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The Tanh Method: II. Perturbation Technique for Conservative Systems

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Abstract

With the aid of the tanh method, nonlinear wave equations are solved in a perturbative way. First, the KdVBurgers equation is investigated in the limit of weak dispersion. As a result, a general shock wave profile, with a perturbative solitary-wave contribution superposed, emerges. For a particular choice of the parameters, a comparison with the exact solution is made. Further, the MKdVBurgers is investigated in the same limit and similar results are obtained.

1. Introduction

When a nonlinear wave equation cannot be solved exactly, one tries various perturbation techniques to solve it approximately. Most of these techniques, however, are based on a linearization procedure, which we prefer not to introduce here. We want to take full account of the nonlinear property of the problem under study. If one deals with conservative systems, where nonlinear waves propagate without change of their shape, the reductive perturbation theory (Ichikawa and Watanabe [1]) does not have that peculiar behaviour and may be used to tackle those kind of problems. An ingenious choice of new variables (stretched coordinates) is introduced so that the nonlinear character of the lowest-order wave profile is taken into account. In principle, dynamical equations for the higher-order perturbation terms can be investigated, but the results should be examined thoroughly, to avoid secular behaviour. This method has been successfully used in plasma physics to investigate nonlinear wave propagation in collisionless plasmas.

In this paper, however, we propose a somewhat different approach, since we only like to deal with travelling-waves. Starting point is the tanh method used in part I (see [2]), where exact solutions of such type are derived with straightforward and (in most cases) simple algebra. The ease of use of the tanh technique to solve nonlinear evolution equations is striking. Consequently, one may ask whether the method can be applied to solve problems for which no exact solutions exist.

The possibility of using this technique in such cases depends on the availability of having an exact solution (in some limiting case) for the problem under study, so that a valid perturbation scheme can be set up. A problem treated in this sense was first investigated by one of us [3].

Two examples of this kind will be discussed. First, the KdVB (Korteweg–de Vries–Burgers) equation and the associated modified KdVB (in short MKdVB) equation. Because the former equation possesses also an exact solution (see part I [2]), the accuracy of this perturbation scheme can easily be tested. The latter equation has no exact solutions except in the limit of zero dispersion, where it reduces to the Burgers equation or the limit of zero dissipation where it becomes the Korteweg–de Vries equation.

2. Perturbation analysis of the KdV–Burgers equation

As mentioned in part I (see [2]), the solution to the KdV–Burgers equation (KdVB) can be viewed as a unique combination of a solitary wave and a shock-wave structure. As a result, the wave number c which relates velocity, amplitude and width of the localized wave form becomes a fixed parameter, in contrast to the individual KdV or Burgers equation where this parameter can be chosen freely. It is thus very unlikely that experimentally observed waves of this type can be matched with this particular, although exact, result.

The aim of the following analysis is to determine whether approximate solutions can be found with the tanh technique, allowing an arbitrary wave number c.

2.1. *Analysis*

If one has the intention of setting up a perturbation scheme for this equation, two parameters are available which can be assumed small and thus used as an expansion parameter: $a \ll 1$ (small dissipation) or $b \ll 1$ (small dispersion), associated with a perturbed shock wave or a perturbed solitary wave respectively. It turns out however, that only the latter case is suitable to perform a perturbation expansion with the tanh technique (see discussion afterwards).

The KdV–Burgers equation is now rewritten as

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} - a\,\frac{\partial^2 u}{\partial x^3} = -b\,\frac{\partial^3 u}{\partial x^3} \sim O(\varepsilon) \quad \text{with } \varepsilon \ll 1, \tag{1}$$

so that in lowest order ($b = 0$) the Burgers equation appears. Transformation to the variable $\xi = c(x - vt)$ gives

$$-cvU(\xi) + \tfrac{1}{2}cU(\xi)^2 + bc^3\,\frac{\mathrm{d}^2U(\xi)}{\mathrm{d}\xi^2} - ac^2\,\frac{\mathrm{d}U(\xi)}{\mathrm{d}\xi} = 0. \tag{2}$$

In terms of the new variable Y $(=\tanh[c(x - vt)] = \tanh \xi)$, we get instead of (1)

$$-vS(Y) + \tfrac{1}{2}S(Y)^2 - ac(1 - Y^2)\,\frac{\mathrm{d}S(Y)}{\mathrm{d}Y} = bc^2(1 - Y^2)\left[-2Y\,\frac{\mathrm{d}S(Y)}{\mathrm{d}Y} + (1 - Y^2)\,\frac{\mathrm{d}^2S(Y)}{\mathrm{d}^2Y}\right]. \tag{3}$$

For $b = 0$ (Burgers equation), the solution is $S_B(Y) = 2ac(1 - Y)$. The associated velocity is $v = 2ac$ and the wave number c is still an arbitrary parameter.

