Involution: The Formal Theory of Differential Equations and its Applications in Computer Algebra. *By Werner M. Seiler.* Springer-Verlag, Berlin Heidelberg, 2010. \$129.00. xxii+650 pp., hardcover: ISBN 978-3-642-01286-0.

Standard books on the theory of differential equations deal with scalar equations and systems of equations in normal or Cauchy-Kovalevskaya form, i.e., systems that are solvable with respect to the highest-order derivative. Traditionally, one considers systems where a distinguished independent variable t exists such that the system can be written in evolution form, $\mathbf{u}_t = \boldsymbol{\phi}(t, \mathbf{x}, \mathbf{u}, \mathbf{u}_{\mathbf{x}})$, where the remaining independent variables are collectively denoted by $\mathbf{x}$.

Despite the popularity of that assumption, *non-normal* systems are also ubiquitous in, e.g., classical and fluid mechanics, gas dynamics, electrodynamics, field theory, and general relativity. Classical examples include the incompressible Navier-Stokes equations, Maxwell's equations, the Yang-Mills equations, and Einstein's equations. In mathematics, non-normal systems of ordinary and partial differential equations (ODEs and PDEs) arise in differential geometry, the study of completely integrable systems, etc. For example, the geometric problems studied by Darboux and Goursat when formulated in local coordinates lead to overdetermined systems of PDEs in non-normal form.

Seiler's book deals with the formal theory of mainly non-normal systems of both ODEs and PDEs. The treatment requires sophisticated tools from differential geometry and commutative abstract algebra, which many applied mathematicians might not be familiar with.

Seiler's book has the catchy title: Involution. Naively, completing a system of differential equations to involutive form compares with bringing a system of linear algebraic equations into triangular form by Gauss elimination. That process eliminates redundant equations and reveals inconsistent equations, if present. Once brought into triangular form, the solvability and dimension of the solution space of the algebraic system become apparent. In the case of differential equations there is more freedom; one may not only perform algebraic operations but also differentiate equations. Yet, the goals of Gauss elimination and the completion process are similar: The *completion* of a system of PDEs will reveal the consistency of the system and allows one to determine the size of the formal solution space.

For example in Lie-point symmetry analysis [3], one has to solve an overdetermined system of linear homogeneous PDEs for the coefficients of the vector field. To design a reliable and powerful integration algorithm for such a system, it needs to be brought into a canonical form (called the involutive or passive form). Roughly speaking, the original system is appended by all its differential consequences. Next, highest-derivatives (often arising from cross-differentiations) are eliminated, and, if they occur, integrability conditions are added to the system. The procedure is then repeated until the new system is in involutive form, which includes all integrability conditions. This type of "reduction" has to be done with extreme care so that no hidden integrability conditions are missed. As expected, the reduced system is easier to analyze and solve. For example, it allows one to determine the size of the symmetry group without having to integrate the equations explicitly.

Thus, the task at hand is "completion of general systems of PDEs to involutive form" and early approaches are due to Riquier and Janet. Nowadays, computer algebra systems can be programmed to do "completion" fast and efficiently. When applied to polynomials, an involutive basis is a Gröbner basis [1], so named by Buchberger in 1960 in honor of his supervisor. All major computer algebra systems use a variant of Buchberger's famous algorithm to compute Gröbner bases. Matters are far more complicated for systems of PDEs and many "differential generalizations" of Buchberger's algorithm are currently available. See, e.g., [3] for a review of early differential Gröbner basis algorithms and their application to symmetry analysis, and [2] for a recent survey of differential Gröbner bases. Obviously, the theory of involutive bases largely parallels the theory of Gröbner bases and Seiler's book draws heavily on that parallelism.

The book has ten chapters (covering 500 pages) and three long appendices (another 100 pages). Chapter 1 gives a short overview of the type of problems that will be treated in the book. In Chapter 2, Seiler introduces jet bundles from two points of view: a "pedestrian" approach based on Taylor series in local coordinates, and a coordinate-independent approach that stresses the intrinsic properties of jet bundles, prolongations, and projections. With jet bundles at hand, Seiler continues with a geometric definition of systems of differential equations and only resorts to (local) coordinates in explicit computations.

No doubt, one has to resort to algebraic methods to get to the heart of involution. Chapter 3 introduces the concept of *involution* – the main theme of the book– in a purely algebraic framework, which at first sight is not at all related to differential equations.

As an interlude, Chapter 4 reviews concrete algorithms for the computation of involutive bases (heavily inspired by work of Gerdt, Blinkov, and Zharkov). In particular, Seiler describes the optimized algorithm that underlies most implementations of involutive bases in computer algebra systems.

