CP10

THE RAMAN-NATH EQUATIONS REVISITED

R. MERTENS, W. HEREMAN

Instituut voor Theoretische Mechanica
Rijksuniversiteit Gent
Krijgslaan 281-S9
B-9000 Gent, Belgium

and J.-P. OTTOY

Seminarie voor Toegepaste Wiskunde en Biometrie
Rijksuniversiteit Gent
Coupure Links 653
B-9000 Gent, Belgium

Proceedings ULTRASONICS INTERNATIONAL 85, pp. 422-428 (1985)

THE RAMAN-NATH EQUATIONS REVISITED

R. Mertens, W. Hereman

Instituut voor Theoretische Mechanica, Rijksuniversiteit Gent,
Krijgslaan 281-S9, B-9000 Gent, Belgium

and J.P. Ottoy

Seminarie voor Toegepaste Wiskunde en Biometrie, Rijksuniversiteit Gent,
Coupure Links 653, B-9000 Gent, Belgium

The N^{th} order approximation (NOA) method is applied to the diffraction of light by ultrasound. A truncated system of Raman-Nath equations is treated by an eigenvalue method and by Heaviside's operational method. Both methods lead to equivalent expressions for the intensities of the diffracted orders, that are easily implemented on computer. Theoretical curves obtained by the NOA method are compared with previous approximations and experimental data.

INTRODUCTION

Recently there was a revived interest in Raman-Nath equations, not only for their relevance to acoustooptical problems[1] and holography[2,3], but also to the study of stimulated Compton scattering[4], stimulated Cherenkov emission, and to the theory of the free electron laser [5]. The present paper is devoted to an approximate solution of the Raman-Nath set of difference-differential equations for the amplitudes of the diffracted light waves due to acoustooptic interaction in an isotropic medium. The procedure used is the N^{th} order approximation (NOA) method, introduced by Nagendra Nath[6] for N=1 (thus restricted to the Bragg diffraction regime) and extended by Mertens[7] in 1962 for the problem of superposed ultrasonic waves, but practically restricted to N=2, according to the computer facilities at that time. In two recent publications[8,9], Blomme and Leroy derived finite analytical expressions for the intensities using the 2OA and 3OA methods. In this paper, we extend their results to arbitrary order N. If one wishes to obtain a solution in the NOA, the amplitudes $\phi_0, \phi_{\pm 1}, \phi_{\pm 2}, \ldots, \phi_{\pm N}$ will be determined by a truncated system of N+1 Raman-Nath equations, neglecting the amplitudes $\phi_{\pm(N+1)}, \phi_{\pm(N+2)}$, etc.. To integrate truncated Raman-Nath systems several methods have been proposed, e.g. an operator technique introduced by Benlarbi and Solymar[10] (extremely useful to treat higher-order Bragg diffraction) and a Laplace-transform method[1]. In this article, we will use two straightforward methods :

(1) an eigenvalue method, leading to the solution of a characteristic equation of degree N+1, with real roots;
(2) the operational method of Heaviside-Jeffreys[11], leading to expressions for the amplitudes in terms of determinants.

The latter method has the advantage of using a finite acoustooptical interaction length L, whereas in the formulation of the Laplace-transform method[1] a physically inadmissible infinite width L had to be introduced. Both our methods give equivalent results that are equally easy to program. The actual computer facilities admit numerical calculations for large values of N, so that nearly exact solutions may be obtained not only in the Raman-Nath or Bragg diffraction regimes but also in the intermediate region. A comparison of our NOA curves (for N=7) is made with the experimental results of Klein and Hiedemann[12] (for the zeroth-order intensity) for different values of the Klein-Cook parameter[13] Q and with the Raman-Nath parameter[13,14] v ranging from 0 to 10. The new theoretical curves fit the experimental points far more better than curves resulting from previous approximate formulae[14,15,16].

