SYMBOLIC METHODS TO FIND EXACT SOLUTIONS OF NONLINEAR PDES

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1. Introduction

Two related methods are presented to find exact solitary wave and soliton solutions of nonlinear partial differential equations (PDEs). The methods are illustrated by solving typical nonlinear PDEs borrowed from fluid dynamics and mathematical biology.

Using the notorious Korteweg-de Vries (KdV) equation from soliton theory, we describe a simplified version of Hirota's direct method in Section 2. In Section 3, we find the exact travelling wave solution of the Fisher equation with convection term, and show why, in contrast to e.g. the KdV equation, the Fisher equation has no soliton solutions. In Section 4, a truncated Laurent expansion is used to derive an auto-Bäcklund transformation and a travelling wave solution to the Fitzhugh-Nagumo (FHN) equation with convection term.

The two closely related methods are applicable to many other nonlinear PDEs and can easily be implemented with any symbolic manipulation package such as Mathematica, Macsyma, and Maple. The calculations in this paper were carried out with Macsyma.

2. Simplified Version of Hirota's Method

To introduce a simplified version of Hirota's method [1], we consider the ubiquitous KdV equation

$$u_t + 6uu_x + u_{xxx} = 0. \tag{1}$$

First, a change of dependent variable,

$$u(x,t) = 2\frac{\partial^2 \ln f(x,t)}{\partial x^2} = 2\frac{(ffxx - f_x^2)}{f^2}, \tag{2}$$

allows us to transform (1) into a quadratic equation in f and its derivatives,

$$f(f_{xt} + f_{xxxx}) - f_x f_t - 4f_x f_{xxx} + 3f_{xx}^2 = 0. \tag{3}$$

To make Hirota's technique applicable to equations that cannot be written in bilinear form, we intentionally do not introduce Hirota's bilinear operators [1], but simply write (3) as $\mathcal{N}(f,f) = 0$,

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where the nonlinear differential operator $\mathcal{N}$ is defined as

$$\mathcal{N}(f,g) = (\mathcal{I}f)(\frac{\partial^2 g}{\partial x \partial t} + \frac{\partial^4 g}{\partial x^4}) - \frac{\partial f}{\partial x}\frac{\partial g}{\partial t} - 4\frac{\partial f}{\partial x}\frac{\partial^3 g}{\partial x^3} + 3\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 g}{\partial x^2}. \tag{4}$$

$\mathcal{I}$ denotes the identity operator, i.e. $\mathcal{I}f(x,t) = f(x,t)$.

We seek a formal solution,

$$f(x,t) = 1 + \sum_{n=1}^{\infty} \epsilon^n \, f^{(n)}(x,t), \tag{5}$$

where ϵ serves as a bookkeeping parameter (not a small quantity). Proceeding as in Hirota's method, we substitute (5) into $\mathcal{N}(f,f) = 0$, and equate to zero the different powers of ϵ,

$$\begin{aligned}
O(\epsilon^1) &: \quad \mathcal{L}f^{(1)} = 0 && (6)\\
O(\epsilon^2) &: \quad \mathcal{L}f^{(2)} = -\mathcal{N}(f^{(1)}, f^{(1)}) && (7)\\
O(\epsilon^3) &: \quad \mathcal{L}f^{(3)} = -\mathcal{N}(f^{(1)}, f^{(2)}) - \mathcal{N}(f^{(2)}, f^{(1)}) && (8)\\
&\vdots && (9)\\
O(\epsilon^n) &: \quad \mathcal{L}f^{(n)} = -\sum_{j=1}^{n-1} \mathcal{N}(f^{(j)}, f^{(n-j)}), && (10)
\end{aligned}$$

where $\mathcal{L}$ denotes the linear differential operator,

$$\mathcal{L}\bullet = \frac{\partial^2 \bullet}{\partial x \partial t} + \frac{\partial^4 \bullet}{\partial x^4}. \tag{11}$$