In the limit of weak dispersion [$b \sim O(\varepsilon)$], we approximate the solution of (1) by the following (infinite series) expansion

$$F(Y) = S_0(Y) + bS_1(Y) + b^2S_2(Y) + \dots . \tag{4}$$

The usual boundary conditions still apply, so that we require

$$F(Y = 1) = 0$$

or

$$S_0(Y = 1) = 0, \quad S_1(Y = 1) = 0, \quad S_2(Y = 1) = 0, \dots . \tag{5}$$

We recall that an exact solution was found with the aid of $F(Y) = (1 - Y)T(Y)$ with $T(1) \neq 0$. We anticipate that the same behaviour occurs in this case, so that all $S_i(Y) \sim (1 - Y)$. Substituting the associated asymptotic form, which goes like $\exp(-2\xi)$ for $\xi \to +\infty$, into (2) gives

$$v = 4bc^2 + 2ac, \tag{6}$$

the same result as in the exact case (see [2]).

In terms of the expansion parameter b, we thus have

$$v = v_0 + bv_1 + b^2v_2 + \dots,$$

$$\text{so that } v_0 = 2ac, \; v_1 = 4c^2, \; v_2 = v_3 = \dots = 0. \tag{7}$$

Remark that the relation between the left-hand boundary value [obtained in the limit for $Y \to -1$ from (3)] and the velocity still exists:

$$v = \tfrac{1}{2}F(Y \to -1) \quad \text{or} \quad 4ac + 8bc^2 = F(Y \to -1). \tag{8}$$

Substitution of (4), (6) and (7) into (3) gives a series expansion in b. To successive orders, we obtain:

$$b^0: \quad -2v_0 S_0(Y) + S_0(Y)^2 - 2ac(1 - Y^2)\frac{dS_0(Y)}{dY} = 0, \tag{9a}$$

$$\begin{aligned} b^1: \quad & -v_1S_0(Y) - v_0 S_1(Y) + S_0(Y)S_1(Y) \\ & - ac(1 - Y^2)\frac{dS_1(Y)}{dY} + c^2(1 - Y^2)^2\frac{d^2S_0(Y)}{dY^2} \\ & + 2c^2Y(1 - Y^2)\frac{dS_0(Y)}{dY} = 0, \end{aligned} \tag{10b}$$

$$\begin{aligned} b^2: \quad & -2v_0 S_2(Y) - 2v_1S_1(Y) + 2S_0(Y)S_2(Y) \\ & + S_1(Y)^2 - 4c^2Y(1 - Y^2)\frac{dS_1(Y)}{dY} \\ & + 2c^2(1 - Y^2)^2\frac{d^2S_1(Y)}{dY^2} \\ & - 2ac(1 - Y^2)\frac{dS_2(Y)}{dY} = 0, \quad \text{etc.}\dots \end{aligned} \tag{10c}$$

The lowest order equation (9a) represents Burgers case. As already mentioned, the corresponding solution reads

$$S_0(Y) = 2ac(1 - Y). \tag{11}$$

Next, we deal with a first-order inhomogeneous differential equation in $S_1(Y)$, which can readily be solved, substituting eqs (11) into (10b). As a result, we get

$$S_1(Y) = (1 - Y^2)\left[-4c^2 \ln(1 + Y) + \frac{4c^2}{1 + Y} - C_1\right] \tag{12a}$$

or

$$S_1(Y) = 4c^2(1 - Y) - (1 - Y^2)[4c^2 \ln(1 + Y) + C_1]. \tag{12b}$$

The first term on the r.h.s. of (12b) contributes to the boundary condition at $Y \to -1$, while the other terms represent a solitary wave correction (which vanishes for $Y \to \pm 1$) to the shock wave.

In (12a, b) the integration constant needs to be determined. In contrast to a perturbation approach within the framework of dispersive waves, one cannot adjust here the integration constant to an initial condition at $t = 0$. It will be argued that one can take $C_1 = 0$.

Finally eq. (10c), again an ODE of first order in $S_2(Y)$, is then solved, it yields

$$\begin{aligned} S_2(Y) = \frac{c}{a}(1 - Y^2)\{&12c^2(1 + Y) \\ &- 8c^2 \ln(1 + Y)[1 - Y \ln(1 + Y)] \\ &- 8c^2(1 - Y)\ln(1 + Y) - a^2C_2\}, \end{aligned} \tag{13}$$

with the aid of (11) and (12a, b). The second-order solution now only contributes as a 'solitary wave' correction to the perturbed shock wave structure. Again, an integration constant C_2 is present.

A perturbed solution of the KdVB equation with weak dispersion is thus obtained, with an arbitrary wave number c and some integration constants. If necessary, higher-order solutions can be found as well.

2.2. *Discussion*

Because an exact solution is known for a particular wave number [$c = (a/10b)$], we are able to compare the results of the perturbation approach with the exact solution. Hence we first choose

$$c = 1, \quad a = 1 \quad \text{and} \quad b = \tfrac{1}{10}, \tag{14a}$$

to cope with the requirement for the exact solution.