THE NOA METHOD

We restrict ourselves to the problem of the diffraction of a light beam by a progressive ultrasonic wave in an isotropic medium. At normal light-sound interaction, the amplitudes $\phi_n(\zeta)$ of the diffracted light waves must satisfy the infinite set of Raman-Nath equations

$$2\frac{d\phi_n}{d\zeta} - (\phi_{n-1} - \phi_{n+1}) = in^2\rho\phi_n \,, \qquad (i=\sqrt{-1}) \tag{1}$$

with boundary conditions

$$\phi_n(0) = \delta_{n0} \quad , \quad n = 0,\pm1,\pm2,\ldots \,. \tag{2}$$

In (1), $\zeta = vz/L$, where the so-called Raman-Nath parameter $v = \pi\varepsilon_1 L/\varepsilon_r\lambda$ physically represents the peak phase shift of the light, over the acoustooptic interaction length L, due to the peak variation ε_1 of the relative permittivity ε_r of the medium; and $\rho = 2\varepsilon_r\lambda^2/\varepsilon_1\lambda^{*2}$, stands for a regime parameter, containing the ratio of wavelengths of light and ultrasound. The Klein-Cook parameter[13] is defined by $Q = \rho v$, and thus independent of the amplitude ε_1 of the disturbing sound wave. In the NOA method one neglects the energy in the diffraction orders higher than N, i.e. $\phi_{\pm(N+1)} = \phi_{\pm(N+2)} = \ldots = 0$. Hence, using the symmetry property[15,17] $\phi_{-n} = (-1)^n\phi_n$, (1) can be replaced by the following truncated system of N+1 equations :

$$\begin{aligned}
&2\frac{d\phi_0}{d\zeta} + 2\phi_1 = 0 \,, \\
&2\frac{d\phi_n}{d\zeta} - \phi_{n-1} + \phi_{n+1} = in^2\rho\phi_n \qquad (n=1,2,\ldots,N-1) \,, \\
&2\frac{d\phi_N}{d\zeta} - \phi_{N-1} = iN^2\rho\phi_N \,,
\end{aligned} \tag{3}$$

with the boundary conditions (2), however, for $n = 0,1,\ldots,N$.

EIGENVALUE PROBLEM

Proposing a solution of (3) of the form

$$\phi_0 = \sqrt{2}a_0 \exp(\tfrac{1}{2}is\zeta) \ , \tag{4}$$

$$\phi_n = a_n \exp(\tfrac{1}{2}is\zeta) \quad , \qquad n = 1,\dots,N \ ;$$

the real constants $a_0, a_1, \dots, a_N$ and the characteristic numbers s will be linked by the matrix equation

$$(\mathbf{M} - s\mathbf{I}).\mathbf{a} = \mathbf{O} \quad , \tag{5}$$

where $\mathbf{I}$ is the $(N+1)\times(N+1)$ unit matrix, $\mathbf{a}^T = (a_0\ a_1\ \dots\ a_N)$ and

$$\mathbf{M} = \begin{bmatrix} 0 & i\sqrt{2} & 0 & \cdot & \cdot\ \cdots & & & & & \cdots\ \cdot & \cdot & \cdot & 0 \\ -i\sqrt{2} & \rho & i & 0 & \cdot\ \cdots & & & & & \cdots\ \cdot & \cdot & \cdot & 0 \\ 0 & -i & 4\rho & i & 0\ \cdots & & & & & \cdots\ \cdot & \cdot & \cdot & 0 \\ \vdots & & & & & & & & & & & & \vdots \\ 0 & \cdot & \cdot & \cdot & \cdot\ \cdots\ 0 & -i & n^2\rho & i & 0 & \cdots\ \cdot & \cdot & \cdot & 0 \\ \vdots & & & & & & & & & & & & \vdots \\ & & & & & & & & & & & & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot\ \cdots & & & & & \cdots\ 0 & -i & (N-1)^2\rho & i \\ 0 & \cdot & \cdot & \cdot & \cdot\ \cdots & & & & & \cdots\ \cdot & 0 & -i & N^2\rho \end{bmatrix} \tag{6}$$

is a $(N+1)\times(N+1)$ Hermitian matrix. For a nonzero solution $\mathbf{a}$ the eigenvalues s will be the real roots[18] of the characteristic equation

