SIMON STEVIN, A Quarterly Journal of Pure and Applied Mathematics
Volume 54 (1980), Number 3 - 4 (July - October 1980)

DIFFRACTION OF LIGHT BY AN AMPLITUDE-MODULATED ULTRASONIC WAVE AT NORMAL AND OBLIQUE INCIDENCE OF THE LIGHT

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communicated by R. Mertens

The intensities of the diffraction pattern of light in a liquid, disturbed by an amplitude-modulated ultrasonic wave, have been calculated on the hand of Raman-Nath's elementary theory. The results obtained by Mertens are extended to the case of a multiple frequency transducer-output. The intensity expressions obtained here for oblique incidence of the light beam, include interference cases neglected by Aggarwal et al. Finally the symmetry of the various types of diffraction lines is investigated : in the large wavelength approximation the principal lines remain symmetric with respect to the zero order central line, but there is no longer symmetry of the satellite lines with respect to a principal line.

1. INTRODUCTION

Raman and Nagendra Nath [14,15] established their so-called "elementary theory" in order to calculate the amplitudes of the diffracted electrical field traversing a liquid medium disturbed by an ordinary progressive ultrasonic wave (with large wavelength). This method, based on the principles of geometrical optics, is independent of the nature of the disturbing sound wave. So it is also applicable to the case of diffraction of light by an amplitude-modulated (AM) ultrasonic wave. Experimental research on that topic has been done by Pancholy and Parthasarathy [1], while more theoretical considerations can be find in papers of Aggarwal et al. [1,2], Phariseau [11,12], Mertens [8,9] and Hereman and Mertens [5]. We refer to publication [9] for a survey of the other papers on this phenomenon. All those authors considered an AM-wave only depending on one modulation frequency and most of them have restricted their treatment to normal incidence of the light beam.

In the first part of the present paper we generalize the most recent results (obtained by Mertens [8,9]) to the case of a AM-output with multiple frequencies. We obtain an intricate Bessel-function expression which is valid at normal incidence of the light. In the calculation of that expression for the intensities we have taken into account the interference cases, not considered by Aggarwal et al. [1,2]. In the second part we treat the same problem, this time for oblique incidence of the light beam. Furthermore we have shown that the symmetry properties of the diffraction pattern, as investigated by Mertens for normal incidence of the light [9], are no longer fulfilled at oblique incidence of the light beam.

2. GENERAL AMPLITUDE-MODULATED (ULTRASONIC) WAVE

Consider an (ultrasonic) wave, with frequency $\nu^* = \frac{\omega^*}{2\pi}$ and wave-length $\lambda^* = \frac{2\pi}{k^*}$, travelling in the x-direction with constant amplitude a_0,

$$a_0 \sin(\omega^* t - k^* x + \delta) \ . \tag{1}$$

This wave is said to be modulated when its amplitude, frequency or phase constant δ is modified by an other wave. We shall not persue the last two cases, which are nearly related and often called frequency modulation (FM), but only treat amplitude modulation (AM) here. In the latter case the constant amplitude a_0 must be replaced by

$$a_0(1+mS(x,t)) \ , \tag{2}$$

where m is called the modulation depth and $S(x,t)$ represents the modulation wave. In the most simple case $S(x,t)$ varies sinusoidally in time (at given x) at the modulation pulsatance ω_1^* and sinusoidally in space (at fixed time) with the modulation wavenumber k_1^*. Hence, the amplitude-modulated wave

$$a_0[1+m\cos(\omega_1^* t-k_1^* x+\delta_1)]\sin(\omega^* t-k^* x+\delta) \ , \tag{3}$$

may be rewritten as

$$a_0\sin(\omega^* t-k^* x+\delta)+\frac{1}{2}a_0 m\sin[(\omega^*+\omega_1^*)t-(k^*+k_1^*)x+\delta+\delta_1]$$
$$+\frac{1}{2}a_0 m\sin[(\omega^*-\omega_1^*)t-(k^*-k_1^*)x+\delta-\delta_1] \ , \tag{4}$$

where δ_1 is a arbitrary phase constant of the modulating wave.

In all practical realizations, the modulation depth $m \ll 1$, the modulation frequency $\nu_1^* = \omega_1^*/2\pi$ is small compared to the carrier frequency ν^* ; further a translation in time is introduced to make $\delta = 0$. The amplitude-modulated (almost sinusoidally travelling) wave (4) is a superposition of a simple travelling wave with pulsatance ω^* called the carrier wave and a sum of two travelling waves with pulsatance $\omega^*+\omega_1^*$ (called the upper sideband) and $\omega^*-\omega_1^*$ (called the lower sideband). In commercial AM we must take into account [4] not just one modulation frequency but a whole range of slightly different modulation frequencies $\nu_1^*,\nu_2^*,\ldots,\nu_N^*$, all small compared with the carrier frequency ν^*.
The AM-wave (4) may then be generalized to

$$a_0 \sin(\omega^* t - k^* x) + \frac{1}{2} \sum_{j=1}^{N} a_j \sin[(\omega^* + \omega_j^*)t - (k^* + k_j^*)x + \delta_j]$$
$$+ \frac{1}{2} \sum_{j=1}^{N} a_j \sin[(\omega^* - \omega_j^*)t - (k^* - k_j^*)x - \delta_j] \quad , \qquad (5)$$

consisting of the carrier wave and a sum of travelling waves with frequencies $\nu^* + \nu_j^*$ in the upper sideband and $\nu^* - \nu_j^*$ in the lower side-band. a_j and δ_j ($j=1,\dots,N$) respectively are the constant amplitude and phase constant belonging to the pair of travelling waves with frequencies $\nu^* \pm \nu_j^*$.