Secondly, we take

$$c = 1, \quad a = 1 \quad \text{and} \; b = \tfrac{1}{5}, \tag{14b}$$

To get an idea of the accuracy of the perturbation, we have substituted the successive approximate solutions into the original wave equation (3), and calculated the remaining terms with relations (14a, b) respectively. These remaining terms are defined as remainder terms. Note that the quantity Y in the figures is taken as $\tanh x$, since time plays no role in the presentation of these wave forms. Hence we define:

remainder term A_0:

$$\text{substitution of } F_0(Y) = (2ac + 4bc^2)(1 - Y); \tag{15a}$$

remainder term A_1:

$$\text{substitution of } F_1(Y) = S_0(Y) + bS_1(Y); \tag{15b}$$

remainder term A_2:

$$\text{substitution of } F_2(Y) = S_0(Y) + bS_1(Y) + b^2S_2(Y). \tag{15c}$$

We have defined $F_0(Y)$ here as the lowest-order solution in (15a) and not $S_0(Y) = 2ac(1 - Y)$, because it satisfies the exact boundary conditions for $Y \to -1$. After substitution of these successive approximations (15a, b and c) into the KdVB equation (3) we observe, as expected, that $A_0 = O(b)$, $A_1 = O(b^2)$ and $A_2 = O(b^3)$ (valid for all values of C_1 and C_2).

However, one problem remains: the determination of the integration constants. In the first-order approximation $S_1(Y)$, we choose $C_1 = 0$. This can be argued as follows. The improved zeroth-order approximation [see eq. (15a)] is equal to $(2ac + 4bc^2)$ at $Y = 0$. The same value at $Y = 0$ is reached by $S_0(Y) + bS_1(Y)$, if $C_1 = 0$. Any integration constant C_1, different from zero, may only deviate from this value by $O(b)$, a small amount, to remain consistent with the perturbation approach. Moreover, from a three-dimensional plot of A_1 (made at $t = 0$) with the relevant values $-10 < C_1 < 10$, $-5 < x < 5$ and relations (14a) for the parameters a, b and c, it is observed that the remainder term $|A_1|$ reaches its lowest values for $C_1 \approx 0$.

On the other hand, the situation for the next order is a little bit different. Using the same argument as before (at $Y = 0$), we get $C_2 = 12(c^2/a^2)$, which equals 12 with the choice (20a) of the parameters. However, a three-dimensional plot of A_2 with $-15 < C_2 < 15$ reveals that $C_2 \approx 9$ eventually will be the best choice. The arbitrariness of C_2 can be used to refine the perturbation approach in that order.

The results of the successive remainder terms are plotted in Fig. 1. The lowest-order solution (with the correct boundary conditions) is represented here by $F_0(Y) = \frac{12}{5}(1 - Y) = \frac{12}{5}(1 - \tanh x)$; the smallness parameter is $b = \frac{1}{10}$. As soon as the perturbation terms are included, a much better agreement with the exact solution is achieved, since the remainder terms become smaller.

It should be noted that Canosa and Gazdag [4] faced the same problem concerning the integration constants, during their analysis of the KdVB equation (see discussion in next section). They used a phase plane analysis to cope with this problem.

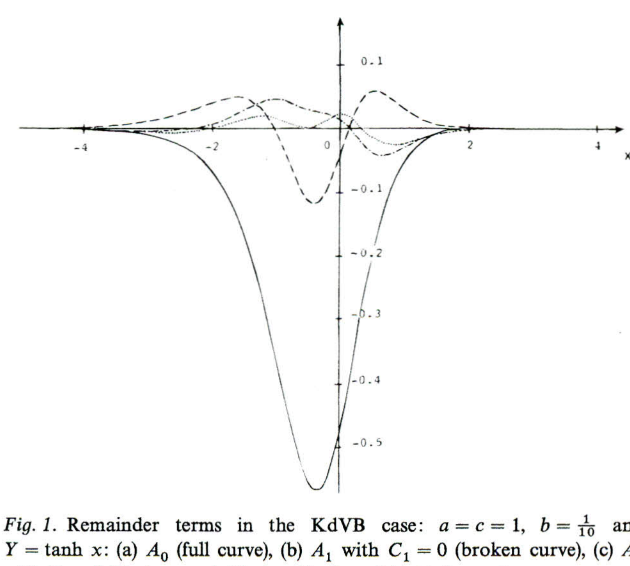

Fig. 1. Remainder terms in the KdVB case: $a = c = 1$, $b = \frac{1}{10}$ and $Y = \tanh x$: (a) A_0 (full curve), (b) A_1 with $C_1 = 0$ (broken curve), (c) A_2 with $C_2 = 0$ (chain curve), (d) A_2 with $C_2 = 9$ (dotted curve).

