

# The Tanh Method: I. Exact Solutions of Nonlinear Evolution and Wave Equations

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Received August 1, 1995; in revised form March 22, 1996; accepted April 28, 1996

## Abstract

A systemized version of the tanh method is used to solve particular evolution and wave equations. If one deals with conservative systems, one seeks travelling wave solutions in the form of a finite series in  $\tanh$ . If present, boundary conditions are implemented in this expansion. The associated velocity can then be determined *a priori*, provided the solution vanishes at infinity. Hence, exact closed form solutions can be obtained easily in various cases.

## 1. Introduction

If one tries to solve nonlinear evolution and wave equations, one first starts to look for travelling-wave solutions. In principle, these waves can be found easily, because the PDE under consideration can immediately be transformed into an ODE.

In conservative systems, solutions are found by direct integration, suitable transformation or substitution, or other *ad hoc* techniques. The original PDE could be solved also with more sophisticated methods such as the Hirota's bilinear technique [1], truncated Painlevé expansion [2], direct algebra methods [3, 4] and the like. Other PDEs are no longer that easy to solve. For instance, ingenious transformations are needed to obtain closed form solutions to the KdV–Burgers equation [5], despite the simplicity of this equation.

As an alternative, the tanh method is introduced to find solutions of travelling-wave type. This technique was used by Huibin and Kelin [6] to solve a higher-order KdV equation and other authors (see for instance B. Liu *et al.* [7]) in a straightforward but not practical manner. They introduced a power series in  $\tanh$  as a possible solution and substituted this expansion directly into the equation under study. As a result, algebraic equations appear from which the coefficients of the power series as well as the velocity are determined.

To avoid algebraic complexity, we had customized this technique [8] by introducing  $\tanh$  as a new variable, since all derivatives of a  $\tanh$  are represented by a  $\tanh$  itself. A straightforward analysis can then be carried out so that the method will be applicable to a large class of equations. In this paper we further refine and systemize this technique through the incorporation of boundary conditions and the *a priori* determination of the velocity of the travelling wave.

A travelling-wave solution  $u(x, t)$  (or stationary wave form) requires one coordinate:

$$\xi = c(x - vt) \quad \text{and thus} \quad u(x, t) = U(\xi), \quad (1)$$

where  $U(\xi)$  represents the (localized) wave solutions, which travels with speed  $v$ . It exemplifies a stationary wave with characteristic width  $L = c^{-1}$ . Usually, the wave number  $c$  is arbitrary but in some cases it assumes particular fixed values [3]. Under the above transformation, the PDE reduces to an ODE in  $U(\xi)$ , which should be successively integrated as many times as possible. Adhering to the boundary conditions

$$U(\xi) \rightarrow 0 \quad \text{and} \quad \frac{d^n U(\xi)}{d\xi^n} \rightarrow 0 \quad (n = 1, 2, \dots) \quad \text{for} \quad \xi \rightarrow \pm\infty, \quad (2)$$

the integration constants, if present, should all be set zero. With the assumption that travelling wave solutions are expressible in terms of  $\tanh(\xi)$ , we introduce  $Y = \tanh(\xi)$  as a new dependent variable. Now conjecture that we deal with solutions of the form

$$u(x, t) = U(\xi) = S(Y) = \sum_{n=0}^N a_n Y^n \quad \text{with} \quad Y = \tanh(\xi) = \tanh[c(x - vt)]. \quad (3)$$

The highest power  $N$  will be determined by balancing the highest degree terms in  $Y$ , upon substitution of eq. (3) into the ODE. It turns out that  $N = 2$  in most cases, so we start with this value to illustrate the procedure.