For technical comment on producing AM, bandwidth, bandpass filters etc., we refer to specialized literature [10,16,17].

3. RAMAN-NATH'S ELEMENTARY THEORY FOR NORMAL INCIDENCE OF LIGHT

In their first paper, Raman and Nagendra Nath [14] built up their theory for the diffraction phenomenon of light by sound in a liquid, at normal incidence of the light wave.

They considered a parallel beam of monochromatic light with wave length λ (in vacuum), circular wavenumber $k = 2\pi/\lambda$ and frequency $\nu = \omega/2\pi$, traversing a homogeneous liquid medium over a distance L. Putting the x-axis along the direction of propagation of the ultrasonic wave and the z-axis (perpendicular to it) in the direction of the incident light beam, the incoming lightwave may be written as

$$\Psi_0(t) = \exp(ikct) \quad , \qquad (6)$$

while the outgoing diffracted lightwave can be expressed as

$$\Psi(x,L,t) = \exp[ik(ct - \sqrt{\varepsilon(x,t)}L)] \quad . \qquad (7)$$

Following Born and Wolf [3] we prefered the use of the relative permittivity $\varepsilon(x,t)$ of the medium instead of the refractive index $\mu(x,t)$ of the liquid. The use of the relative permittivity $\varepsilon(x,t)$, undergoing periodic fluctuations due to the ultrasonic wave, is more in accordance with the physics of the problem. Furthermore it presents no more difficulties in the mathematical treatment since both are related by the simple expression

$$\varepsilon(x,t) = \mu^2(x,t) \ . \tag{8}$$

From the considerations of Section 1, the relative permittivity at a point of the disturbed liquid medium is given by

$$\begin{aligned}\varepsilon(x,t) = \varepsilon_0 &+ \varepsilon_1 \sin(\omega^* t - k^* x) \\ &+ \frac{1}{2}\sum_{j=1}^{N} \varepsilon_{j+1} \sin[(\omega^* + \omega_j^*)t - (k^* + k_j^*)x + \delta_j] \\ &+ \frac{1}{2}\sum_{j=1}^{N} \varepsilon_{j+1} \sin[(\omega^* - \omega_j^*)t - (k^* - k_j^*)x - \delta_j] \ ,\end{aligned} \tag{9}$$

where we have taken into account that the quartz crystal, producing the modulated ultrasonic field, has several slightly different modulation frequencies. ε_0 is the dielectric constant of the medium in its undisturbed state, ε_1 the maximum variation of the relative permittivity for the carrier wave, ε_{j+1} $(j=1,2,..,N)$ the maximum variations for the travelling waves at frequency $\nu^* \pm \nu_j^*$. Since $\varepsilon_1,\ldots,\varepsilon_{N+1}$ are always much smaller than ε_0, we approximately have

$$\begin{aligned}[\varepsilon(x,t)]^{1/2} = \sqrt{\varepsilon_0} &+ \frac{1}{2}\frac{\varepsilon_1}{\sqrt{\varepsilon_0}} \sin(\omega^* t - k^* x) \\ &+ \frac{1}{4\sqrt{\varepsilon_0}}\sum_{j=1}^{N} \varepsilon_{j+1} \sin[(\omega^* + \omega_j^*)t - (k^* + k_j^*)x + \delta_j]\end{aligned}$$

$$+\frac{1}{4\sqrt{\varepsilon_0}}\sum_{j=1}^{N}\varepsilon_{j+1}\sin[(\omega^*-\omega_j^*)t-(k^*-k_j^*)x-\delta_j]\ . \qquad (10)$$

Substituting this expression into (7) we obtain

$$\Psi(x,L,t)=\exp(i\omega t)\exp(-ik\sqrt{\varepsilon_0}L)\ \exp\left[\frac{-ikL\varepsilon_1}{2\sqrt{\varepsilon_0}}\sin(\omega^*t-k^*x)\right]$$

$$\prod_{j=1}^{N}\exp\left\{\frac{-ikL\varepsilon_{j+1}}{4\sqrt{\varepsilon_0}}\sin[(\omega^*+\omega_j^*)t-(k^*+k_j^*)x+\delta_j]\right\}$$

$$\prod_{j=1}^{N}\exp\left\{\frac{-ikL\varepsilon_{j+1}}{4\sqrt{\varepsilon_0}}\sin[(\omega^*-\omega_j^*)t-(k^*-k_j^*)x-\delta_j]\right\}\ . \qquad (11)$$

