The Korteweg–de Vries Equation

Willy A. Hereman

1 Historical Perspective

In 1895 Diederik Korteweg (1848–1941) and Gustav de Vries (1866–1934) derived a partial differential equation (PDE) that models the "great wave of translation" that naval engineer John Scott Russell had observed in the Union Canal in 1834.

Assuming that the wave propagates in the X direction, the evolution of the surface elevation $\eta(X,T)$ above the undisturbed water depth h at time T can be modeled by the Korteweg–de Vries (KdV) equation:

$$\frac{\partial \eta}{\partial T} + \sqrt{gh}\frac{\partial \eta}{\partial X} + \frac{3}{2}\frac{\sqrt{gh}}{h}\eta\frac{\partial \eta}{\partial X} + \tfrac{1}{2}h^2\sqrt{gh}\Big(\frac{1}{3} - \frac{\mathcal{T}}{\rho g h^2}\Big)\frac{\partial^3 \eta}{\partial X^3} = 0, \quad (1)$$

where g is the gravitational acceleration, ρ is the density, and $\mathcal{T}$ is the surface tension. The dimensionless parameter $\mathcal{T}/\rho g h^2$, called the *Bond number*, measures the relative strengths of surface tension and the gravitational force. Equation (1) is valid for long waves of relatively small amplitude, $|\eta|/h \ll 1$.

In dimensionless variables, (1) can be written as

$$u_t + \alpha u u_x + u_{xxx} = 0, \quad (2)$$

where subscripts denote partial derivatives. The term $\sqrt{gh}\eta_X$ in (1) has been removed by an elementary transformation. Conversely, a linear term in u_x can be added to (2). The parameter α can be scaled to any real number. Commonly used values are $\alpha = \pm 1$ and $\alpha = \pm 6$.

The term u_t describes the time evolution of the wave. Therefore, (2) is called an *evolution* equation. The nonlinear term $\alpha u u_x$ accounts for steepening of the wave. The linear dispersive term u_{xxx} describes spreading of the wave.

It is worth noting that the KdV equation had already appeared in seminal work on water waves published by Joseph Boussinesq about twenty years earlier.

2 Solitary Waves and Periodic Solutions

The balance of the steepening and spreading effects gives rises to a stable solitary wave,

$$u(x,t) = \frac{\omega - 4k^3}{\alpha k} + \frac{12k^2}{\alpha}\,\mathrm{sech}^2(kx - \omega t + \delta), \quad (3)$$

where the wave number k, the angular frequency ω, and the phase δ are arbitrary constants. Requiring that $\lim_{x\to\pm\infty} u(x,t) = 0$ for all t leads to $\omega = 4k^3$, in which case (3) reduces to

$$u(x,t) = 12(k^2/\alpha)\,\mathrm{sech}^2(kx - 4k^3 t + \delta). \quad (4)$$

This hump-shaped solitary wave of finite amplitude $12k^2/\alpha$ travels to the right at constant phase speed $v = \omega/k = 4k^2$, and it models Scott Russell's "great wave of translation" that traveled without change of shape over a fairly long distance.

As shown by Korteweg and de Vries, (2) also has a periodic solution:

$$u(x,t) = \frac{\omega - 4k^3(2m-1)}{\alpha k} + 12(k^2/\alpha)m\,\mathrm{cn}^2(kx - \omega t + \delta; m). \quad (5)$$

They called this the *cnoidal wave* solution because it involves Jacobi's elliptic cosine function, cn, with modulus m, $0 < m < 1$. In the limit $m \to 1$, $\mathrm{cn}(\xi; m) \to \mathrm{sech}\,\xi$ and (5) reduces to (3).

3 Modern Developments

The solitary wave was, for many years, considered an unimportant curiosity in the field of nonlinear waves. That changed in 1965, when Zabusky and Kruskal realized that the KdV equation arises as the continuum limit of a one-dimensional anharmonic lattice used by Fermi, Pasta, and Ulam in 1955 to investigate how energy is distributed among the many possible oscillations in the lattice. Since taller solitary waves travel faster than shorter ones, Zabusky and Kruskal simulated the collision of two waves in a nonlinear crystal lattice and observed that each retains its shape and speed after collision. Interacting solitary waves merely experience a phase shift, advancing the faster wave and retarding the slower one. In analogy with colliding particles, they coined the word "solitons" to describe these elastically colliding waves.

To model water waves that are weakly nonlinear, weakly dispersive, and weakly two-dimensional, with all three effects being comparable, Kadomtsev and Petviashvili (KP) derived a two-dimensional version of (2) in 1970:

$$(u_t + 6uu_x + u_{xxx})_x + 3\sigma^2 u_{yy} = 0, \quad (6)$$

where $\sigma^2 = \pm 1$ and the y-axis is perpendicular to the direction of propagation of the wave (along the x-axis).

The KdV and KP equations, and the nonlinear Schrödinger equation

$$\mathrm{i}u_t + u_{xx} + \kappa|u|^2 u = 0 \tag{7}$$

(where κ is a constant and $u(x,t)$ is a complex-valued function), are famous examples of so-called completely integrable nonlinear PDEs. This means that they can be solved with the inverse scattering transform, a nonlinear analogue of the Fourier transform.

The inverse scattering transform is not applied to (2) directly but to an auxiliary system of linear PDEs,

$$\psi_{xx} + (\lambda + \tfrac{1}{6}\alpha u)\psi = 0, \tag{8}$$

$$\psi_t + \tfrac{1}{2}\alpha u_x \psi + \alpha u \psi_x + 4\psi_{xxx} = 0, \tag{9}$$

which is called the *Lax pair* for the KdV equation. Equation (8) is a linear Schrödinger equation for an eigenfunction ψ, a constant eigenvalue λ, and a potential $(-\alpha u)/6$. Equation (9) governs the time evolution of ψ. The two equations are compatible, i.e., $\psi_{xxt} = \psi_{txx}$, if and only if $u(x,t)$ satisfies (2). For given $u(x,0)$ decaying sufficiently fast as $|x| \to \infty$, the inverse scattering transform solves (8) and (9) and finally determines $u(x,t)$.

4 Properties and Applications

Scientists remain intrigued by the rich mathematical structure of completely integrable nonlinear PDEs. These PDEs can be written as infinite-dimensional bi-Hamiltonian systems and have additional, remarkable features. For example, they have an associated Lax pair, they can be written in Hirota's bilinear form, they admit Bäcklund transformations, and they have the Painlevé property. They have an infinite number of conserved quantities, infinitely many higher-order symmetries, and an infinite number of soliton solutions.

As well as being applicable to shallow-water waves, the KdV equation is ubiquitous in applied science. It describes, for example, ion-acoustic waves in a plasma, elastic waves in a rod, and internal waves in the atmosphere or ocean. The KP equation models, for example, water waves, acoustic waves, and magnetoelastic waves in anti-ferromagnetic materials. The nonlinear Schrödinger equation describes weakly nonlinear and dispersive wave packets in physical systems, e.g., light pulses in optical fibers, surface waves in deep water, Langmuir waves in a plasma, and high-frequency vibrations in a crystal lattice. Equation (7) with an extra linear term $V(x)u$ to account for the external potential $V(x)$ also arises in the study of Bose–Einstein condensates, where it is referred to as the time-dependent Gross–Pitaevskii equation.

Further Reading

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