A direct comparison of the exact and the approximate solution confirms the accuracy of the above results. For a fair comparison, however, one has to keep in mind that the successive (and thus better) approximations cause a phase shift, due to a steepening of the shock wave. So we introduce $Y' = \tanh(x + d)$ into the exact solution (which of course remains an exact solution under translation) and $Y = \tanh x$ into the perturbed solution. As a reference point, we choose d in such a way that the difference between the two solutions vanishes at $x = 0$ $(Y = 0)$. We then define

$$D = F_{\mathrm{KdVB}}(Y') - [S_0(Y) + bS_1(Y) + b^2 S_2(Y)], \tag{16}$$

as the difference between the exact and the approximate solution. We have thus chosen relations (14a) and two representatives for the approximate solution: both with $C_1 = 0$, one with $C_2 = 0$ and the other with $C_2 = 9$. As expected, the second order approximation with $C_2 = 9$ gives remarkably good results, as one can observe from Fig. 2, since the difference D satisfies $|D| < 0.008$.

Finally, in Fig. 3, the different perturbation solutions are plotted, using the parameters of the exact solution: $c = 1$, $a = 1$ and $b = \frac{1}{10}$ [i.e. eq. (14a)]. As expected, a relatively

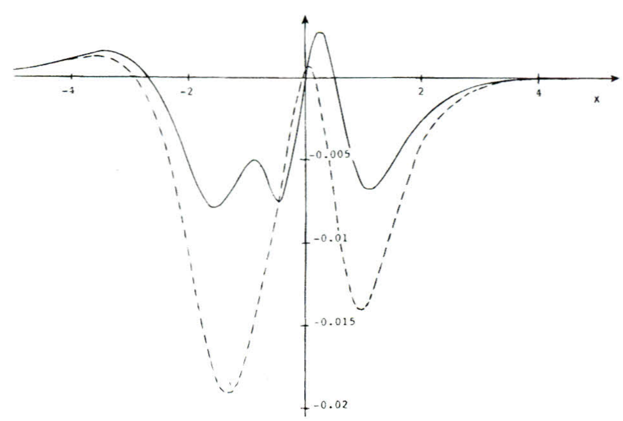

Fig. 2. Plot of the difference D between exact [$Y = \tanh(x + d)$] and perturbed solution of the KdVB equation for $a = c = 1$, $b = \frac{1}{10}$ and $Y = \tanh x$: (a) with $C_2 = 0$ in $S_2(Y)$ (dotted curve); (b) with $C_2 = 9$ in $S_2(Y)$ (full curve).

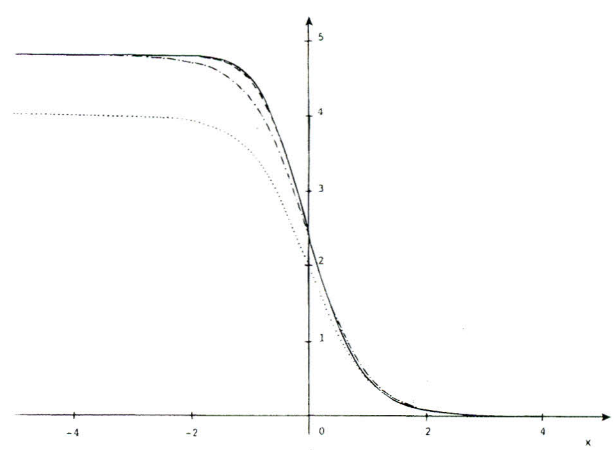

Fig. 3. Perturbed solutions of the KdVB equation for $a = c = 1$ and $b = \frac{1}{10}$ with $Y = \tanh x$: (a) $S_0(Y)$ (dotted curve); (b) $S_0(Y) + 4bc^2(1 - Y)$ (chain curve); (c) $S_0(Y) + bS_1(Y)$ (broken curve); (d) $S_0(Y) + bS_1(Y) + b^2S_2(Y)$ with $C_2 = 9$ in $S_2(Y)$ (full curve).

large difference between $S_0(Y)$ [defined in eq. (11)] and the improved lowest order solution $F_0(Y) = S_0(Y) + 4bc^2(1 - Y)$ is observed. Only the latter approximation has the correct boundary condition. Adding more perturbation terms, it is observed that the shock wave tends to steepen.

For other values of the parameters [see eq. (14b)], i.e.

$$c = 1, \quad a = 1 \quad \text{and} \quad b = \tfrac{1}{5}$$

we have plotted in Fig. 4 the same approximate solutions. Due to an increase of dispersion (large value for b), one now clearly observes the solitary-wave correction on top of the shock-wave structure, before the steepening sets in. In general, steepening of the shock-wave is more pronounced than in the case without dispersion.