Taking into account Jacobi's formula

$$e^{-iz\sin\theta}=\sum_{n=-\infty}^{+\infty}(-1)^n J_n(z)e^{in\theta}\ , \qquad (12)$$

the foregoing expression can be transformed into a product of absolutely convergent series, and becomes after some rearrangements

$$\Psi(x,L,t) = \exp(i\omega t)\ \exp(-ik\sqrt{\varepsilon_0}L)$$

$$\sum_{n_0,n_1,\ldots,n_{2N}=-\infty}^{+\infty} J_{n_0}\left(\frac{kL\varepsilon_1}{2\sqrt{\varepsilon_0}}\right)\left[\prod_{j=1}^{N} J_{n_{2j-1}}\left(\frac{kL\varepsilon_{j+1}}{4\sqrt{\varepsilon_0}}\right)J_{n_{2j}}\left(\frac{kL\varepsilon_{j+1}}{4\sqrt{\varepsilon_0}}\right)\right]$$

$$\exp\{-i[(n_0+\sum_{j=1}^{2N}n_j)\omega^*+\sum_{j=1}^{N}(n_{2j-1}-n_{2j})\omega_j^*]t\}$$

$$\exp\{i[(n_0+\sum_{j=1}^{2N}n_j)k^*+\sum_{j=1}^{N}(n_{2j-1}-n_{2j})k_j^*]x\}$$

$$\exp\{-i\sum_{j=1}^{N}(n_{2j-1}-n_{2j})\delta_j\}\ , \qquad (13)$$

wherein $J_n(z)$ is the Besselfunction of the first kind of order n.

In order to give an interpretation of the diffracted lightwave emerging at the boundary plane z = L, we shall change the 2N+1 summation indices as follows :

Putting

$$\ell = n_0 + \sum_{j=1}^{2N} n_j \ , \tag{14}$$

$$m_j = n_{2j-1} - n_{2j} \qquad (j=1,\ldots,N) \ , \tag{15}$$

we define in this way already N+1 new summation indices $\ell, m_1, \ldots, m_N$. Consequently we have to choose another N summation indices, among the given ones $n_0, n_1, n_2, \ldots, n_{2N}$. With regard to (15), it seems justified to take $n_1, n_3, \ldots, n_{2N-1}$ as the N remaining summation indices, and to express $n_0, n_2, n_4, \ldots, n_{2N}$ in terms of the new indices $\ell, m_1, m_2, \ldots, m_N$, $n_1, n_3, \ldots, n_{2N-1}$.
On the hand of (14)-(15) it is immediately clear that

$$n_{2j} = n_{2j-1} - m_j \qquad (j=1,\ldots,N) \ , \tag{16}$$

and

$$n_0 = \ell + \sum_{j=1}^{N} (m_j - 2n_{2j-1}) \ . \tag{17}$$

For notational convenience we put $\alpha_j = n_{2j-1}$ $(j=1,\ldots,N)$, so that we finally obtain as an equivalent form of (13),

$$\begin{aligned}\Psi(x,L,t) = {} & \exp(i\omega t) \exp(-ik\sqrt{\varepsilon_0}L) \\ & \sum_{\ell, m_1, m_N, \alpha_1, \alpha_2, \ldots, \alpha_N = -\infty}^{+\infty} J_{\ell + \sum_{j=1}^{N} (m_j - 2\alpha_j)} \left(\frac{kL\varepsilon_1}{2\sqrt{\varepsilon_0}}\right) \\ & \left[\prod_{j=1}^{N} J_{\alpha_j}\left(\frac{kL\varepsilon_{j+1}}{4\sqrt{\varepsilon_0}}\right) J_{\alpha_j - m_j}\left(\frac{kL\varepsilon_{j+1}}{4\sqrt{\varepsilon_0}}\right)\right] \\ & \exp[-i(\ell\omega^* + \sum_{j=1}^{N} m_j\omega_j^*)t] \exp[i(\ell k^* + \sum_{j=1}^{N} m_j k_j^*)x] \\ & \exp[-i \sum_{j=1}^{N} m_j \delta_j] \ . \end{aligned} \tag{18}$$

So it is clear that the lightwave is split into different subwaves.

Each subwave represents a diffracted beam of light, making an angle $\theta_{\ell,m_1,\ldots,m_N}$ with the z-axis, given by

$$\sin\theta_{\ell,m_1,\ldots,m_N} = -\left(\frac{\ell}{\lambda^*} + \frac{m_1}{\lambda_1^*} + \frac{m_2}{\lambda_2^*} + \ldots + \frac{m_n}{\lambda_N^*}\right) \frac{\lambda}{\sqrt{\varepsilon_0}} , \tag{19}$$

and having the frequency-shift

$$\Delta\nu_{\ell,m_1,\ldots,m_N} = -(\ell\nu^* + m_1\nu_1^* + m_2\nu_2^* + \ldots + m_N\nu_N^*) . \tag{20}$$