The usual boundary condition  $U(\xi) \rightarrow 0$  for  $\xi \rightarrow +\infty$  or  $\xi \rightarrow -\infty$  implies that  $S(Y) \rightarrow 0$  for  $Y \rightarrow +1$  or  $Y \rightarrow -1$ . Without loss of generalization we only consider the limit  $Y \rightarrow 1$ . Two possible solutions arise in this case:

$$1. \quad S(Y) = F(Y) = b_0(1 - Y)(1 + b_1 Y) \\ = (1 - Y)T(Y) \quad \text{with} \quad T(1) \neq 0 \quad (4a)$$

or

$$2. \quad S(Y) = G(Y) = d_0(1 - Y)^2, \quad (4b)$$

since one does not know how fast the solution decays. In the first case,  $S(Y)$  decays as  $\exp(-2\xi)$ , whereas in the second case  $S(Y) \approx \exp(-4\xi)$  as  $\xi \rightarrow +\infty$ .



In general,  $N$  types of expansions could occur, so that eq. (3) transforms into:

$$u(x, t) = U(\xi) = (1 - Y)^m \sum_{n=0}^{N-m} a_n Y^n$$

with  $Y = \tanh(\xi)$ , (4c)

since a solution may behave as  $(1 - Y)^m$  with  $m = 1, 2, \dots$ . One then separately investigates the different cases  $m = 1, m = 2, \dots$ . This procedure then leads to a further systemization of the method.

Furthermore, these built-in restrictions on the form  $S(Y)$  will allow us to determine the velocity of the travelling wave *a priori*. This knowledge of the velocity plays also a considerable role in the use of a perturbation approach (see Part 2 [13]). To illustrate this procedure we treat some particular examples in more detail. In some cases, new results are obtained.

## 2. Examples

As a tutorial example, we first treat the well-known Korteweg de Vries–Burgers equation.

### 2.1. KdV–Burgers (KdVB) equation

This basic equation is written as

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + b \frac{\partial^3 u}{\partial x^3} - a \frac{\partial^2 u}{\partial x^2} = 0, \quad (5)$$

where  $u(x, t)$  is a conserved quantity [ $d/dt \int_{-\infty}^{+\infty} u(x, t) dx = 0$ , i.e. the area under  $u(x, t)$  is conserved for all  $t$ , or  $\int_{-\infty}^{+\infty} u(x, t) dx$  is a constant of the motion]. This equation is familiar in fluid mechanics. It describes for instance shallow water waves in an elastic tube with dispersion and dissipation [9]. After changing the variables and one integration, we obtain:

$$-cvU(\xi) + \frac{1}{2}cU(\xi)^2 + bc^2 \frac{d^2 U(\xi)}{d\xi^2} - ac \frac{dU(\xi)}{d\xi} = C. \quad (6)$$

Requiring

$$U(\xi), \quad \frac{dU(\xi)}{d\xi} \quad \text{and} \quad \frac{d^2 U(\xi)}{d\xi^2} \rightarrow 0 \quad \text{as} \quad \xi \rightarrow \infty, \quad (7)$$

the integration constant  $C$  is set to zero. Equation (6) can be expressed in the new variable  $Y$  as:

$$\begin{aligned} & -vS(Y) + \frac{1}{2}S(Y)^2 + bc^2(1 - Y^2) \\ & \times \left( -2Y \frac{dS(Y)}{dY} + (1 - Y^2) \frac{d^2 S(Y)}{d^2 Y} \right) \\ & - ac(1 - Y^2) \frac{dS(Y)}{dY} = 0. \end{aligned} \quad (8)$$

Substitution of the expansion (3) into eq. (8) and balancing the highest degree in  $Y$ , yields  $N = 2$ . As already stated in eq. (4) two solutions defined in eq. (4a, b) are possible. We first take  $F(Y) = (1 - Y)T(Y)$  with  $T(Y) = b_0(1 + b_1 Y)$ . Upon substitution in eq. (8) and subsequent cancellation of a common factor  $(1 - Y)$ , we take the limit  $Y \rightarrow 1$ , which results in an expression for the velocity:

$$v = 4bc^2 + 2ac. \quad (9)$$

Alternatively, this expression for the velocity could also be obtained from substitution of the asymptotic form  $U(\xi) \approx \exp(-2\xi)$  into eq. (6) (with  $C = 0$ ).