$$\det\ (\mathbf{M} - s\mathbf{I}) = 0 \ . \tag{7}$$

The eigenvector $\mathbf{a}^{(k)}$, $\mathbf{a}^{(k)T} = (a_0^{(k)}\ a_1^{(k)} \dots\ a_N^{(k)})$, associated with the eigenvalue s_k $(k=1,2,\dots,N+1)$ will be determined by

$$(\mathbf{M} - s_k\mathbf{I}).\mathbf{a}^{(k)} = \mathbf{O} \quad . \tag{8}$$

Regarding the structure of the system (8), we may see that $a_0^{(k)} \neq 0$ and that $a_n^{(k)}$ may be chosen real (respectively purely imaginary) if n is even (respectively odd). Furthermore, we can choose $a_0^{(k)} = \sqrt{2}/2$ $(k=1,2,\dots,N+1)$ since the eigenvectors are only determined up to an arbitrary factor. The general solution of the truncated linear system (3), may then be written[18] as

$$\phi_0 = \sum_{k=1}^{N+1} c_k \exp(\tfrac{1}{2} i s_k \zeta) \ , \tag{9}$$

$$\phi_n = \sum_{k=1}^{N+1} c_k a_n^{(k)} \exp(\tfrac{1}{2} i s_k \zeta) \ , \quad n = 1,2,\ldots,N \ ; \tag{10}$$

where the N+1 constants c_k are real (as a consequence of taking $a_0^{(k)}$ real). These constants c_k follow from

$$\sum_{k=1}^{N+1} c_k = 1 \quad , \quad \sum_{k=1}^{N+1} c_k a_n^{(k)} = 0 \quad , \quad n = 1,2,\ldots,N \ ; \tag{11}$$

obtained by applying the boundary conditions to (9) and (10).
Finally, one can calculate the intensities[7], which in z=L may be expressed as

$$I_0(v) = \phi_0 \overline{\phi_0} = 1 - 4 \sum_{\substack{j,k=1 \\ j<k}}^{N+1} c_j c_k \sin^2(s_j - s_k)\frac{v}{4} \ , \tag{12}$$

$$I_{\pm n}(v) = \phi_{\pm n}(v)\overline{\phi_{\pm n}}(v) = -4 \sum_{\substack{j,k=1 \\ j<k}}^{N+1} c_j c_k a_n^{(j)} \overline{a_n^{(k)}} \sin^2(s_j - s_k)\frac{v}{4}, \quad n=1,\ldots,N. \tag{13}$$

Needless to say that the characteristic equation (7) of degree N+1 in s, can only be solved analytically for N≦3. In the latter cases explicit analytical expressions for the intensities can be obtained[6,7,8,9]; otherwise the problem has to be treated numerically in the following steps : (i) determine the eigenvalues and eigenvectors of matrix $\mathbf{M}$; (ii) next solve the linear system (11) for c_k; (iii) finally substitute these results in (12) and (13).

HEAVISIDE'S OPERATIONAL METHOD

Now, we will apply Heaviside's operational method[11] to the truncated system (3). After Jeffreys[11] (p.237), we write p for d/dζ and interpret p^{-1} as the operation of definite integration

$$p^{-1} f(\zeta) = \int_0^{\zeta} f(z)dz \ . \tag{14}$$

The resulting subsidiary equations

$$\begin{aligned} &-s\phi_0 + 2i\phi_1 = -s \ , \\ &(\rho n^2 - s)\phi_n - i\phi_{n-1} + i\phi_{n+1} = 0 \qquad (n=1,2,\ldots,N-1) \ , \\ &(\rho N^2 - s)\phi_N - i\phi_{N-1} = 0 \ , \end{aligned} \tag{15}$$

with s=-2ip, are to be solved as if p (or s) was a number. In compact notation (15) can be rewritten as

$$\mathbf{D}\boldsymbol{\phi} = -s\mathbf{E} \ , \tag{16}$$

where the almost Hermitian $(N+1)\times(N+1)$ matrix $\mathbf{D}$ is related to $\mathbf{M}$ by