2.3. *Comparison with other theories and conclusion*

Other perturbation approaches appear in the literature for the KdVB case. We first mention the series solution carried out by Gagliardi *et al.* [5], based on the Rosales method (in fact Padé-approximants). Xin *et al.* [6] define a series solution in three distinct intervals with some matching properties. They arrive at a nonlinear system of algebraic equations which can be investigated numerically. Numerical results in both cases show a bump-like behaviour, although with a slight oscillatory character in the case of weak dispersion. On the contrary, pure numerical results performed by Grad and Hu [7], confirm our analytical results.

Due to the analogy between the Fisher equation and the KdVB equation (assuming a slow change of the wave form so that the second derivative is small and thus neglected in zeroth-order), Canosa [8], and the same author with Gazdag [4] treated also this KdVB case with weak dispersion. In fact, their analysis was based on earlier work of Johnson [9]. They first imposed fixed boundary conditions

$$U(\xi) \to 1 \quad \text{as} \quad \xi \to -\infty \quad \text{and} \quad U(\xi) \to 0 \quad \text{as} \quad \xi \to \infty$$

as steady states, so that the (unknown !) velocity in their case is normalized and defined as $v = 1$. Moreover, they had to transform the Fisher equation in terms of the KdVB equation.

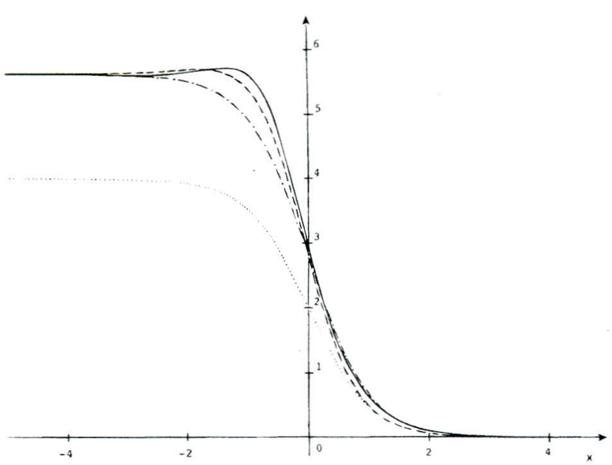

Fig. 4. Perturbed solutions of the KdVB equation for $a = c = 1$ and $b = \frac{1}{5}$ with $Y = \tanh x$: (a) $S_0(Y)$ (dotted curve); (b) $S_0(Y) + 4bc^2(1 - Y)$ (chain curve); (c) $S_0(Y) + bS_1(Y)$ (broken curve); (d) $S_0(Y) + bS_1(Y) + b^2S_2(Y)$ with $C_2 = 9$ in $S_2(Y)$ (full curve).

If we transform their results to our Y-variable, the zeroth and first order solutions of their approximate solution are:

$$S_c(Y) = \tfrac{1}{2}(1 - Y) - \frac{b'}{4}(1 - Y^2)[1 - \ln(1 - Y^2)] + O(b'^2), \tag{17}$$

with $b' = b/a^2$. These results differ slightly from ours.

With the aid of phase plane analysis, they introduced special (initial) conditions at $Y = 0$, the inflection point of a shock wave with profile $\frac{1}{2}(1 - Y)$:

$$\text{zeroth order solution } (b' = 0)\text{:} \quad \tfrac{1}{2} \quad \text{at} \quad Y = 0, \tag{18a}$$

$$\text{first order solution } [O(b')]\text{:} \quad \frac{b'}{4} \quad \text{at} \quad Y = 0. \tag{18b}$$

In this approach, no attempt was made to derive a second-order solution.

A perturbation scheme for the KdVB case with weak dispersion is carried out. As a result, we have again a free parameter c as in the KdV or Burgers case. This is in contrast to the exact solution, which has limited use since the boundary condition for $Y \to -1$ (or $\xi = -\infty$) and the wave profile are completely determined by the constant parameters a and b.

In the next example, we treat the MKdVB(+) equation (the + sign refers to the positive nonlinear term) for which no analytical result is known. We are again forced to consider the limit of weak dispersion, taking the dispersionless case as starting point for our perturbation approach.

3. Approximate solution of a MKdVB equation

3.1. *Analysis*

In this second example, we treat a combination of a modified KdV equation with Burgers equation, a natural extension as in the previous case. But now, the nonlinear term can be positive or negative. In the case of a negative nonlinear term [referred to as MKdVB(−)], an exact solution has already been found (Huang *et al.* [10]), similar to the KdVB case. In the case where the nonlinear term has the opposite sign [MKdVB(+)], no exact solutions are known to us.