The remaining constants  $b_0$  and  $b_1$  in  $T(Y)$  are now easily found through simple algebra. As a result we get:

$$c = \frac{a}{10b}: \quad v = 24bc^2, \quad F(Y) = 36bc^2(1 - Y)(1 + \frac{1}{3}Y) \quad (10a)$$

where  $Y = \tanh[c(x - vt)]$ .

This KdVB solution can also be cast in the following form:

$$F(Y) = 12bc^2(1 - Y)(1 + Y) + 24bc^2(1 - Y) \quad \text{or} \quad (10b)$$

$$= 12bc^2 \operatorname{sech}^2 \xi + 24bc^2(1 - \tanh \xi), \quad (10c)$$

with  $c = a/10b$ .

It represents a particular combination of a solitary wave [first term on the r.h.s. of eq. (10c)] with a Burgers shock-wave (second term). This KdVB solution resembles the form  $A \operatorname{sech}^n \xi + B \tanh^m \xi + D$ , proposed by Jeffrey and Mohamad [10], to get a solution to the KdVB equation. They had to determine seven parameters to obtain a solution. This result is also derived by other authors with other but rather involved methods [5, 11].

For  $a = 0$  (KdV case) we get

$$v = 4bc^2, \quad F(Y) = 12bc^2(1 - Y)(1 + Y) \quad (11)$$

the familiar bell-shaped form, since  $(1 - Y^2) = \operatorname{sech}^2$ .

For  $b = 0$ , eq. (5) reduces to Burgers' equation. In this case  $N = 1$ , so that now the unique solution  $b_0(1 - Y)$  must be proposed. We immediately obtain  $b_0 = v = 2ac$  and  $b_1 = 0$ . Hence

$$b = 0 \text{ (Burgers case): } v = 2ac, \quad F(Y) = 2ac(1 - Y), \quad (12)$$

the familiar shock-wave profile is obtained. With one and the same method we have solved three related cases (KdV, Burgers and KdVB) simultaneously. Note that in the KdV case, as well as Burgers case, the parameter  $c$  is arbitrary, while in the combined case it must admit a particular value. This fact can be interpreted as a subtle balance between a solitary wave (KdV) and a shock wave (Burgers) to form the combined solution [eq. (10c)].

Remarkably, the second possible solution  $G(Y) = d_0(1 - Y)^2$  leads also to a real solution. Using a similar asymptotic procedure [by replacing  $U(\xi)$  in eq. (6) by  $\exp(-4\xi)$ ], we get for the velocity

$$v = 16bc^2 + 4ac. \quad (13)$$

Note that the transformation of  $c$  by  $2c$  in eq. (9) gives the same answer. The only variable left is found to be

$$d_0 = -12bc^2 \quad \text{if} \quad c = -\frac{a}{10b} \quad \text{and thus} \quad v = -24bc^2. \quad (14)$$

With  $c = -k$ , the solution is then written as:

$$k = \frac{a}{10b}: \quad v = -24bk^2, \quad G(Y) = -12bk^2(1 + Y)^2 \quad (15)$$

with  $Y = \tanh[k(x - vt)]$ .

The latter solution is not new either: it can easily be obtained from symmetry considerations. Indeed, replace  $v$  by  $-v$  and  $S(Y)$  by  $S(Y) - 2v$ , which clearly leaves eq. (8) invariant. Application of the same transformation to eq. (10)



yields eq. (14). The results in eq. (14) were also obtained by the same authors [5, 8].

In the case where the integration constant is different from zero, the same analysis can be replaced since  $U(\xi) \neq 0$  as  $\xi \rightarrow \infty$ . If the r.h.s. of eq. (6) is taken different from zero ( $C \neq 0$ ), one introduces the linear transformation

$$U(\xi) = W(\xi) + v - V, \quad (16)$$

to get rid of the constant term  $C$ . The velocity  $V$  in the case is then given by

$$V^2 = v^2 + 2C. \quad (17)$$

The velocity  $v$  then represents the velocity in the case  $C = 0$ . The quantity  $W(\xi)$  obeys the nonlinear wave equation (6) and results similar to eqs (10) or (14) are easily obtained.