$$\mathbf{D}(s) = \mathbf{P}(\mathbf{M}-s\mathbf{I})\mathbf{P}^{-1} \ , \tag{17}$$

with $\mathbf{P} = \mathbf{I}+\mathbf{R}$, such that all elements r_{ij} of $\mathbf{R}$ vanish, except $r_{11} = \sqrt{2}-1$. Furthermore, $\boldsymbol{\phi}^T = (\phi_0\ \phi_1\ \phi_2\ \ldots\boldsymbol{\phi}_N)$ and $\mathbf{E}^T = (1\ 0\ \ldots\ 0)$, the latter vector expresses the boundary conditions. From (16), we obtain the formal solution of the problem :

$$\boldsymbol{\phi} = -s\mathbf{D}^{-1}\mathbf{E} \ , \tag{18}$$

where $\mathbf{D}^{-1}$ is the inverse of $\mathbf{D}$ thus explicitly one has

$$\phi_n = \frac{-s\ D_{1,n+1}}{\det \mathbf{D}} \ , \qquad n=0,1,\ldots,N \ ; \tag{19}$$

where $D_{1,n+1}$ stands for the cofactor of element $d_{1,n+1}$ of $\mathbf{D}$ $(n=0,1,\ldots,N)$. Substituting $s=-2ip$ in (17) and (19) we find

$$\phi_n = \frac{2p\ F_{1,n+1}(p)}{F(p)} \quad (n=0,1,\ldots,N-1) \quad , \quad \phi_N = \frac{2p}{F(p)} \ , \tag{20}$$

where $F_{1,n+1}(p)$ is the following determinant :

$$\begin{vmatrix} 2p-i(n+1)^2\rho & 1 & 0 & \cdot & \cdots & \cdot & \cdot & 0 \\ -1 & 2p-i(n+2)^2\rho & 1 & 0 & \cdots & \cdot & \cdot & 0 \\ 0 & -1 & \cdot & \cdot & \cdots & \cdot & \cdot & 0 \\ \vdots & & & & & & & \vdots \\ 0 & & & & & & & 0 \\ 0 & \cdot & \cdot & \cdot & \cdots & 0 \quad -1 & 2p-i(N-1)^2\rho & 1 \\ 0 & \cdot & \cdot & \cdot & \cdots & \cdot \quad 0 & -1 & 2p-iN^2\rho \end{vmatrix} \tag{21}$$

and $F(p) = (-i)^{N+1} \det \mathbf{D}(p)$.

Next, we will apply Heaviside's expansion theorem[11] (p.238), i.e.

$$\frac{A(p)}{B(p)} = \frac{A(0)}{B(0)} + \sum_{k=1}^{N+1} \frac{A(\alpha_k)}{\alpha_k B'(\alpha_k)} \exp(\alpha_k\zeta) \ , \tag{22}$$

where $A(p)$ is a polynomial in p of the same degree as $B(p)$ or lower; α_k $(k=1,\ldots,N+1)$ are the simple zeros of $B(p)$ and $B' = dB/dp$. Hence, (20) can be replaced by

$$\phi_n = 2\sum_{k=1}^{N+1} \frac{F_{1,n+1}(\alpha_k)}{F'(\alpha_k)} \exp(\alpha_k\zeta) \ (n=0,1,\ldots,N-1); \ \phi_N = 2\sum_{k=1}^{N+1} \frac{\exp(\alpha_k\zeta)}{F'(\alpha_k)} \ . \tag{23}$$

