$

$$f^4{}_2 = c_9 + 2c_{14} t - 2c_{16} y = f^2{}_4 , \qquad f^3{}_4 = c_{10} + 2c_{15} t - 2c_{16} z = f^4{}_3 ,$$

and

$$f = -c_{11} - 2c_{13} x - 2c_{14} y - 2c_{15} z + 2c_{16} t . \qquad (9)$$

We have written x, y, z and t for x^1, x^2, x^3 and x^4; the c_i stand for arbitrary constants. From (5f) and (5g), we get

$$g^a{}_a = f, \quad g^a{}_b + g^b{}_a = 0 \quad (a \neq b), \quad g^a(x^\lambda) = 0, \quad C = 0. \tag{10}$$

Knowing that $F = -f$, the functions H^μ follow from (6) upon integration. We get

$$H^1 = c_1 + c_{11}x + c_{13}x^2 + 2c_{14}xy + 2c_{15}xz - 2c_{16}xt + c_5y - c_{13}y^2 + c_6z - c_{13}z^2 + c_8t + c_{13}t^2,$$

$$H^2 = c_2 + c_{11}y + 2c_{13}xy + c_{14}y^2 + 2c_{15}zy - 2c_{16}ty - c_5x - c_{14}x^2 + c_7z - c_{14}z^2 + c_9t + c_{14}t^2,$$

$$H^3 = c_3 + c_{11}z + 2c_{13}xz + 2c_{14}yz + c_{15}z^2 - 2c_{16}tz - c_6x - c_{15}x^2 - c_7y - c_{15}y^2 + c_{10}t + c_{15}t^2, \tag{11}$$

$$H^4 = c_4 + c_{11}t + 2c_{13}xt + 2c_{14}yt + 2c_{15}zt - c_{16}t^2 + c_8x - c_{16}x^2 + c_9y - c_{16}y^2 + c_{10}z - c_{16}z^2.$$

Eq. (5h) is now satisfied identically. Furthermore, (5i) implies that

$$f^\mu(x^\lambda) = \frac{1}{e}\frac{\partial}{\partial x_\mu} g^5{}_6. \tag{12}$$

Eq. (5j) becomes an identity, and (5k) implies that $M^2 f = 0$. Summarizing, the explicit forms of Φ^μ and Φ^a are

$$\Phi^\mu = \sum_\nu f^\mu{}_\nu A^\nu + f A^\mu - c_{12}\frac{\partial \chi}{\partial x_\mu}, \tag{13}$$

$$\Phi^5 = -e c_{12} \chi \phi^6 + f \phi^5, \quad \Phi^6 = e c_{12} \chi \phi^5 + f \phi^6,$$

where we wrote $g^5{}_6$ as $-e c_{12}\chi$, with χ an arbitrary function of x^λ.

It is not difficult to show that the constants c_1 to c_{10} correspond to generators of the Poincaré group, c_{11} corresponds to a uniform dilatation, c_{12} corresponds to a gauge transformation, and c_{13} to c_{16} correspond to special conformal transformations. The system (1), therefore, is invariant under the Poincaré group and gauge transformations when $M \neq 0$ (i.e., f then vanishes), and under the conformal group and gauge transformations when $M = 0$.

This completes the analysis of the Lie point symmetries of Eqs. (1). The analysis of Eqs. (2) can be carried out along similar lines. Here the program SYMMGRP.MAX, after elimination of dependencies, yielded 200 determining equations. Much as in the previous case, they can be written in a simplified form using relativistic notation. A complete solution has again been obtained, which agrees with a computer-aided solution obtained independently. Lie point symmetries, in the case of Eqs. (2), generate the conformal group with gauge transformations.

To conclude, we point out that it was well known [4] that Eqs. (1) and Eqs. (2) admit the Lie point symmetries we found. Our analysis provides evidence that these are the only ones. The method we used for solving the determining equations should be applicable whenever the original equations are Lorentz invariant.

References

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