An important and striking feature coming out of this analysis is the close relationship between the velocity on one hand and the boundary condition on the other hand. From eq. (8) we directly get the relation

$$vS(Y) = \frac{1}{2}S(Y)^2 \quad \text{for } Y \rightarrow \pm 1, \quad (18)$$

so that  $S(\pm 1) = 0$  or  $S(\pm 1) = 2v$ . We generally deal with the following choices:

- for  $Y \rightarrow +1$  we have  $S(Y \rightarrow +1) \rightarrow 0$ , in view of the required boundary conditions; (19a)

- for  $Y \rightarrow -1$  we have either

$$S(Y \rightarrow -1) \rightarrow 0 \quad (\text{KdV case}) \quad (19b)$$

or

$$S(Y \rightarrow -1) = 2v \quad (\text{Burgers' and KdVB case}). \quad (19c)$$

In the latter case (shock wave type of solutions), this relation is established. This scheme can be applied to all examples under study. Although in the KdV case [or other cases where  $S(\pm 1) = 0$ ] the velocity seems unaffected by the boundary condition, it is still determined by the asymptotic behaviour of  $S(Y)$  for  $Y \rightarrow +1$ .

It is interesting to note that Canosa and Gazdag [12], in a study of a perturbative KdVB-like equation (see Part 2 [13]) observed that the propagation speed of the wave is linearly proportional to its thickness, using eq. (6) (with  $C = 0$ ) and the limit  $\xi \rightarrow \infty$ . Note that these authors had to put the boundary value equal to 1, so that the velocity equals  $\frac{1}{2}$ , since they were not able to find an analytical expression for the velocity.

Another important issue is the stability of such a wave form. It is believed (Bona and Schonbeck [14]) that travelling-wave solutions are stable, but a definite (analytical) proof is still lacking. However, numerical results remain stable for a sufficiently long time and moreover, the Burgers case is known to be asymptotically stable (Jeffrey and Kakutani [15], Peletier [16]). This latter result is also of some importance, considering the perturbative solution of KdVB in the limit of weak dispersion we shall treat in the next paper (see Part 2 [13]).

## 2.2. The DD equation

Another nonlinear dissipative-dispersion equation was derived by Kakutani and Kawahara [17]. They analysed a two-fluid plasma model, consisting of cold ions and warm electrons. In the limit of long wavelengths, the excitation of

ion-acoustic waves was governed by a wave equation in which dispersive (due to charge separation) as well as dissipative effects (due to electron and ion collisions), are present. It is actually a KdV-like equation, which reads

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + b \frac{\partial^3 u}{\partial x^3} - a \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \right) = 0. \quad (20)$$

The quantity  $u(x, t)$  represents the (perturbed) ion velocity or density. Both  $a$  and  $b$  are positive quantities.

The third term represents the dispersive effect, whereas the last term between brackets incorporates some dissipative effects (proportional to the frequency of electron-ion collisions). The same boundary conditions as before apply.

Since no analytical solution of eq. (20) exists, we neglect the last term in eq. (20) which is allowed if  $a$  is relatively small. Hence we deal with the following KdV-like equation:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + b \frac{\partial^3 u}{\partial x^3} - a \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial t} \right) = 0, \quad (21)$$

which we for convenience call the DD (dispersion and dissipation) equation. Repeating the same steps as before and taking the integration constant equal to zero, we get:

$$-cvU(\xi) + \frac{1}{2}cU(\xi)^2 + c^3b \frac{d^2U(\xi)}{d\xi^2} + avc^2 \frac{dU(\xi)}{d\xi} = 0. \quad (22)$$

After introduction of the  $Y$  variable we arrive at

$$\begin{aligned} & -vS(Y) + \frac{1}{2}S^2(Y) + bc^2(1 - Y^2) \\ & \times \left( -2Y \frac{dS(Y)}{dY} + (1 - Y^2) \frac{d^2S(Y)}{dY^2} \right) \\ & + acv(1 - Y^2) \left( \frac{dS(Y)}{dY} \right) = 0. \end{aligned} \quad (23)$$

In this case, we again have  $N = 2$ , equating the orders of  $Y$  in both the highest derivative and the nonlinear term.

