

Als zweites Beispiel werden harmonisch erzwungene Schwingungen derselben Schwinger untersucht. In Bild 2 werden numerische Ergebnisse bzw. Näherungslösungen und experimentelle Resultate zusammengestellt. Man erkennt die stark veränderlichen Schwingungsmodes und Superresonanzen. Näherungslösungen können allerdings nur in einem sehr beschränkten Maß angewendet werden.

Im dritten Beispiel handelt es sich um die Kombinationsresonanz eines Schwingers mit zwei Freiheitsgraden und quadratischer bzw. kubischer Nichtlinearität. Im Gegensatz zum zweiten Beispiel tritt eine solche Resonanz jeweils nur bei einer Schwingungsmasse auf, wie in Bild 3 gezeigt wird. Ebenfalls in Bild 3 sieht man, daß Näherungslösungen [5] dieses komplizierten Phänomen kaum beschreiben können. Mit der Galerkin-Methode kann auch hohe Genauigkeit erreicht werden; nur ist sie sehr zeitaufwendig.

Einzelheiten dieser numerischen Methode sowie weitere Beispiele sind in [6] aufgeführt.

Literatur

- 1 STOER, J.; BULIRSCH, R., Einführung in die numerische Mathematik II, Springer, Berlin-Heidelberg-New York 1978.
- 2 NAYFEH, A. H.; MOK, D. T., Nonlinear Oscillations, John Wiley & Sons, New York-Chichester-Brisbane-Toronto 1979.
- 3 ROSENBERG, R. M.; ATKINSON, C. P., On the Natural Modes and their Stability in Nonlinear Two-Degree-of-Freedom Systems, *J. Appl. Mech.*, **26** (1959), 377–385.
- 4 SETHNA, P. R., Steady-State Undamped Vibrations of a Class of Nonlinear Discrete Systems, *J. Appl. Mech.*, **27** (1960), 187–195.
- 5 VAN DOOREN, R., Differential Tones in a Damped Mechanical System with Quadratic and Cubic Non-Linearities, *Int. J. Non-linear Mech.*, **8** (1973), 575–583.
- 6 LING, F.-H., Numerische Berechnung periodischer Lösungen einiger nichtlinearer Schwingungssysteme, Dissertation, Universität Stuttgart 1981.

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On the Exact Theory of Tops Rising by Friction

The tippe top consists of a spherical segment upon whose planar surface is mounted a short rod. Moving on a rough horizontal plane with sufficiently large initial rotation, the stem will touch this plane after some time (Fig. 1). Experimentally it is seen that the tippe top will rise on its stem. HUGENHOLTZ [1] explained this rising effect by introducing the hypothesis of friction; the solution of his equations of motion were obtained graphically in the case $A = C$ ($A (= B)$ being the moment of inertia about any axis through the centre of mass, normal to the axis of revolution, C the moment of inertia about the axis of revolution). With the same hypotheses MERTENS and DE CORTE [2] gave an exact mathematical solution of the problem. The more realistic case $A \neq C$ will be discussed here. The general equations of motion in the situation of Fig. 1 have been established by HUGENHOLTZ, using the angular momentum equation with respect to the centre of mass of the tippe top:

$$Ka + K'a' = -C\dot{\varphi}\dot{\psi} + (A - C) \cos \theta_0 \dot{\varphi}^2, \quad (1)$$

$$W(a - r \cos \theta_0) + W'(a' - r' \cos \theta_0) = -A \sin \theta_0 \ddot{\varphi}, \quad (2)$$

$$-Wr \sin \theta_0 - Wr' \sin \theta_0 = C \cos \theta_0 \ddot{\varphi} + C\ddot{\psi}, \quad (3)$$