Since each subwave has a different direction (°) and a different frequency-shift, it is meaningful to speak about a diffracted wave of order $(\ell,m_1,m_2,\ldots,m_N)$ with the amplitude $\phi_{\ell,m_1,m_2,\ldots,m_N}$ obtained from (18) :

$$\phi_{\ell,m_1,\ldots,m_N} = \sum_{\alpha_1,\alpha_2,\ldots,\alpha_N=-\infty}^{+\infty} J_{\ell+\sum_{j=1}^{N}(m_j-2\alpha_j)}\left(\frac{kL\varepsilon_1}{2\sqrt{\varepsilon_0}}\right)$$

$$\times \left(\prod_{j=1}^{N} J_{\alpha_j}\left(\frac{kL\varepsilon_{j+1}}{4\sqrt{\varepsilon_0}}\right) J_{\alpha_j-m_j}\left(\frac{kL\varepsilon_{j+1}}{4\sqrt{\varepsilon_0}}\right)\right)$$

$$\times \exp\left(-i \sum_{j=1}^{N} m_j\delta_j\right) . \tag{19}$$

The intensity of order $(\ell,m_1,m_2,\ldots,m_N)$ will then be given by

$$I_{\ell,m_1,\ldots,m_N} = \phi_{\ell,m_1,\ldots,m_N} \cdot \phi^*_{\ell,m_1,\ldots,m_N} \tag{20}$$

where the asterisk stands for complex conjugation.

(°) We suppose the different frequencies $\nu^*,\nu_1^*,\ldots,\nu_N^*$ to be global incommensurable, which means here that
$\ell\nu^* + m_1\nu_1^* + m_2\nu_2^* + \ldots + m_N\nu_N^* = \tilde{\ell}\nu^* + \tilde{m}_1\nu_1^* + \tilde{m}_2\nu_2^* + \ldots + \tilde{m}_N\nu_N^*$
iff $\ell = \tilde{\ell}$, $m_j = \tilde{m}_j$ $(\ell,\tilde{\ell},m_j,\tilde{m}_j \in Z$, $j=1,\ldots,N)$.

So, finally we obtain

$$I_{\ell,m_1,\ldots,m_N} =$$

$$\sum_{\alpha_1,\alpha_2,\ldots,\alpha_N=-\infty}^{+\infty} \; \sum_{\beta_1,\beta_2,\ldots,\beta_N=-\infty}^{+\infty} J_{\ell+\sum\limits_{j=1}^{N}(m_j-2\alpha_j)}\left(\frac{kL\varepsilon_1}{2\sqrt{\varepsilon_0}}\right)$$

$$\times J_{\ell+\sum\limits_{j=1}^{N}(m_j-2\beta_j)}\left(\frac{kL\varepsilon_1}{2\sqrt{\varepsilon_0}}\right)$$

$$\times \left[\prod_{j=1}^{N} J_{\alpha_j}\left(\frac{kL\varepsilon_{j+1}}{4\sqrt{\alpha_0}}\right) J_{\alpha_j-m_j}\left(\frac{kL\varepsilon_{j+1}}{4\sqrt{\varepsilon_0}}\right)\right.$$

$$\left. \times J_{\beta_j}\left(\frac{kL\varepsilon_{j+1}}{4\sqrt{\varepsilon_0}}\right) J_{\beta_j-m_j}\left(\frac{kL\varepsilon_{j+1}}{4\sqrt{\varepsilon_0}}\right)\right] . \qquad (21)$$

From the formulae (19) and (20) it is seen that the direction of the diffraction lines and the corresponding frequency-shift generally depend on the carrier frequency and all the other modulation frequencies.

For $m_1 = m_2 = \ldots = m_N = 0$ we find for each value of ℓ diffraction lines having the same directions and frequencies as in the case where the ultrasonic is not modulated [7], but with different intensities [6]. We shall call them "principal lines".

In the case where $m_1, m_2, \ldots, m_N$ are not equal to zero altogether, we may vary those indices for each fixed value of ℓ, so that for each principal line, associated "satellite lines" are observed.

Now we pay attention to the special case N = 1, where only one modulation frequency ν_1^* acts upon the carrier wave.

In that simplified case we obtain diffraction lines of order (ℓ, m_1), defined by the direction

$$\sin\theta_{\ell,m_1} = -\left(\frac{\ell}{\lambda^*} + \frac{m_1}{\lambda_1^*}\right)\frac{\lambda}{\sqrt{\varepsilon_0}} , \qquad (23)$$

with frequency-shift

$$\Delta\nu_{\ell,m_1} = -(\ell^* \nu^* + m_1 \nu_1^*) \ , \tag{24}$$

and having the intensity (of order (ℓ, m_1))

$$I_{\ell,m_1} = \sum_{\alpha_1,\beta_1=-\infty}^{+\infty} J_{\ell+m_1-2\alpha_1}\left(\frac{kL\varepsilon_1}{2\sqrt{\varepsilon_0}}\right) J_{\ell+m_1-2\beta_1}\left(\frac{kL\varepsilon_1}{2\sqrt{\varepsilon_0}}\right) J_{\alpha_1}\left(\frac{kL\varepsilon_2}{4\sqrt{\varepsilon_0}}\right) J_{\alpha_1-m_1}\left(\frac{kL\varepsilon_2}{4\sqrt{\varepsilon_0}}\right) J_{\beta_1}\left(\frac{kL\varepsilon_2}{4\sqrt{\varepsilon_0}}\right) J_{\beta_1-m_1}\left(\frac{kL\varepsilon_2}{4\sqrt{\varepsilon_0}}\right) . \tag{25}$$