As usual, the first solution which we propose is

$$S(Y) = F(Y) = b_0(1 - Y)(1 + b_1 Y) = (1 - Y)T(Y). \quad (24)$$

From the asymptotic behaviour, i.e.  $U(\xi) \rightarrow \exp(-2\xi)$  in eq. (22), we get

$$v = \frac{4bc^2}{1 + 2ac}. \quad (25)$$

Note that this velocity has the correct limiting behaviour for  $a \rightarrow 0$  (KdV case). Putting eqs (24) and (25) into eq. (23), we get after some algebra the following values for the unknowns:

$$c = -\frac{5}{12}a, \quad b_0 = 36bc^2, \quad b_1 = \frac{1}{3}, \quad v = 24bc^2. \quad (26)$$

We define again  $k = -c$ , so that in terms of the original variables, we get:

$$\begin{aligned} & k = \frac{5}{12}a: \quad v = 24bk^2 \\ & \text{and } F(Y) = 36bk^2(1 + Y)(1 - \frac{1}{3}Y) \end{aligned} \quad (27)$$

with  $Y = \tanh[k(x - vt)]$ , a new result. The second trial  $G(Y)$  leads to no solution.

Due to the presence of the last term in eq. (21), the original solitary wave [emanating from  $a = 0$  in eq. (21), i.e. the KdV equation], disappears and instead, a remarkable shock-wave arises, which moves from the left to the right.



The general case, i.e. eq. (20), is currently under study with the aid of a perturbative approach.

### 2.3. The combined KdV-MKdV equation

This equation reads:

$$\frac{\partial u}{\partial t} - 2Bu \frac{\partial u}{\partial x} - 3Cu^2 \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0. \quad (28)$$

It serves as a model equation in DNA dynamics [18] and in planetary plasmas [19–21]. It is not necessary to specify the quantities  $B$  and  $C$ , though some values cannot be allowed. After the customary transformation and integration, we get

$$-cvU(\xi) - BcU(\xi)^2 - CcU(\xi)^3 + c^2 \frac{d^2 U(\xi)}{d\xi^2} = 0. \quad (29)$$

As before, the integration constant has been put equal to zero. Next, we have

$$-vS(Y) - BS(Y)^2 - CS(Y)^3 + c^2(1 - Y^2) \times \left( -2Y \frac{dS(Y)}{dY} + (1 - Y^2) \frac{d^2 S(Y)}{d^2 Y} \right) = 0. \quad (30)$$

The velocity certainly will be related to the KdV velocity [see eq. (11) with  $b = 1$ ], since they shear equal linear properties.

From the balancing operation we get  $N = 1$ , so we now deal only with

$$L(Y) = a_0(1 - Y) \quad \text{and} \quad v = 4c^2 \quad (\text{the KdV velocity}). \quad (31)$$

After some algebra, we get the following result

$$c = \frac{\sqrt{2}}{6} \frac{B}{\sqrt{C}}: \quad v = 4c^2, \quad L(Y) = -\frac{B}{3C}(1 - Y), \quad (32)$$

a negative ( $B > 0$  and  $C > 0$ ) shock-wave, which moves from the left to the right.

Another solution can be obtained, which is more involved. Take the fraction

$$D(Y) = \frac{(1 - Y)(1 + dY)}{(a + bY^2)} \quad (33)$$

which is actually of the form  $(1 - Y)T(Y)$ . The velocity is thus the same as in the previous case. After some algebra, we find  $d = 1$  and the solution

$$\begin{aligned} S(Y) &= 12c^2 \frac{(1 - Y)(1 + Y)}{(a_+ + a_- Y^2)} \\ &\quad \text{with } a_{\pm} = B \pm \sqrt{B^2 - 18c^2 C}, \\ &= 12c^2 \frac{(1 - Y^2)}{(a_+ + a_- Y^2)} \\ &\quad \text{with } Y = \tanh[c(x - vt)] \text{ and } v = 4c^2. \end{aligned} \quad (34)$$

We thus deal with a remarkable solitary wave since  $(1 - Y^2) = \text{sech}^2$ . Such a solution is likely to exist since the last term in eq. (30) contains the factor  $(1 - Y^2)$  and the remaining terms are proportional to the solution  $S(Y)$ .