The N-soliton solution of the KdV is then generated from

$$f^{(1)} = \sum_{i=1}^{N} f_i = \sum_{i=1}^{N} \exp(\theta_i) = \sum_{i=1}^{N} \exp\left(k_i\, x - \omega_i\, t + \delta_i\right), \tag{12}$$

where k_i, ω_i and δ_i are constants. Substitution of (12) into (6) yields $P(k_i, \omega_i) = 0$, where

$$P(k_i, \omega_i) = -\omega_i k_i + {k_i}^4, \qquad i = 1, 2, ..., N. \tag{13}$$

The dispersion law is thus given by $\omega_i = k_i^3$. Using (12), we then compute the right hand side of (7):

$$-\sum_{i,j=1}^{N} 3k_i k_j^2 (k_i - k_j) f_i f_j = \sum_{1 \le i < j \le N} 3k_i k_j (k_i - k_j)^2 f_i f_j. \tag{14}$$

Observe that the terms in f_i^2 drop out, which is typical for equations admitting solitons. Furthermore, (14) determines the form of $f^{(2)}$:

$$f^{(2)} = \sum_{1 \le i < j \le N} a_{ij} f_i f_j = \sum_{1 \le i < j \le N} a_{ij} \exp[(k_i + k_j)x - (\omega_i + \omega_j)t + (\delta_i + \delta_j)]. \tag{15}$$

With (11), (13) and (15), the left hand side of (7) is readily computed

$$\mathcal{L}f^{(2)} = \sum_{1\le i<j\le N} P(k_i+k_j,\omega_i+\omega_j)\,a_{ij}\,f_if_j = \sum_{1\le i<j\le N} 3k_ik_j(k_i+k_j)^2\,a_{ij}\,f_if_j. \tag{16}$$

Equating (14) and (16) determines

$$a_{ij} = \frac{(k_i-k_j)^2}{(k_i+k_j)^2}, \qquad 1\le i<j\le N. \tag{17}$$

Proceeding in a similar fashion with (8) leads to the explicit form of $f^{(3)}$. For example, for $N=3$ we have

$$f^{(3)} = b_{123}\exp(\theta_1+\theta_2+\theta_3) = b_{123}\exp\left[(k_1+k_2+k_3)x-(\omega_1+\omega_2+\omega_3)t+(\delta_1+\delta_2+\delta_3)\right], \tag{18}$$

with

$$b_{123} = a_{12}\,a_{13}\,a_{23} = \frac{(k_1-k_2)^2\,(k_1-k_3)^2\,(k_2-k_3)^2}{(k_1+k_2)^2\,(k_1+k_3)^2\,(k_2+k_3)^2}. \tag{19}$$

Note that $f^{(3)}$ has no terms in $f_i^2f_j$, with $i,j=1,...,N,\ i\neq j$. For $N=3$, one can also show that $f^{(n)}=0$ for $n>3$, and therefore the expansion in (5) truncates:

$$\begin{aligned} f &= 1+\exp\theta_1+\exp\theta_2+\exp\theta_3 && (20)\\ &+ a_{12}\exp(\theta_1+\theta_2)+a_{13}\exp(\theta_1+\theta_3)+a_{23}\exp(\theta_2+\theta_3) && (21)\\ &+ b_{123}\exp(\theta_1+\theta_2+\theta_3), \end{aligned}$$

with $\epsilon=1$. Upon substitution of (20) into (2) the well-known three soliton solution of the KdV equation follows. The N-soliton solution for any $N>3$ can be constructed in a similar way.