Chapter 5 probes deeper into the theory of involutive bases. Emphasis is on the properties of Pommaret bases which appear in subsequent chapters where Seiler gives a constructive definition of involution for differential equations. Readers interested in special properties of Pommaret bases, such as $\delta-$regularity, Rees decomposition, and syzygy theory, will find what they need. Starting with Spencer cohomology and the dual Koszul homology, Chapter 6 discusses the homological interpretation of Pommaret bases.

In Chapter 7, Seiler returns to differential equations and applies the algebraic theory to the analysis of symbols, which allows him to give a rigorous definition of under- and overdetermined equations. That chapter also addresses the fundamental question "how does one bring an arbitrary differential equation into involutive form?" Chapter 8 is devoted to determining an abstract measure for the size of the formal solution space. Applications include the computation of differential relations (viz., Bäcklund transformations) between two differential equations, and removing the effect of gauge symmetries.

Chapter 9 deals with the existence and uniqueness of solutions of differential equations. The challenge is to extend the Cauchy-Kovalevskaya Theorem (applicable to analytic normal systems) to involutive systems. That subject is not new and early efforts resulted in theories due to Riquier and Janet and Cartan and Kähler, the latter used differential forms instead of the jet bundle formalism. Seiler considers various alternatives from the vast literature on "existence and uniqueness" of solutions of normal systems. Chapter 9 also has a novel presentation of Vessiot's dual version of Cartan's theory of exterior differential systems. Seiler shows that equations must be in involutive form (indeed, formal integrability is insufficient) for a successful application of the approaches due to Janet-Riquier, Cartan-Kähler, and Vessiot.

Finally, in Chapter 10, Seiler considers *linear* systems of differential equations from an algebraic point of view. He shows that the algorithms for Cartan-Kuranishi completion and involutive completion may be merged into a new algebraic algorithm that is faster and more efficient than either one separately.

This book grew out of Seiler's habilitation thesis, defended at the University of Mannheim in 2002. Despite interesting notes with pointers to the literature and tidbits of history at the end of each chapter, the book is highly technical, it reads like a research paper, and it would be hard to use as a textbook on the subject.

Half of the book deals with the geometry behind the formal theory of differential equations; the other half is concerned with commutative algebra. The book targets the mathematically mature reader who can handle the frequent jumps between geometric and algebraic approaches. It is written at the post-graduate level and requires a deep understanding of jet bundles, differential forms, and differential geometry on the one hand and commutative and homological algebra on the other hand. Few mathematicians will have expertise in both and, therefore, Seiler added two long appendices with background material.

The appendices merely review the basic results that are used in the main text; they are certainly no substitute for introductory books on abstract algebra or differential geometry. Yet, the appendices helped me better understand the notation. The two-page glossary provided by Seiler is inadequate for it lacks a brief description of the symbols. Since involutive bases are a special form of Gröbner bases, a specialist in computer algebra who wants to implement algorithms for involution should have some familiarity with Buchberger's algorithm (and its variants). To help with that, one of the appendices covers the basics of Gröbner bases.

The few existing books on the subject of involution are often written in incomprehensible mathematical language that only appeals to aficionados of a particular "school." Although Seiler is strongly influenced by the work of Pommaret and his followers, the notations and terminology in Seiler's book are standard and Seiler stresses the similarities (rather than the differences) between the various theories.

Seiler strikes a good balance between theory, algorithms, and applications. Nonetheless, I would have liked to see an early chapter with carefully chosen examples that illustrate what the formal theory of differential equations wants to accomplish. The historical notes are delightful reading, but they could have been integrated with an introductory chapter that showed (again via examples) how Janet's approach to involution actually works.

By clarifying the relations between differential equations and commutative algebra, Seiler's book hopes to bridge two research communities: one with differential geometers concerned with the formal analysis of differential equations, the other with (computer) algebraists interested in computations with polynomials modules. Only time (and success in sales) will tell if Seiler's book accomplished this lofty goal.

References

[1] B. Buchberger, M. Kauers, *Groebner basis*, Scholarpedia, 5 (10):7763, October 2010; `http://www.scholarpedia.org/article/Groebner_basis`.

[2] G. Carrà Ferro, *A survey on differential Gröbner bases*, in Gröbner Bases in Symbolic Analysis, M. Rosenkranz and D. Wang, eds., Radon Series on Computation and Applied Mathematics, vol. 2, Walter de Gruyter, Berlin, 2007, pp. 43-73.

[3] W. Hereman, *Symbolic software for Lie symmetry analysis*, in CRC Handbook of Lie Group Analysis of Differential Equations, vol. 3: New Trends in Theoretical Developments and Computational Methods, N. H. Ibragimov, ed., CRC Press, Boca Raton, Florida, 1996, pp. 367-413.

Willy Hereman
Colorado School of Mines