Putting

$$\left.\begin{aligned} \ell+m_1-2\alpha_1 &= p \\ \ell+m_1-2\beta_1 &= q \end{aligned}\right. , \tag{26}$$

from which

$$\begin{aligned} \alpha_1 &= \frac{\ell+m_1-p}{2} \ , & \alpha_1-m_1 &= \frac{\ell-m_1-p}{2} \ , \\ \beta_1 &= \frac{\ell+m_1-q}{2} \ , & \beta_1-m_1 &= \frac{\ell-m_1-q}{2} \ , \end{aligned} \tag{27}$$

it is clear that (25) is equivalent to

$$I_{\ell,m_1} = \sum_{\ell,m_1=-\infty}^{+\infty} \sum_{\substack{p,q=-\infty \\ \text{both even} \\ \text{or odd}}}^{+\infty} J_p\left(\frac{kL\varepsilon_1}{2\sqrt{\varepsilon_0}}\right) J_q\left(\frac{kL\varepsilon_1}{2\sqrt{\varepsilon_0}}\right) \times J_{\frac{\ell+m_1-p}{2}}\left(\frac{kL\varepsilon_2}{4\sqrt{\varepsilon_0}}\right) J_{\frac{\ell+m_1-q}{2}}\left(\frac{kL\varepsilon_2}{4\sqrt{\varepsilon_0}}\right) \times J_{\frac{\ell-m_1-p}{2}}\left(\frac{kL\varepsilon_2}{4\sqrt{\varepsilon_0}}\right) J_{\frac{\ell-m_1-q}{2}}\left(\frac{kL\varepsilon_2}{4\sqrt{\varepsilon_0}}\right) . \tag{28}$$

Remark that α_1 and β_1 being integers, $\ell+m_1$ (or $\ell-m_1$) and p (resp. q) must be both even or both odd. Consequently the summation indices p and q must be even or odd at the same time.

On the one side, this result is in accordance to the intensity expression derivable from the generalized Raman-Nath theory in the special case $\rho = 0$ (long wavelength approximation) [5, paragraph 5].

On the other hand, up to some trivial modifications, (28) corresponds to the intensity formulae obtained by Mertens [9, Eq.(11)]. In that paper a discussion of the diffraction pattern, concerning the symmetry and the relative intensities of the diffraction lines, can be found.

Before extending these considerations to the case of N modulation frequencies, we generalize the foregoing results to the case of oblique incidence of the light.

4. RAMAN-NATH'S ELEMENTARY THEORY FOR OBLIQUE INCIDENCE OF LIGHT

In a second paper [15] Raman and Nagendra Nath extended their elementary theory to the case of oblique incidence of the light beam. They proved that the optical length of a path in the medium between z=0 and z=L and parallel to the direction of the incident light (making this time an angle φ with the z-axis) is equal to

$$[0, L\sec\varphi] = \int_0^{L\sec\varphi} \sqrt{\varepsilon(s)}\,ds , \qquad (29)$$

where $\varepsilon(s) = \varepsilon(x-s\sin\varphi,t)$ is the periodic relative permittivity of the considered medium. Once the integral (29) is calculated, the diffracted lightwave coming out in the boundary plane z=L, may be represented by

$$\Psi(x,L,t) = \exp\{ik(ct-x\sin\varphi-[0,L\sec\varphi])\} . \qquad (30)$$

Although it is possible to consider from the beginning a transducer with an AM-output consisting of N modulation frequencies, we confine the treatment for the sake of notational simplicity to the case of only one modulation frequency ν_1^*. The generalization of the results will go without saying. Employing (10) (for N=1) we get

$$
\begin{aligned}
[0,L\sec\varphi] = \sqrt{\varepsilon_0}L\sec\varphi &+ A_1\sin(\omega^* t-k^* x) - B_1\cos(\omega^* t-k^* x) \\
&+ A_2\sin[(\omega^*+\omega_1^*)t-(k^*+k_1^*)x] - B_2\cos[(\omega^*+\omega_1^*)t-(k^*+k_1^*)x+\delta_1] \\
&+ A_3\sin[(\omega^*-\omega_1^*)t-(k^*-k_1^*)x] - B_3\cos[(\omega^*-\omega_1^*)t-(k^*-k_1^*)x-\delta_1]
\end{aligned}
\tag{31}
$$

with

$$
\begin{aligned}
A_1 &= \frac{1}{2}\,\frac{\varepsilon_1}{\sqrt{\varepsilon_0}}\,\frac{1}{k^*\sin\varphi}\,\sin(k^* L\tan\varphi) \\
B_1 &= \frac{1}{2}\,\frac{\varepsilon_1}{\sqrt{\varepsilon_0}}\,\frac{1}{k^*\sin\varphi}\,[\cos(k^* L\tan\varphi)-1] \quad ,
\end{aligned}
\tag{32}
$$