The form of the MKdVB equation to be investigated is:

$$\frac{\partial u}{\partial t} + 6u^2 \frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial x^2} + \beta \frac{\partial^3 u}{\partial x^3} = 0. \tag{19a}$$

Here β and ν are positive constants measuring dispersion and dissipation respectively. Applying the tanh method to this equation affords no exact solutions, except in the limiting case $\beta = 0$ (no dispersion, i.e. a Burgers equation) or $\nu = 0$ (no dissipation, i.e. the MKdV equation). Again, as explained in the earlier, we only develop a perturbation approach in the case of weak dispersion. Therefore, we rewrite (19a) as

$$\frac{\partial u}{\partial t} + 6u^2 \frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial x^2} = -\beta \frac{\partial^3 u}{\partial x^3} \sim O(\varepsilon) \quad \text{with } \varepsilon \ll 1. \tag{19b}$$

With the customary transformation $\zeta = c(x - vt)$ with $U(\zeta)$ [$= u(x, t)$], we get:

$$-vU(\zeta) + 2U(\zeta)^3 - \nu c \frac{dU(\zeta)}{d\zeta} = -\beta c^2 \frac{d^2U(\zeta)}{d\zeta^2}, \tag{20}$$

assuming a vanishing integration constant (the usual boundary conditions still apply) to get a localized solution.

Next, we perform the transformation

$$U(\zeta) = \sqrt{h(\zeta)} \tag{21}$$

to remove the cubic nonlinearity. Hence

$$-4vh(\zeta) + 8h(\zeta)^2 - 2vc \frac{dh(\zeta)}{d\zeta} = \beta c^2 \frac{1}{h(\zeta)} \left(\frac{dh(\zeta)}{d\zeta}\right)^2 - 2\beta c^2 \frac{d^2h(\zeta)}{d\zeta^2}. \tag{22}$$

Remark that the l.h.s. of eq. (22) is closely related to eq. (2) with $b = 0$, actually the Burgers equation in the Y variable. We are thus confident that we shall deal with a stable solution in the equilibrium regions (for $\zeta \to \pm\infty$) since stable solutions exist in Burgers case (proved by Peletier [11]).

The next step is $Y = \tanh \zeta$ so that $h(\zeta) = R(Y)$, which necessarily must be a positive quantity. Equation (22) then becomes

$$-4vR(Y) + 8R(Y)^2 - 2vc(1 - Y^2) \frac{dR(Y)}{dY} = \beta c^2 \frac{1}{R(Y)} \left[(1 - Y^2)^2 \left(\frac{dR(Y)}{dY}\right)^2\right] - 2\beta c^2(1 - Y^2) \times \left[-2Y \frac{dR(Y)}{dY} + (1 - Y^2) \frac{d^2R(Y)}{dY^2}\right]. \tag{23}$$

We again propose the following series expansion

$$R(Y) = R_0(Y) + \beta R_1(Y) + \beta^2 R_2(Y) + \cdots \tag{24}$$

with the boundary conditions

$$R(Y = 1) = 0$$

or

$$R_0(Y = 1) = 0, \quad R_1(Y = 1) = 0, \quad R_2(Y = 1) = 0, \ldots \tag{25}$$

keeping in mind that we again assume that $R(Y)$ and $R_i(Y)$ ($i = 1, 2, 3, \ldots$) go to zero for $Y \to 1$ as $(1 - Y)$. Similarly, the velocity is developed in a series expansion

$$v = v_0 + \beta v_1 + \beta^2 v_2 + \cdots. \tag{26}$$

Note that the quantity $1/R(Y)$ in the first term on the r.h.s. is replaced by

$$\frac{1}{R_0(Y)} \left[1 - \beta \frac{R_1(Y)}{R_0(Y)} - \beta^2 \frac{R_2(Y)}{R_0(Y)} + \beta^2 \frac{R_1^2(Y)}{R_0^2(Y)} + O(\beta^3) + \cdots\right]. \tag{27}$$

As usual, the velocity can be determined by the asymptotic behaviour of $R(Y)$ so that the solution of (23) behaves as exp (-2ζ) for $\zeta \to +\infty$. Substituting this asymptotic behaviour into (22), we get for the velocity

$$v = vc + \beta c^2 = v_0 + \beta v_1 \tag{28}$$

so that

$$v_0 = vc, \quad v_1 = c^2 \quad \text{and} \quad v_2 = \cdots = 0. \tag{29}$$

Note that the boundary condition at the l.h.s. (the limit for $Y \to -1$) is now determined by the relation

$$v = \tfrac{1}{2}R(Y \to -1) \quad \text{or} \quad 2vc + 2\beta c^2 = R(Y \to -1). \tag{30}$$