3. Example: The Fisher Equation with Convection

We apply the method in Section 2 to the Fisher equation with convection term [2],

$$u_t + kuu_x - u_{xx} - u(1-u) = 0, \tag{22}$$

which describes the propagation of nerve pulses. Painlevé analysis (see Section 4) reveals that

$$u(x,t) = -\frac{2}{k}\frac{\partial \ln f(x,t)}{\partial x} = -\frac{2}{k}(\frac{f_x}{f}) \tag{23}$$

transforms (22) into the quadratic equation

$$\mathcal{N}(f,f) = f(f_{xxx}+f_x-f_{xt}) + f_x(f_t-f_{xx}+\frac{2}{k}f_x) = 0. \tag{24}$$

Here, the nonlinear operator is defined as

$$\mathcal{N}(f,g) = (\mathcal{I}f)(\frac{\partial^3 g}{\partial x^3} + \frac{\partial g}{\partial x} - \frac{\partial^2 g}{\partial x \partial t}) + (\frac{\partial f}{\partial x})(\frac{\partial g}{\partial t} - \frac{\partial^2 g}{\partial x^2} + \frac{2}{k}\frac{\partial g}{\partial x}). \tag{25}$$

Seeking a solution of type (5), the scheme (6)-(10) remains the same, but $\mathcal{L}$ is replaced by

$$\mathcal{L}\bullet = \frac{\partial^3 \bullet}{\partial x^3} + \frac{\partial \bullet}{\partial x} - \frac{\partial^2 \bullet}{\partial x \partial t}. \tag{26}$$

From (6), one obtains $\omega_i = -(1+k_i^2), i = 1, 2, ..., N$, and consequently, (7) reduces to

$$\mathcal{L}f^{(2)} = -\sum_{i=1}^{N} k_i(1 + \frac{2}{k}k_i)f_i^2 - \sum_{1 \le i < j \le N} \frac{4}{k} k_i k_j (k_i + k_j) f_i f_j. \tag{27}$$

An essential difference with the KdV case is the presence of terms in f_i^2. It is no longer possible to derive a two or three soliton solution, much less a N-soliton solution. Inclusion of terms in f_i^2 in $f^{(2)}$ would not help because then $f^{(n)} \neq 0$, for all n, and f would be an infinite series.

The alternative is to get rid of the first sum in the right hand side of (27) by setting $k_i = -\frac{k}{2}$. Rejecting equal values for k_i we are led to the case $N = 1$. Taking into account (5), (12) and the dispersion law, we find that

$$f(x,t) = 1 + \exp\theta = 1 + \exp[-\frac{k}{2}x + \frac{1}{4}(4+k^2)t + \delta]. \tag{28}$$

Note that cf, with c constant, is also a solution of (24) and therefore, with (23),

$$u(x,t) = \frac{\exp\theta}{c + \exp\theta} = \frac{\exp[-\frac{k}{2}x + \frac{1}{4}(4+k^2)t + \delta]}{c + \exp[-\frac{k}{2}x + \frac{1}{4}(4+k^2)t + \delta]}, \tag{29}$$

where we have absorbed $|c|$ in the phase δ. The final solution can be written in a more pleasing form as

$$u(x,t) = \frac{1 + \tanh\frac{1}{2}\theta}{(1+c) + (1-c)\tanh\frac{1}{2}\theta}, \tag{30}$$

The special case corresponding to $c = 1$ was first derived by Malfliet via a tanh-method [3].

The solution of the Fisher equation without convection ($k = 0$) can be obtained in a similar way. From Painlevé analysis (Section 4) one determines the transformation,

$$u(x,t) = -6\frac{\partial^2 \ln f(x,t)}{\partial x^2} + \frac{6}{5}\frac{\partial \ln f(x,t)}{\partial t} = -6\frac{(f f_{xx} - f_x^2)}{f^2} + \frac{6}{5}(\frac{f_t}{f}), \tag{31}$$

which allows to obtain a quadratic equation in f and its derivatives. Using the method in this Section, one gets

$$u(x,t) = \frac{[1 - \tanh\frac{1}{2}\theta]^2}{[(1+c) - (1-d)\tanh\frac{1}{2}\theta]^2}, \tag{32}$$

with either $d = c$ or $d = c + 4$, c any constant, and with $\theta = \frac{1}{\sqrt{6}}x - \frac{5}{6}t + \delta$. Note that (32) does not follow from (30) in the limit for $k \to 0$. The solution (32) was first obtained by Ablowitz and Zeppetella [4].