and wherein A_2, B_2 respectively A_3, B_3 stand for analogous expressions where ε_1 is replaced by $\varepsilon_2/2$ and k^* respectively by $k^*+k_1^*$ and $k^*-k_1^*$. It is straightforward to verify that the optical length can be rewritten as

$$
\begin{aligned}
[0,L\sec\varphi] = \sqrt{\varepsilon_0}L\sec\varphi &+ \frac{\varepsilon_1}{\sqrt{\varepsilon_0}k^*\sin\varphi}\sin\left(\frac{k^* L\tan\varphi}{2}\right)\sin\left(\omega^* t-k^* x+\frac{k^* L\tan\varphi}{2}\right) \\
&+ \frac{\varepsilon_2}{2\sqrt{\varepsilon_0}(k^*+k_1^*)\sin\varphi}\sin\left[\frac{(k^*+k_1^*)L\tan\varphi}{2}\right]\sin\left[(\omega^*+\omega_1^*)t-(k^*+k_1^*)x+\delta_1 + \frac{(k^*+k_1^*)L\tan\varphi}{2}\right] \\
&+ \frac{\varepsilon_2}{2\sqrt{\varepsilon_0}(k^*-k_1^*)\sin\varphi}\sin\left[\frac{(k^*-k_1^*)L\tan\varphi}{2}\right]\sin\left[(\omega^*-\omega_1^*)t-(k^*-k_1^*)x-\delta_1 + \frac{(k^*-k_1^*)L\tan\varphi}{2}\right] .
\end{aligned}
\tag{33}
$$

Substituting this final expression into (30) and using the

Jacobi expansion (12), we obtain after some trivial operations

$$\Psi(x,L,t) = \exp(i\omega t)\ \exp[-ik(x\sin\varphi+\sqrt{\varepsilon_0}L\sec\varphi)]$$

$$\sum_{n_0,n_1,n_2=-\infty}^{+\infty} J_{n_0}\left[\frac{k\varepsilon_1\sin(\frac{k^*L\tan\varphi}{2})}{\sqrt{\varepsilon_0}k^*\sin\varphi}\right] J_{n_1}\left[\frac{k\varepsilon_2\sin[\frac{(k^*+k_1^*)L\tan\varphi}{2}]}{2\sqrt{\varepsilon_0}(k^*+k_1^*)\sin\varphi}\right]$$

$$\times J_{n_2}\left[\frac{k\varepsilon_2\sin[\frac{(k^*-k_1^*)L\tan\varphi}{2}]}{2\sqrt{\varepsilon_0}(k^*-k_1^*)\sin\varphi}\right]$$

$$\times\exp\{-i[(n_0+n_1+n_2)\omega^*+(n_1-n_2)\omega_1^*]t\}$$

$$\times\exp\{+i[(n_0+n_1+n_2)k^*+(n_1-n_2)k_1^*]x\}$$

$$\times\exp\{-i[(n_1-n_2)\delta_1+(n_0+n_1+n_2)\frac{k^*L\tan\varphi}{2}$$

$$+ (n_1-n_2)\frac{k_1^*L\tan\varphi}{2}]\}\ . \qquad (34)$$

According to (14) and (15) we put

$$\ell = n_0+n_1+n_2\ ,$$
$$m_1 = n_1-n_2\ , \qquad (35)$$

from which it follows that

$$n_2 = n_1-m_1\ ,$$
$$n_0 = \ell+m_1-2n_1\ . \qquad (36)$$

Hence, the diffracted lightwave in the case of oblique incidence of light becomes

$$\Psi(x,L,t) = \sum_{\ell,m_1,\alpha_1=-\infty}^{+\infty} J_{\ell+m_1-2\alpha_1}\left[\frac{k\varepsilon_1 \sin(\frac{1}{2}k^* L\tan\varphi)}{\sqrt{\varepsilon_0}k^*\sin\varphi}\right]$$

$$\times J_{\alpha_1}\left[\frac{k\varepsilon_2 \sin[\frac{1}{2}(k^*+k_1^*)L\tan\varphi]}{2\sqrt{\varepsilon_0}(k^*+k_1^*)\sin\varphi}\right] J_{\alpha_1-m_1}\left[\frac{k\varepsilon_2 \sin[\frac{1}{2}(k^*-k_1^*)L\tan\varphi]}{2\sqrt{\varepsilon_0}(k^*-k_1^*)\sin\varphi}\right]$$

$$\times \exp(i\omega t)\ \exp[-ik(x\sin\varphi+\sqrt{\varepsilon_0}L\sec\varphi)]$$

$$\times \exp[-i(\ell\omega^*+m_1\omega_1^*)t]$$

$$\times \exp[i(\ell k^*+m_1k_1^*)x]$$

$$\times \exp[-i(m_1\delta_1+\frac{1}{2}\ell k^* L\tan\varphi+\frac{1}{2}m_1k_1^*L\tan\varphi)] \quad , \qquad (37)$$