Substitution of eqs (24), (28) and (29) into eq. (23) gives a series expansion in β:

$$\beta^0: \quad -4v_0 R_0(Y) + 8R_0(Y)^2 - 2cv(1 - Y^2) \frac{dR_0(Y)}{dY} = 0, \tag{31a}$$

$$\beta^1: \quad -4v_1 R_0(Y) - 4v_0 R_1(Y) + 16R_0(Y)R_1(Y) - 2vc(1 - Y^2) \frac{dR_1(Y)}{dY} - \frac{c^2}{R_0(Y)} (1 - Y^2)^2 \left(\frac{dR_0(Y)}{dY}\right)^2 + 2c^2(1 - Y^2)\left[-2Y \frac{dR_0(Y)}{dY} + (1 - Y^2) \frac{d^2R_0(Y)}{dY^2}\right] = 0, \tag{32b}$$

$$\beta^2: \quad -4v_1 R_1(Y) - 4v_0 R_2(Y) + 8R_1(Y)^2 - 2vc(1 - Y^2) \frac{dR_2(Y)}{dY} - \frac{c^2}{R_0(Y)} (1 - Y^2)^2 \times \left[2 \frac{dR_0(R)}{dY} \frac{dR_1(Y)}{dY} - \frac{R_1(Y)}{R_0(Y)} \left(\frac{dR_0(Y)}{dY}\right)^2\right] + 2c^2(1 - Y^2)\left[-2Y \frac{dR_1(Y)}{dY} + (1 - Y^2) \frac{d^2R_1(Y)}{dY^2}\right] = 0. \tag{33c}$$

The zeroth-order solution ($\beta = 0$) is found by direct application of the balancing procedure to the linear term of highest order and the nonlinear term. Then $M = 1$, so that $R_0(Y) = r_0(1 - Y)$. After some simple algebra, we get

$$R_0(Y) = \frac{cv}{4} (1 - Y) \quad \text{with} \quad v(= v_0) = vc, \tag{34}$$

a solution, quite similar to the Burgers shock wave.

The first-order equation is also readily solved

$$R_1(Y) = \tfrac{1}{8}(1 - Y^2)\left[-3c^2 \ln (1 + Y) + \frac{2c^2}{1 + Y} - 16cvD_1\right], \tag{35}$$

which represents again a "solitary-wave" correction, quite similar to the KdVB case. The second term on the r.h.s. again contributes to the boundary condition, while the last term represents the solution of the homogeneous part of the ODE of first order. With the same arguments as in the previous case, the integration constant D_1 can be neglected. Even for relatively small values of D_1, the perturbative solution obviously (35) will diverge quickly.

After substitution of (34) and (35) with $D_1 = 0$ into (33c), we are able to solve the resulting first-order ODE. This

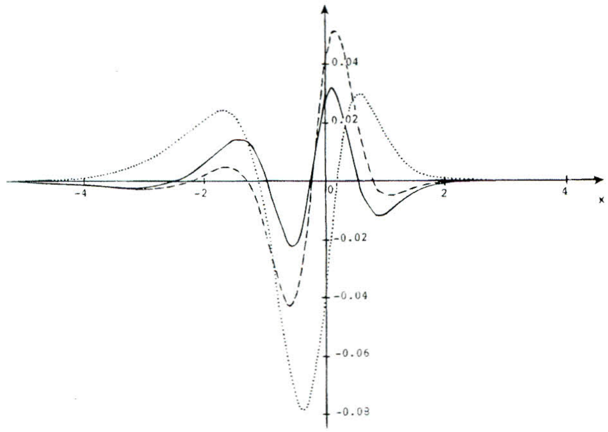

Fig. 5. Remainder terms in the MKdVB case, for $v = c = 1$, $\beta = \frac{1}{10}$ and $Y = \tanh x$: (a) B_0 (dotted curve); (b) B_1 with $D_1 = 0$ (broken curve); (c) B_2 with $D_2 = -\frac{1}{30}$ (full curve).

second-order solution is

$$R_2(Y) = \frac{3c^2}{16v}(1 - Y^2)\left[(5Y - 3) + \ln(1 + Y) \times [3Y\ln(1 + Y) + 3Y - 7] - \frac{256v^2}{3c^2} D_2\right]. \quad (36)$$

Obviously, the integration constant D_2 cannot differ much from zero either. If one considers a zero contribution at $Y = 0$ (following same discussion as before), one simply gets $D_2 = -(9c^3/256v^2)$.

3.2. *Discussion*

Again, like in the KdVB case, one can define remainder terms, representing the remaining contribution after substitution of the different approaches into the MKdVB(+) equation. Hence we find the remainder term

B_0: substitution of $R(Y) = \frac{1}{4}(cv + \beta c^2)(1 - Y)$; (37)

B_1: substitution of $R(Y) = R_0(Y) + \beta R_1(Y)$; (38)

B_2: substitution of $R(Y) = R_0(Y) + \beta R_1(Y) + \beta^2 R_2(Y)$. (39)

For certain representative values of the parameters, these remainder terms are plotted in Fig. 5.