Of course, some restrictions are required on  $B$ ,  $C$  and  $c$  because the denominator may not vanish ( $0 \leq Y^2 \leq 1$ ); moreover  $B^2 \geq 18c^2 C$  for any value of  $c$  to keep the solutions real. We deal with a more general solution than the previous one, because  $c$  now remains a free parameter.

These results were previously found by Khan and coworkers [18] by direct substitution of a trial function.

The fact that  $T(Y)$  can be represented by a fraction also appears in other situations [8].

Again the velocity is related to the boundary condition. Equation (30) yields

$$-vS(-1) - BS(-1)^2 - CS(-1)^3 = 0 \quad (35a)$$

or

$$v = -BS(-1) - CS(-1)^2 = 0 \quad \text{if } S(-1) \neq 0. \quad (35b)$$

As expected, these relations, which connect the velocity with the boundary conditions, are satisfied by both results.

### 2.4. An extended MKdV-KdV-Burgers equation

Some years ago, Mohamad [22] solved a KdV-MKdV equation, similar to eq. (28). We added one more term and examined this equation, to see whether the tanh method was capable of generating a solution. We start thus with

$$\frac{\partial u}{\partial t} + 6u \frac{\partial u}{\partial x} - 6u^2 \frac{\partial u}{\partial x} + b \frac{\partial^3 u}{\partial x^3} - a \frac{\partial^2 u}{\partial x^2} = 0. \quad (36)$$

Imposing the usual boundary conditions [see eq. (2)], which give an expression for the velocity, we get after some algebra the following shock-wave profile

$$\begin{aligned} u(x, t) &= c \sqrt{b} \{1 - \tanh[c(x - vt)]\} \\ &\quad \text{with } c = \frac{1}{6b} (3\sqrt{b} - a) \\ &\quad \text{and } v = \frac{2c}{3} (3\sqrt{b} - a). \end{aligned} \quad (37)$$

A limiting case can be found setting  $a = 0$  (no dissipation or diffusion). For a thorough discussion of this solution, as well as two-dimensional cases, see [23].

### 2.5. The Fisher equation with nonlinear convection

We deal with a Fisher equation in which a term, describing nonlinear convection, is added. Murray [24] showed that wave solutions really exist in this case. However, no solution was obtained.

We start thus with

$$\frac{\partial u}{\partial t} + Ku \frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} + u(1 - u). \quad (38)$$

Introducing the  $\xi$  variable, we find

$$\begin{aligned} -cv \frac{dU(\xi)}{d\xi} + cKU(\xi) \frac{dU(\xi)}{d\xi} \\ = c^2 \frac{d^2 U(\xi)}{d\xi^2} + U(\xi)[1 - U(\xi)]. \end{aligned} \quad (39)$$

As boundary conditions we assume that, in analogy with Fisher's case,

$$U(\xi), \quad \frac{dU}{d\xi} \quad \text{and} \quad \frac{d^2 U(\xi)}{d\xi^2} \rightarrow 0 \quad \text{as } \xi \rightarrow \infty. \quad (40)$$

and

$$\begin{aligned} U(\xi) \rightarrow 1, \quad \text{while } \frac{dU(\xi)}{d\xi} \\ \text{and } \frac{d^2 U(\xi)}{d\xi^2} \rightarrow 0 \quad \text{as } \xi \rightarrow -\infty. \end{aligned} \quad (41)$$