built up of subwaves of order (ℓ,m_1) with direction

$$\sin\theta_{\ell,m_1} = \sin\varphi - \left(\frac{\ell}{\lambda^*}+\frac{m_1}{\lambda_1^*}\right)\frac{\lambda}{\sqrt{\varepsilon_0}} \quad , \qquad (38)$$

frequency-shift (24) as in the case of normal incidence of the light, but with more complicated intensities

$$I_{\ell,m_1}(\varphi) = \sum_{\alpha_1,\beta_1=-\infty}^{+\infty} J_{\ell+m_1-2\alpha_1}\left[\frac{k\varepsilon_1 \sin(\frac{1}{2}k^* L\tan\varphi)}{\sqrt{\varepsilon_0}k^*\sin\varphi}\right]$$

$$\times J_{\ell+m_1-2\beta_1}\left[\frac{k\varepsilon_1 \sin(\frac{1}{2}k^* L\tan\varphi)}{\sqrt{\varepsilon_0}k^*\sin\varphi}\right] J_{\alpha_1}\left[\frac{k\varepsilon_2 \sin[\frac{1}{2}(k^*+k_1^*)L\tan\varphi]}{2\sqrt{\varepsilon_0}(k^*+k_1^*)\sin\varphi}\right]$$

$$\times J_{\beta_1}\left[\frac{k\varepsilon_2 \sin[\frac{1}{2}(k^*+k_1^*)L\tan\varphi]}{2\sqrt{\varepsilon_0}(k^*+k_1^*)\sin\varphi}\right] J_{\alpha_1-m_1}\left[\frac{k\varepsilon_2 \sin[\frac{1}{2}(k^*-k_1^*)L\tan\varphi]}{2\sqrt{\varepsilon_0}(k^*-k_1^*)\sin\varphi}\right]$$

$$\times J_{\beta_1-m_1}\left[\frac{k\varepsilon_2 \sin[\frac{1}{2}(k^*-k_1^*)L\tan\varphi]}{2\sqrt{\varepsilon_0}(k^*-k_1^*)\sin\varphi}\right] \quad . \qquad (39)$$

It is easy to show that this result is equivalent to the result already found as a special case of the generalized Raman-Nath theory [5, Eq.(52)].

We also remark that it is possible to obtain the intensities (25) corresponding to normal incidence of light, from (39) as a limit for φ tending to zero. Furthermore comparing the results (25) and (39) it is clear that

$$I_{\ell,m_1,m_2,\ldots,m_N}(\varphi) = \sum_{\alpha_1,\alpha_2,\ldots,\alpha_N=-\infty}^{+\infty} \; \sum_{\beta_1,\beta_2,\ldots,\beta_N=-\infty}^{+\infty}$$

$$J_{\ell+\sum_{j=1}^{N}(m_j-2\alpha_j)}\left[\frac{k\varepsilon_1\sin(\frac{1}{2}k^*L\tan\varphi)}{\sqrt{\varepsilon_0}k^*\sin\varphi}\right] J_{\ell+\sum_{j=1}^{N}(m_j-2\beta_j)}\left[\frac{k\varepsilon_1\sin(\frac{1}{2}k^*L\tan\varphi)}{\sqrt{\varepsilon_0}k^*\sin\varphi}\right]$$

$$\times\left\{\prod_{j=1}^{N} J_{\alpha_j}\left[\frac{k\varepsilon_{j+1}\sin[\frac{1}{2}(k^*+k_j^*)L\tan\varphi]}{2\sqrt{\varepsilon_0}(k^*+k_j^*)\sin\varphi}\right] J_{\alpha_j-m_j}\left[\frac{k\varepsilon_{j+1}\sin[\frac{1}{2}(k^*-k_j^*)L\tan\varphi]}{2\sqrt{\varepsilon_0}(k^*-k_j^*)\sin\varphi}\right]\right.$$

$$\left. J_{\beta_j}\left[\frac{k\varepsilon_{j+1}\sin[\frac{1}{2}(k^*+k_j^*)L\tan\varphi]}{2\sqrt{\varepsilon_0}(k^*+k_j^*)\sin\varphi}\right] J_{\beta_j-m_j}\left[\frac{k\varepsilon_{j+1}\sin[\frac{1}{2}(k^*-k_j^*)L\tan\varphi]}{2\sqrt{\varepsilon_0}(k^*-k_j^*)\sin\varphi}\right]\right\} \tag{40}$$

will be the intensity of order $(\ell,m_1,\ldots,m_N)$ (in the case of oblique incidence) when the AM-output contains N modulation frequencies.

Anyhow we get expressions for the intensities different from those obtained by Aggarwal et al. [1,2]. Those authors considered any line of the diffraction pattern as the consequence of one single subwave, part of (9), and excluded all possible interference.
A further investigation of this point of discussion and of the relative intensities of the various types of lines is given by Mertens [9].

5. SYMMETRY PROPERTIES OF THE DIFFRACTION SPECTRA

In this final section we investigate the symmetry properties of the diffraction spectrum for normal incidence as well as for oblique incidence of the light.