Regarding (17), the matrices $\mathbf{D}$ and $\mathbf{M}\text{-}s\mathbf{I}$ are similar[18]. So, one can prove that $\alpha_k = \frac{1}{2}is_k$; s_k being the real eigenvalues of $\mathbf{M}$. Paying special attention to the case n=0, we obtain after some calculation[7]

$$I_0(v) = \phi_0(v)\overline{\phi_0}(v) = 1 - 16 \sum_{j,k=1}^{N+1} \frac{F_{1,1}(\frac{1}{2}is_j)F_{1,1}(\frac{1}{2}is_k)}{F'(\frac{1}{2}is_j)F'(\frac{1}{2}is_k)}\sin^2(s_j-s_k)\frac{v}{4} \quad (j<k) , \tag{24}$$

which is equivalent to (12). Similar expressions for I_n (n=1,...,N) are readily obtained. To simplify the calculations of the determinants one can use the following recursion relations :

$$\begin{aligned} &F_{1,j}(p) = (2p-ij^2\rho)F_{1,j+1}(p) + F_{1,j+2}(p) \quad , \quad F_{1,N+1} = 1 \quad , \quad F_{1,N+2} = 0 \ ; \\ &F'(p) = 2F_{1,1}(p) + 2p\,F'_{1,1}(p) + 2\,F'_{1,2}(p) \ ; \\ &F'_{1,j}(p) = 2F_{1,j+1}(p) + (2p-ij^2\rho)F'_{1,j+1}(p) + F'_{1,j+2}(p) \ (j=1,\dots,N) \ . \end{aligned} \tag{25}$$

DISCUSSION

In Fig. 1 (Q=1.26) and Fig. 2 (Q=1.48) we compare curves for I_0 obtained from Raman-Nath's geometrical theory[14], Mertens' perturbation method[15,16] and the NOA method (for N=7) with the experimental results of Klein and Hiedemann[12]. After numerical calculation, $\phi_j \cong 0$ for $j \geq 7$, hence, we have computed (12) and (24) for N=7. These theoretical curves, which perfectly coincide, fit the experimental points even for v>4.5 when Q=1.26 and for v>3.5 for Q=1.48, where the other approximate formulae clearly failed. For a profound discussion of the accuracy of the 20A and 30A methods, in a wide range of the parameters ρ and v, we refer to papers of Blomme and Leroy[8,9]. Concerning computertime, using Heaviside's operational method is 25% faster than using the eigenvalue method.

ACKNOWLEDGEMENTS

One of the authors (R.M.) wishes to thank the Belgian National Science Foundation for research grants.

REFERENCES

1. Leroy O. and Claeys J.M., Wave Motion, vol 6 (1984), pp. 33-39.
2. Lewis J.W. and Solymar L., Proc. R. Soc. Lond., vol A 398 (1985), pp. 45-80.
3. Hariharan P.,'Optical Holography', Cambridge University Press, Cambridge UK(1984).
4. Bosco P., Gallardo J. and Dattoli G., J. Phys. A : Math. Gen., vol 17 (1984), pp. 2739-2742.
5. Dattoli G., Richetta M. and Pinto I., Il Nuovo Cimento, vol 4 D (1984),pp. 293-311.
6. Nagendra Nath N.S., Proc. Indian Acad. Sci., vol 8A (1938), pp. 499-503.
7. Mertens R., Proc. Indian Acad. Sci., vol 55A (1962), pp. 63-98.
8. Blomme E. and Leroy O., J. Acoust. Soc. India, vol 11 (1983), pp. 1-6.
9. Blomme E. and Leroy O., Acustica, vol 57 (1985), pp. 168-174.
10. Benlarbi B. and Solymar L., Int. J. Electron., vol 48 (1980), pp. 361-368.
11. Jeffreys H. and Jeffreys B.,'Methods of Mathematical Physics',3rd edition, Cambridge University Press, Cambridge UK (1966), Chapter 7 and 8.
12. Klein W.R. and Hiedemann E.A., Physica, vol 29 (1963), pp. 981-986.
13. Klein W.R. and Cook B.D., IEEE Trans. Son. Ultrason., vol SU-14 (1967), pp.123-134.
14. Raman C.V. and Nagendra Nath N.S., Proc. Indian Acad. Sci., vol 2A (1935), pp. 406-412.
15. Mertens R., Med. Kon. Vl. Acad. Wet. België, n° 12 (1950), pp. 1-37.
16. Kuliasko F., Mertens R. and Leroy O., Proc. Indian Acad.Sci., vol 68A (1968),pp. 295-302.
17. Mertens R. and Leroy O., Acustica, vol 28 (1973), pp. 182-185.
18. Franklin J.N., 'Matrix Theory', Prentice-Hall Corp., New Jersey (1968).