The solution will thus develop between the two end states 1 and 0. With the aid of the new variable  $Y$ , we get instead of eq. (39)

$$cv(1 - Y^2) \frac{dS(Y)}{dY} + c^2(1 - Y^2) \times \left( -2Y \frac{dS(Y)}{dY} + (1 - Y^2) \frac{d^2S(Y)}{d^2Y} \right) - cK(1 - Y^2)S(Y) \frac{dS(Y)}{dY} + S(Y) - S(Y)^2 = 0. \quad (42)$$

Performing the same analysis of balancing linear terms *vs.* nonlinear ones, we find  $M = 1$ . A possible solution will be  $S(Y) = \frac{1}{2}(1 - Y)$ , in view of the boundary conditions (40) and (41). As before, the velocity is then easily calculated with the aid of the asymptotic behaviour of  $S(Y)$ . This yields

$$v = \frac{4c^2 + 1}{2c}. \quad (43)$$

After substitution of this expression and  $S(Y) = \frac{1}{2}(1 - Y)$  into eq. (42), one obtains

$$-\frac{c}{4}(1 + Y)(1 - Y^2)(4c - K) \quad (44)$$

so that  $c = K/4$  leads to the exact solution:

$$u(x, t) = \frac{1}{2}(1 - \tanh \xi) \quad (45)$$

with

$$\xi = \frac{K}{4} \left[ x - \left( \frac{K^2 + 4}{2K} \right) t \right]. \quad (46)$$

Note that in the limit  $K \rightarrow 0$ , eq. (48) transforms into a Fisher equation without linear diffusion:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1 - u), \quad (47)$$

while the argument in eq. (46) becomes  $\xi = -t/4$ . Since this argument is independent of  $x$  we no longer deal with the Fisher equation but with Verhulst equation (see Murray [24]):

$$\frac{du}{dt} = u(1 - u), \quad (48)$$

which describes logistic growth without diffusion. Note that eq. (48) is easily extended to

$$\frac{\partial u}{\partial t} + Ku \frac{\partial u}{\partial x} = D \frac{\partial^2 u}{\partial x^2} + pu(1 - u), \quad (49)$$

with  $D$  and  $p$  positive parameters. By a suitable (but not trivial) transformation, it can be rewritten in its original form, or alternatively solved with the tanh technique. As a result, we get

$$c = \frac{K}{4} \quad \text{and} \quad v = \frac{8K^2 + 2Dp}{K}, \quad (50)$$

to be substituted into the variable  $\xi = c(x - vt)$ . The other examples can be generalized and treated in the same sense.

Although a definite proof is lacking, we expect these waveforms to be stable. In the case of the Fisher equation one knows that general waveforms, defined at  $t = 0$  and

with a velocity less than 2, lead to unstable waves [24]. This result does not apply here since the velocity, given by eq. (43), is always larger than or equal to  $2(c > 0)$ .

## 2.6. Model equation describing foam drainage

Recent Verbist and Weaire [25] derived the following nonlinear wave equation:

$$\frac{\partial \alpha}{\partial t} = -\frac{\partial}{\partial x} \left( \alpha^2 - \frac{\sqrt{\alpha}}{2} \frac{\partial \alpha}{\partial x} \right), \quad (51)$$

which describes foam drainage through a cross-sectional area  $A [\sim \alpha(x, t)]$ .

The tanh method (without boundary conditions) can be used again to solve this remarkable nonlinear diffusion equation. We thus first introduce  $u(\xi) = u[c(x - vt)] = \alpha(x, t)$ , a stationary wave profile. Hence, eq. (51) transforms into

$$-v \frac{du(\xi)}{d\xi} = -\frac{d}{d\xi} \left( u(\xi)^2 - c \frac{\sqrt{u(\xi)}}{2} \frac{du(\xi)}{d\xi} \right). \quad (52)$$