Following Mertens [9] we consider symmetry of the principal lines with respect to the zero-order central line, by changing ℓ into $-\ell$ and m_j into $-m_j$ $(j=1,\ldots,N)$ in the formula (40). Replacing the summation indices α_j, β_j by their opposites and making repeatedly use of the property of the Besselfunctions

$$J_{-n}(z) = (-1)^n J_n(z) , \tag{41}$$

we obtain that

$$I_{-\ell,-m_1,-m_2,\ldots,-m_N}(\varphi) = I_{\ell,m_1,m_2,\ldots,m_N}(\varphi) . \tag{42}$$

We thus may conclude that even for oblique incidence of light the whole diffraction spectrum is symmetric with respect to the zero-order line ($\ell=m_j=0$, $j=1,\ldots,N$). Of course this satement is only valid in the long wavelength approximation ($\rho=0$) treated here [5].

Regarding the invariance of all the Besselfunctions in (40) for the transformation $\varphi \to -\varphi$, another property may be obtained :

$$I_{\ell,m_1,\ldots,m_N}(-\varphi) = I_{\ell,m_1,\ldots,m_N}(\varphi) , \tag{43}$$

which signifies that the diffraction spectrum remains invariant, if the angle φ between the direction of the incident light beam and the z-axis is reversed (see also [13]).

In order to investigate the symmetry of the satellite lines with respect to the corresponding ℓ^{th} order principal line, it seems better to use another equivalent form of the intensities as a generalization of (28), instead of (25). Putting

$$\ell+(m_j-2\alpha_j)N = p_j \ , \tag{44}$$

$$\ell+(m_j-2\beta_j)N = q_j \qquad (j=1,\dots,N) \ , \tag{45}$$

we find that

$$\ell + \sum_{j=1}^{N} (m_j-2\alpha_j) = \frac{1}{N}\sum_{j=1}^{N} p_j \ , \tag{46}$$

$$\alpha_j = \frac{\ell+Nm_j-p_j}{2N} \ , \tag{47}$$

$$\alpha_j-m_j = \frac{\ell-Nm_j-p_j}{2N} \ , \tag{48}$$

and analogous formulae relating β_j and q_j.

In the transformed expression of the intensities

$$I_{\ell,m_1,\dots,m_N}(\varphi) = \sum_{p_1,p_2,\dots,p_N=-\infty}^{+\infty} \ \sum_{q_1,q_2,\dots,q_N=-\infty}^{+\infty}$$

$$J_{\frac{1}{N}\sum_{j=1}^{N}p_j}\left[\frac{k\varepsilon_1\sin(\frac{1}{2}k^* L\tan\varphi)}{\sqrt{\varepsilon_0}k^*\sin\varphi}\right] J_{\frac{1}{N}\sum_{j=1}^{N}q_j}\left[\frac{k\varepsilon_1\sin(\frac{1}{2}k^* L\tan\varphi)}{\sqrt{\varepsilon_0}k^*\sin\varphi}\right]$$

$$\times \left\{ \prod_{j=1}^{N} J_{\frac{\ell+Nm_j-p_j}{2N}}\left[\frac{k\varepsilon_{j+1}\sin(\frac{1}{2}(k^*+k_j^*)L\tan\varphi)}{2\sqrt{\varepsilon_0}(k^*+k_j^*)\sin\varphi}\right]\right.$$

$$\times J_{\frac{\ell-Nm_j-p_j}{2N}}\left[\frac{k\varepsilon_{j+1}\sin(\frac{1}{2}(k^*-k_j^*)L\tan\varphi)}{2\sqrt{\varepsilon_0}(k^*-k_j^*)\sin\varphi}\right]$$

$$\times J_{\frac{\ell+Nm_j-p_j}{2N}}\left[\frac{k\varepsilon_{j+1}\sin(\frac{1}{2}(k^*+k_j^*)L\tan\varphi)}{2\sqrt{\varepsilon_0}(k^*+k_j^*)\sin\varphi}\right]$$

$$\left.\times J_{\frac{\ell-Nm_j-p_j}{2N}}\left[\frac{k\varepsilon_{j+1}\sin(\frac{1}{2}(k^*-k_j^*)L\tan\varphi)}{2\sqrt{\varepsilon_0}(k^*-k_j^*)\sin\varphi}\right]\right\} \ , \tag{49}$$

the summation indices p_j and q_j should be taken so that

$\frac{\ell \pm Nm_j - p_j}{2N}$ and $\frac{\ell \pm Nm_j - q_j}{2N}$ remain integers. On the hand of (49) it is very easy to see that

$$I_{\ell,m_1,\ldots,-m_i,\ldots,m_N}(\varphi) = I_{\ell,m_1,\ldots,m_i,\ldots,m_N}(\varphi) \qquad i=1,2,\ldots,N \, , \tag{50}$$

iff $\varphi = 0$.

Consequently one ore more summation indices may be replaced by their opposite, from which we may conclude that at normal incidence of the light the intensity of the satellite lines is symmetric with respect to a corresponding principal line.

At oblique incidence of the light, this symmetry property will no longer hold.

ACKNOWLEDGEMENT

We wish to thank Professor R. Mertens for stimulating and clarifying discussions on the present generalizations of the theory.

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(received July 1979)