4. Truncated Laurent Expansion

A PDE passes the Painlevé test [1,5] if its solution $u(x,t)$, expressed as a Laurent series in the complex plane,

$$u(x,t) = f^{\alpha} \sum_{k=0}^{\infty} u_k f^k, \tag{33}$$

has no worse singularities than movable poles. First, the negative integer α and the coefficient $u_0(x,t)$ are computed by substituting (33) into the given PDE and requiring that the most singular terms vanish. Second, one computes the particular values of k (called resonances, denoted by r) for which arbitrary coefficients $u_r(x,t)$ enter the series. Third, one checks the compatibility conditions at all the resonance levels.

Let us consider the FHN equation with convection term [2],

$$u_t + kuu_x - u_{xx} - u(1-u)(a-u) = 0. \tag{34}$$

For (34), $\alpha = -1$ and u_0 must satisfy $u_0^2 - ku_0 f_x - 2f_x^2 = 0$. The resonances are $r = -1$ and $r = 4 + k(\frac{u_0}{f_x}) = 2 + (\frac{u_0}{f_x})^2$. In order to have integer resonances, we must have $u_0(x,t) = \sqrt{m} f_x$ and $k = \frac{m-2}{\sqrt{m}}$, for any positive integer m.

To find a traveling wave solution, we substitute the Laurent series, truncated at the constant level term,

$$u(x,t) = \sqrt{m}\frac{\partial \ln f(x,t)}{\partial x} + u_1(x,t) = \sqrt{m}\frac{f_x(x,t)}{f(x,t)} + u_1(x,t) \tag{35}$$

into (34) with $k = \frac{m-2}{\sqrt{m}}$, and, in contrast to the method in Section 2, we set the different power terms in f separately equal to zero. This results in an overdetermined system in $f(x,t)$ and $u_1(x,t)$:

$$f_t - (1+m) f_{xx} - \sqrt{m}\,[\frac{2}{m}(1+m)\,u_1 - 1 - a]\,f_x = 0, \tag{36}$$

$$f_{xt} - f_{xxx} + \frac{1}{\sqrt{m}}(m-2)\,u_1\,f_{xx} + [3{u_1}^2 - 2(1+a)u_1 + a + \frac{1}{\sqrt{m}}(m-2)\,(u_1)_x]\,f_x = 0, \tag{37}$$

$$(u_1)_t + \frac{1}{\sqrt{m}}(m-2)u_1(u_1)_x - (u_1)_{xx} + u_1(1-u_1)(a-u_1) = 0. \tag{38}$$

The above system is linear if u_1 is any of the constant solutions of the FHN equation (38), i.e. $u_1 = 0, 1,$ or a.
For $u_1 = 0$ and (12), we find that $N = 2$ and $k_1 = \frac{1}{\sqrt{m}}$, $\omega_1 = \frac{am-1}{m}$, $k_2 = \frac{a}{\sqrt{m}}$, and $\omega_2 = \frac{a(m-a)}{m}$. Hence,

$$f = c + exp[\frac{1}{\sqrt{m}}x + \frac{a(a-m)}{m}t + \delta_1] + exp[\frac{a}{\sqrt{m}}x + \frac{(1-am)}{m}t + \delta_2]. \tag{39}$$

Returning to $u(x,t)$ via (35) yields

$$u(x,t) = \frac{exp[\frac{1}{\sqrt{m}}x + \frac{(1-am)}{m}t + \delta_1] + a\, exp[\frac{a}{\sqrt{m}}x + \frac{a(a-m)}{m}t + \delta_2]}{c + exp[\frac{1}{\sqrt{m}}x + \frac{(1-am)}{m}t + \delta_1] + exp[\frac{a}{\sqrt{m}}x + \frac{a(a-m)}{m}t + \delta_2]}. \tag{40}$$

This solution reduces for $m = 2$ to the known solution [6] of the classical FHN equation without convection.

5. References

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