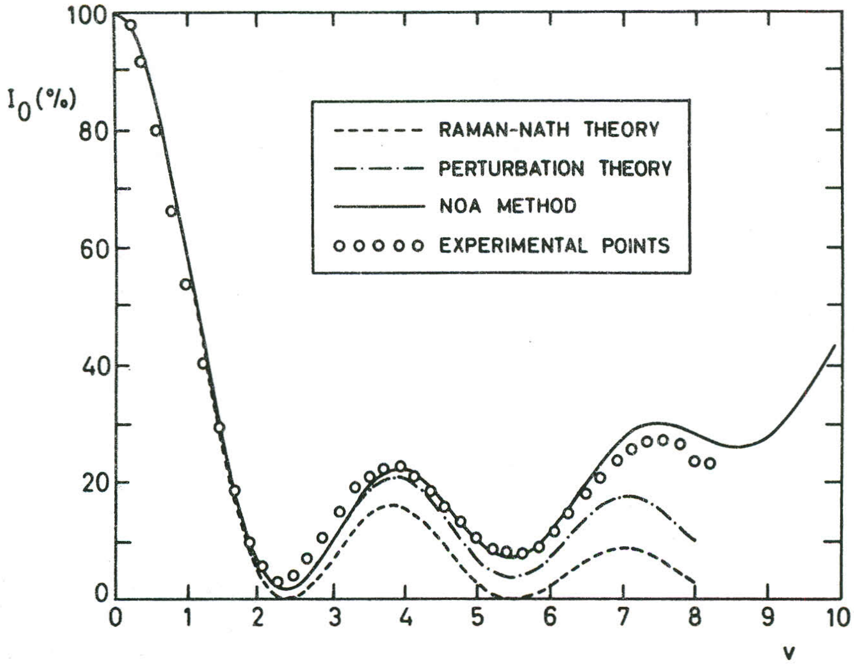

Fig. 1 Zeroth-order intensity versus Raman-Nath parameter for Q=1.26

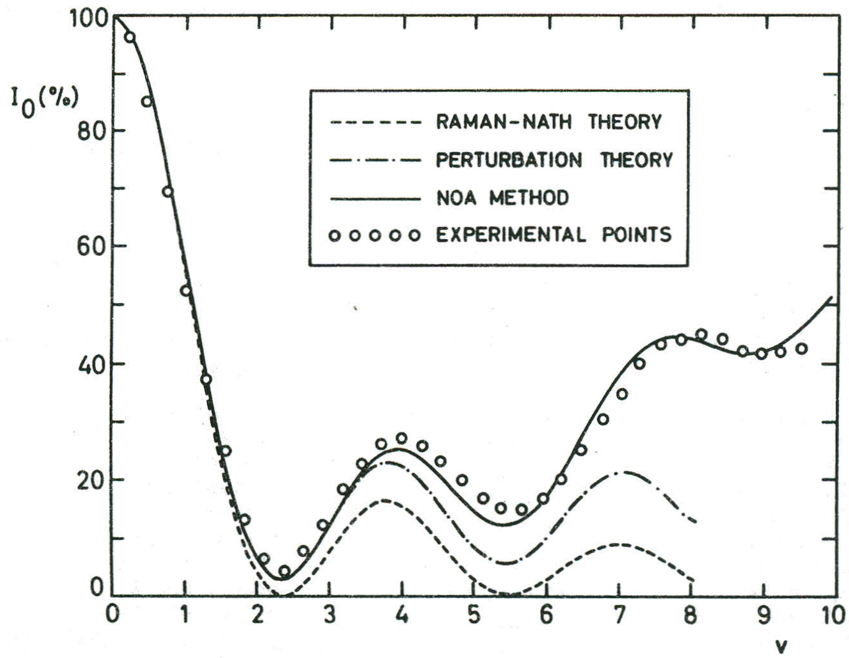

Fig. 2 Zeroth-order intensity versus Raman-Nath parameter for Q=1.48