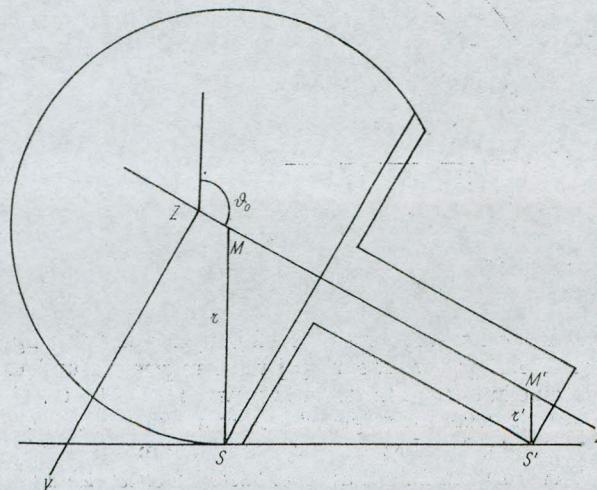


Fig. 1. The tippe top with spherical surface and stem in contact with the horizontal rough plane

where φ, ψ, θ are EULER's angles (θ_0 being the constant value of θ in the configuration of Fig. 1); W and W' the frictional forces in S and S' respectively, having the magnitudes fK and fK' , K and K' being the normal components of the reactions in S and S' respectively, f the coefficient of friction; $a = ZM$, $a' = ZM'$, Z being the centre of mass, M the centre of the spherical part of the top.

Taking into account the equation of motion of the centre of mass, leading to $K + K' = mg$ (m being the total mass of the top), together with equation (1), we obtain

$$K = [C\dot{\varphi}\dot{\psi} - (A - C) \cos \theta_0 \dot{\varphi}^2 + mga]/(a' - a), \quad K' = -[C\dot{\varphi}\dot{\psi} - (A - C) \cos \theta_0 \dot{\varphi}^2 + mga]/(a' - a). \quad (4)$$

The spherical part of the top only shall touch the floor, when $K' = 0$, corresponding with points in the $(\dot{\varphi}, \dot{\psi})$ -plane on the hyperbola $C\dot{\varphi}\dot{\psi} - (A - C) \cos \theta_0 \dot{\varphi}^2 + mga = 0$; the stem only shall touch the floor, when $K = 0$, corresponding with the hyperbola $C\dot{\varphi}\dot{\psi} - (A - C) \cos \theta_0 \dot{\varphi}^2 + mga' = 0$. Both hyperbolas have the same centre (the origin) and the same asymptotes $(\dot{\varphi} = 0, \dot{\psi} = (A - C) \cos \theta_0 \dot{\varphi}/C)$. As long as the spherical segment and the stem are in contact with the floor, the points in the $(\dot{\varphi}, \dot{\psi})$ -diagram will move in the space between those hyperbolas. In the general case we also have to consider three separate areas, defined by

- I $\dot{\psi} > -a\dot{\varphi}/r$, corresponding with $W = fK$, $W' = fK'$,
- II $-a'\dot{\varphi}/r' < \dot{\psi} < -a\dot{\varphi}/r$, corresponding with $W = -fK$, $W' = fK'$,
- III $\dot{\psi} < -a'\dot{\varphi}/r'$, corresponding with $W = -fK$, $W' = -fK'$.

In all three cases it was possible [3] to reduce the system (2), (3) with the help of (4) to RICCATI equations in $\dot{\varphi}$, which in turn could be transformed into WEBER differential equations. So $\dot{\varphi}$ can be expressed by WEBER functions [4].

There is, however, a more elegant method to solve the problem in the general case ($A \neq C$), by reducing it to the more simple special case, $A = C$, with the help of the linear transformation

$$\dot{\tilde{\varphi}} = \dot{\varphi}, \quad \dot{\tilde{\psi}} = \frac{C}{A} \left(\dot{\psi} - \frac{A - C}{C} \cos \theta_0 \dot{\varphi} \right). \quad (5), (6)$$

So, for instance, the equations of motion (1), (2), (3) can be written as the equations (1), (2), (3) in the paper of MERTENS and DE CORTE [2], with $\dot{\varphi}$ and $\dot{\psi}$ replaced by $\dot{\tilde{\varphi}}$ and $\dot{\tilde{\psi}}$ respectively. The hyperbolas, in whose points K' and K are zero, respectively become $A\dot{\tilde{\varphi}}\dot{\tilde{\psi}} + mga = 0$ and $A\dot{\tilde{\varphi}}\dot{\tilde{\psi}} + mga' = 0$. Only the equations of the two straight lines $\dot{\psi} = -a\dot{\varphi}/r$ and $\dot{\psi} = -a'\dot{\varphi}/r'$ delimiting the different areas become slightly more complicated, but still represent straight lines through the origin.