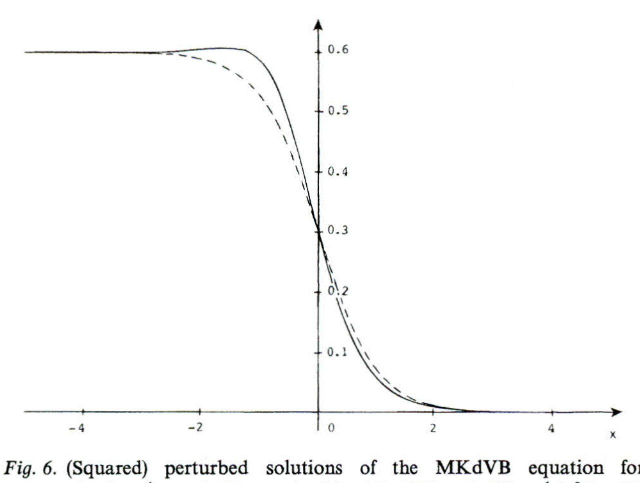

Fig. 6. (Squared) perturbed solutions of the MKdVB equation for $v = c = 1$, $\beta = \frac{1}{10}$ and $Y = \tanh(x)$: (a) $R(Y) = R_0(Y) + \frac{1}{4}\beta c^2(1 - Y)$ (broken curve); (b) $R(Y) = R_0(Y) + \beta R_1(Y) + \beta^2 R_2(Y)$ with $D_0 = 0$, $D_1 = 0$ and $D_2 = -\frac{1}{30}$ (full curve).

Finally, the perturbed solutions of (3) are pictured in Fig. 6. Again, a solitary-wave structure appears on top of the shock-wave. Through relation (21), one simply has to take the square root of the obtained approximate results of (24) to get the final solutions, which of course leads to the same conclusions.

4. General discussion and conclusions

With the aid of the tanh technique, we are able to establish a perturbation procedure to solve the KdVB and the MKdVB(+) equations approximately in the limit of small dispersion (b or $\beta \to 0$). It is clear from the analysis that the use of a new independent variable $Y = \tanh$ makes the calculations transparent and straightforward.

Unfortunately, the other limiting case (small dissipation) cannot be carried out in this way for both equations under study. It limits the use of the tanh technique. Such failure is due to the value the solution takes at the left boundary ($\xi \to -\infty$). One expects here a solution, closely related to a KdV solitary wave, with a tail of small thickness at $\xi \to -\infty$, which should effectively disappear in the limit $a \to 0$. However, due to the close relationship between the velocity and the boundary condition at that point [see eqs (8) and (30)], this requirement cannot be fulfilled. As an example, take the KdVB case for instance. From (3) we get

$$vS(Y \to -1) = \tfrac{1}{2}S(Y \to -1)^2, \quad (40)$$

which gives for $S(Y = -1) \neq 0$ the relation

$$2v = 4ac + 8bc^2 = S(Y = -1), \quad (41)$$

using eq. (4). Even for small values of a, the boundary value $S(Y = -1)$ has finite thickness and will never vanish in the limit $a \to 0$.

Nevertheless, in the cases where the tanh technique can be used, the ease of use is striking. Other applications, such as nonlinear diffusion equations, are under study. In combination with a phase plane analysis and numerical simulation real insight is gained in the structure of the associated solutions.

We are aware of the fact that the relevant perturbed equations cannot be solved directly in some cases. One can then try at least an infinite expansion in tanh. If for instance the desired solution decays for $\xi \to \pm\infty$ (or $Y \to \pm 1$), the successive terms in such an expansion are becoming less important because the variable Y satisfies the inequality relation $-1 \leqslant Y = \tanh \xi \leqslant 1$. Such an approach was already used in [3], to solve approximately a coupled set of reaction-diffusion equations, originating from the domain of chemical reaction kinetics and population dynamics. The only difficulty left is then the determination of the different coefficients of the expansion, because the recurrence relations generally show a (nonlinear) coupling between these coefficients. A subtle and careful analysis of limiting cases will be needed and the knowledge of some numerical results proves necessary for deriving valid results. The latter case, however, falls beyond our present analysis.

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References

1. Ichikawa, Y. H. and Watanabe, S., J. de Physique **12**, C6 (1977).
2. Malfliet, W. and Hereman, W., Physica Scripta **54**, 563 (1996).
3. Malfliet, W., J. Phys. A.: Math. Gen. **24**, 5499 (1991).
4. Canosa, J. and Gazdag, J., J. Comp. Phys. **23**, 393 (1977).
5. Gagliardi, G., Mainardi, F. and Turchetti, G., Proc. 10th International Conf. on Nonlinear Acoustics 1984.
6. Xin, X. and Zhao, Y., Science in China **35**, 1066 (1992).
7. Grad, H. and Hu, P., Phys. Fluids **10**, 2596 (1967).
8. Canosa, J., J. Math. Phys. **10**, 1862 (1969).
9. Johnson, R., J. Fluid Mech. **86**, 415 (1973).
10. Huang, G., Lou, S. and Dai, X., Sing. J. Phys. **8**, 123 (1991).
11. Peletier, L., "Proc. IUTAM Symposium on Instability of Continuous Systems" (Edited by H. Leipholz) (Springer-Verlag, Berlin 1971).