To eliminate the square root in the r.h.s. of this equation, we define

$$u(\xi) = w^2(\xi). \quad (53)$$

Next, using the tanh formalism, we get with  $w(\xi) = S(\tanh \xi) = S(Y)$  the equation

$$2v \frac{dS(Y)}{dY} - 4S^2(Y) \frac{dS(Y)}{dY} + 2c(1 - Y^2) \left( \frac{dS(Y)}{dY} \right)^2 + cS(Y) \frac{d}{dY} \left( (1 - Y^2) \frac{dS(Y)}{dY} \right) = 0. \quad (54)$$

The balancing procedure to determine  $M$ , is now different: one has to compare the second term [cubic in  $S(Y)$ ] with the other quadratic terms in eq. (54). The value  $N = 1$  is then easily obtained. Hence

$$S(Y) = b_0 + b_1 Y. \quad (55)$$

After substitution of eq. (65) into eq. (64), we get the following recurrence relations from the coefficients of the quadratic function in  $Y$ , which must vanish completely:

$$Y^2 \text{ coeff.: } 2b_1(c + b_1) = 0, \quad (56a)$$

$$Y^1 \text{ coeff.: } b_0(c + 4b_1) = 0, \quad (56b)$$

$$Y^0 \text{ coeff.: } -b_1c - v + 2b_0^2 = 0. \quad (56c)$$

Obviously,  $b_1 \neq 0$  so that  $b_1 = -c$  from eq. (56a). Simple algebra then leads to

$$b_1 = -c, \quad a_0 = 0 \quad \text{and} \quad v = c^2 \quad (57)$$

$$\text{so that } S(Y) = -cY \quad \text{or} \quad w(\xi) = -c \tanh \xi.$$

From eq. (53), we finally get the general solution

$$u(\xi) = c^2 \tanh^2 \xi \quad \text{or} \quad \alpha(x, t) = c^2 \tanh^2 [c(x - c^2t)]. \quad (58)$$

If we take the required boundary condition (zero flow at  $x \rightarrow +\infty$ ) into account, we get their result

$$\alpha(x, t) = c^2 \tanh^2 [c(x - c^2t)], \quad x \leq c^2t \\ = 0, \quad x \geq c^2t, \quad (59)$$



which was presented without any derivation.

This shock wave [eq. (59)] describes the transition between wet foam (by continuous addition of liquid) and dry foam.

### 3. Discussion and conclusion

It is shown that the tanh method in its present form is a powerful technique for investigating nonlinear wave equations, in particular those where diffusion is involved. The main feature of this approach is based on the hypothesis that the travelling-wave solutions we are looking for may be found and expressed in terms of a tanh. This hyperbolic function is then used as an independent variable. Moreover, the embedding of the boundary conditions within the proposed solutions (whenever possible) and the *a priori* determination of the velocity through asymptotes has a strong impact on the ease of use of the method, so that tedious algebra is avoided. Closed-form solutions are then derived in an elegant and straightforward way since we deal with polynomial expressions.

One observes from the result that the wave number  $c$  (inversely proportional to the width of the wave form) is sometimes an arbitrary parameter (more often when dealing with simple nonlinear equations), while in some cases it is not (the more complicated ones). This latter case definitely excludes any possible soliton behaviour (interaction of solitary waves), since the velocity depends on  $c$ . Adding still more terms to these equations, one obviously deals with equations that generally cannot be solved exactly.

Numerous other examples can be treated as well, even KdV–Burgers and MKdV–Burgers in two dimensions, as well as coupled equations [26]. We have chosen some selected problems to underline the generality of this technique. Known solutions are now derived with this alternative technique in an elegant and much shorter way. Moreover, in some cases, new results are also found. Of course, we are aware that some classes of equations are not suited for the tanh technique. For instance, we should mention the nonlinear Sine–Gordon equation and related equations.

Moreover, the applicability of the presented method will be greatly enhanced by its use as a perturbation technique.

In the following paper (Part 2), we will examine different approximate solutions of problems that are “tanh”-like.

### Acknowledgements

W.M. thanks the Belgian National Fund for Scientific Research (NFWO) for a travel grant to the U.S.A. Part of this work was performed at the Colorado School of Mines. Their hospitality is greatly appreciated. Thanks are due to Professor Frank Verheest for fruitful discussions.

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