In the cases I and III the solution may be written as (cf. equation (5) in [2]),

$$\dot{\tilde{\varphi}} = \dot{\tilde{\varphi}}_0 \exp [-(\xi^2 - \xi_0^2)] - (2/N)^{1/2} \cos \theta_0 \{ F(\xi) - \exp [-(\xi^2 - \xi_0^2)] F(\xi_0) \}, \quad (7)$$

where

$$F(x) = e^{-x^2} \int_0^x e^{t^2} dt$$

represents DAWSON's integral [5], $\xi = (N/2)^{1/2} \dot{\tilde{\psi}}$, $N = (a' - a) A \sin^2 \theta_0 / mg(a'r - ar')$; $\dot{\tilde{\varphi}}_0$, $\dot{\tilde{\psi}}_0$ and ξ_0 are the initial values of $\dot{\tilde{\varphi}}$, $\dot{\tilde{\psi}}$ and ξ respectively on the hyperbola $A\dot{\tilde{\varphi}}\dot{\tilde{\psi}} + mga = 0$.

In the case II the solution of the modified system (2), (3) goes along the same lines as in reference [2], leading to a RICCATI equation for $\dot{\tilde{\varphi}}$, the solution of which reads

$$\dot{\tilde{\varphi}} = -\frac{D^{1/2} \left[\frac{PS}{D} U \left(\frac{1}{2} - \frac{PS}{D}, x \right) + C' V \left(\frac{1}{2} - \frac{PS}{D}, x \right) \right]}{P \left[U \left(-\frac{PS}{D} - \frac{1}{2}, x \right) + C' V \left(-\frac{PS}{D} - \frac{1}{2}, x \right) \right]}, \quad (8)$$

with $x = D^{1/2}[t - (E/D)]$ and $D = QR - PS (> 0)$; $U(a, x)$ and $V(a, x)$ are the fundamental solutions of WEBER's differential equation, $u'' - [(x^2/4) + a] u = 0$ [4]; the constants E, P, Q, R, S have the same significance as in reference [2]; the integration constant C' can be determined by means of the initial condition for $\dot{\tilde{\varphi}}$ (on one of the straight lines, separating the different cases in the $(\dot{\tilde{\varphi}}, \dot{\tilde{\psi}})$ -plane. Further we have the relationship between $\dot{\tilde{\psi}}$ and $\dot{\tilde{\varphi}}$:

$$\dot{\tilde{\psi}} = (P/R) \dot{\tilde{\varphi}} + (D/R) t + \alpha, \quad (9)$$

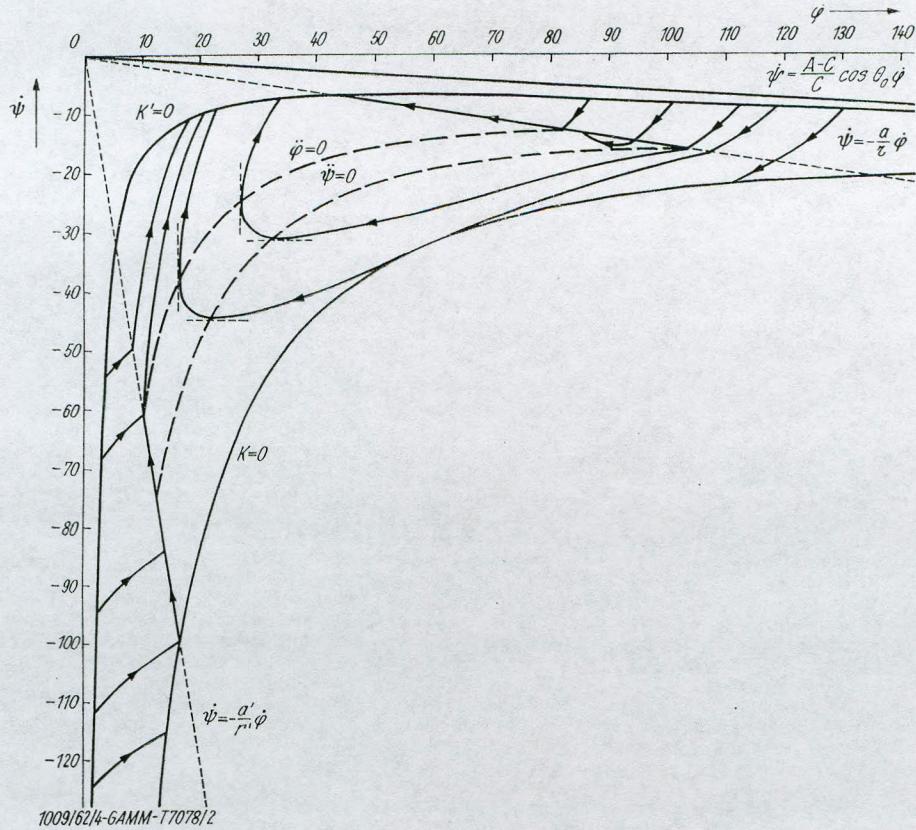
where α is an integration constant to be determined by the initial conditions:

$$\alpha = \dot{\tilde{\psi}}_1 - (P/R) \dot{\tilde{\varphi}}_1, \quad (10)$$

where $\dot{\tilde{\varphi}}_1, \dot{\tilde{\psi}}_1$ are the coordinates of the point of intersection of the curve (7) and the straight line $\dot{\tilde{\psi}} = -(C/A) \times \times [(A - C)/C] \cos \theta_0 + a/r \dot{\tilde{\varphi}}$, which is the transformed line $\dot{\psi} = -(a/r) \dot{\varphi}$ (the new time origin has been chosen at this intersection point).

As an example of numerical calculation we take the following parameters for the tippe top: $a = 3 \cdot 10^{-3}$ m, $a' = 3 \cdot 10^{-2}$ m, $r = 2 \cdot 10^{-2}$ m, $r' = 5 \cdot 10^{-3}$ m, $m = 10^{-2}$ kg, $A = 20 \cdot 10^{-7}$ kgm², $C = 18 \cdot 10^{-7}$ kgm².

In Fig. 2 some paths in the $(\dot{\varphi}, \dot{\psi})$ -diagram are sketched. Next to the hyperbolas corresponding with $K = 0$ and $K' = 0$ (cf. equations (4)) the two straight lines separating the cases I, II, III are drawn. In the region II

Fig. 2. Paths in the $(\dot{\phi}, \dot{\psi})$ -diagram

(between the two straight lines) the arcs of the hyperbolas $\ddot{\phi} = 0$ and $\ddot{\psi} = 0$ are also represented. The figure shows that the minimum initial value $\dot{\phi}_0$ will be larger than the corresponding value in the case $A = C = 20 \cdot 10^{-7} \text{ kgm}^2$, where it took a value about 91 rad/s [2]. It is easy to see that initial motions with small $\dot{\phi}_0$ and large $\dot{\psi}_0$ also may lead to a motion where the top rests on its stem.

References

- 1 HUGENHOLTZ, N. M., On tops rising by friction, *Physica* **18**, 515–527 (1952).
- 2 MERTENS, R.; DE CORTE, L., An exact mathematical solution of the problem of tops rising by friction, *ZAMM* **58**, T 116–T 118 (1978).
- 3 DE SPIEGELEERE, R., De Tippe Top, M. Sc. Dissertation, Gent 1980.
- 4 National Physical Laboratory, Tables of WEBER Parabolic Cylinder Functions, HMSO, London 1955.
- 5 ABRAMOWITZ, M.; STEGUN, I. A., Handbook of Mathematical Functions, Dover Publ., New York 1968, p. 298.

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Stabilität bei Systemen mit Parametererregung

Zur Stabilitätsuntersuchung an parametererregten Systemen können hinreichende Stabilitätskriterien herangezogen werden, die mit weniger Aufwand als exakte Verfahren (z. B. FLOQUET-Theorie) sichere Stabilitätsaussagen erlauben. Oft ist damit aber nur ein relativ kleines Teilgebiet des wirklichen Stabilitätsgebiets zu ermitteln. In dieser Arbeit wird ein Frequenzbereichsverfahren angegeben, das sich für die Behandlung periodischer Systeme besonders gut eignet und das bessere Stabilitätsergebnisse ermöglicht.

1. Systembeschreibung

Gegeben sei ein lineares, periodisch zeitvariables System

$$\dot{x} = A(t)x + b, \quad (1)$$