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# THE RAMAN-NATH EQUATIONS REVISITED. II.

## Oblique incidence of the light - Bragg reflection

R.A. MERTENS

Instituut Theoretische Mechanica
Rijksuniversiteit Gent
Krijgslaan 281
B-9000 Gent, Belgium

W. HEREMAN

Mathematics Department and Center for the
Mathematical Sciences
University of Wisconsin-Madison
Madison, WI 53706
USA

J.-P. OTTOY

Seminarie Toegepaste Wiskunde en Biometrie
Rijksuniversiteit Gent
Coupure Links 653
B-9000 Gent, Belgium

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# THE RAMAN-NATH EQUATIONS REVISITED. II. OBLIQUE INCIDENCE OF THE LIGHT - BRAGG REFLECTION

R.A. Mertens[•], W. Hereman[♦], J.-P. Ottoy[▲]

[•] Inst. Theor. Mechanica, Rijksuniversiteit Gent, Gent, Belgium
[♦] Math. Dept. and MRC, Univ. Wisconsin, Madison, WI 53706, USA
[▲] Semin. Toeg. Wiskunde & Biometrie, Rijksuniversiteit Gent, Gent, Belgium

The N-th order approximation (NOA) method is applied to the diffraction of light by ultrasound, in cases of oblique incidence of the light. A truncated system of Raman-Nath equations is integrated numerically by means of an eigenvalue method. The thus obtained theoretical curves for the light intensities, with varying Raman-Nath parameter v and different Klein-Cook parameters Q, are compared with previous approximations and experimental data. The theoretical predicted symmetries of the diffraction spectra with respect to the various Bragg angles are verified.

## INTRODUCTION

Recently the intensities of the diffracted lightwaves in acoustooptical problems were successfully determined by analytical-numerical solution of the Raman-Nath (RN) equations in the simplified case of normal incidence of light[1,2,3,4]. For that, the authors used a N-th order approximation (NOA) method, introduced by Nagendra Nath[5] for N=1 and extended by Mertens[6]. In the present paper we treat the approximate solution of a similar RN system, however comprising all cases of oblique light incidence. A solution in the NOA may be obtained from a truncated RN system of 2N+1 equations, relating the amplitudes $\phi_0, \phi_{\pm 1}, \ldots, \phi_{\pm N}$, thus ignoring the amplitudes $\phi_{\pm(N+1)}, \phi_{\pm(N+2)}, \ldots$ The solution of this finite system is then reduced in a classical way to an eigenvalue problem, distinctly suited for numerical treatment. The case N=1 is already long due to Nagabhushana Rao[7]; actual computer facilities however admit calculations for large values of N, yielding nearly exact solutions not only in the RN and Bragg regimes but also in the intermediate region. In the latter two cases the diffraction spectrum is asymmetric with respect to the zeroth order line. This asymmetry with regard to the intensities clearly results in an unequal number of positive and negative lines, determined by the computer routine itself. A comparison is made with the experiments of Mayer[8] and Klein et al [9]. The light intensities of zeroth and first order are shown versus the angle of incidence for different values of the Klein-Cook parameter[10] Q and with the RN parameter[11] v=2 or v=3. The fitting of the theoretical curves with the experimental points is excellent.

## THE NOA METHOD FOR OBLIQUE INCIDENCE

Restricting ourselves to the problem of a light beam diffracted by a progressive ultrasonic wave in an isotropic medium, where the direction of propagation of the light beam makes an angle $\varphi$ with the ultrasonic wave fronts, the amplitudes $\phi_n(\zeta)$ of the diffracted light waves satisfy the infinite set of RN equations[12,13]

$$2\frac{d\phi_n}{d\zeta} - (\phi_{n-1}-\phi_{n+1}) = i\rho n(n+\beta)\phi_n \ , \tag{1}$$

with boundary conditions

$$\phi_n(0) = \delta_{n0} \quad , \quad n = 0,\pm1,\pm2,\ldots \tag{2}$$

In (1), $\zeta = \pi\varepsilon_1 z/\sqrt{\varepsilon_r}\lambda_0\cos\varphi$, $\rho = 2\lambda_0^2/\varepsilon_1\lambda^{*2}$, $\beta = -(2\sqrt{\varepsilon_r}\lambda^*/\lambda_0)\sin\varphi$, where $\varepsilon_1$ is the peak variation of the relative permittivity $\varepsilon_r$ of the medium, $\lambda_0$ the wave length of light in vacuum, $\lambda^*$ the wave length of ultrasound. If $\beta = -p$, with p integer, then $p(\lambda_0/2\sqrt{\varepsilon_r}\lambda^*) = \sin\varphi_{BR}^{(p)}$, where $\varphi_{BR}^{(p)}$ is called the Bragg angle of order p. The z-axis being parallel with the sound wave fronts; z=0 and z=L correspond with the boundaries of the ultrasonic field. In what follows we shall make use of the RN parameter $v = \zeta L/z$ and the Klein-Cook parameter $Q = \rho v$, the latter being independent of $\varepsilon_1$. Since in practice $\varphi$ is always very small we set $\cos\varphi \cong 1$. In the NOA method one neglects the energy in the diffraction orders higher than N, i.e. $\phi_{\pm(N+1)}=\phi_{\pm(N+2)}=\ldots=0$. Hence (1) is replaced by the truncated system of 2N+1 equations

$$2d\phi_n/d\zeta - \phi_{n-1}(1-\delta_{n,-N}) + \phi_{n+1}(1-\delta_{nN}) = i\rho n(n+\beta)\phi_n \ , \tag{3}$$

with $\phi_n(0) = \delta_{n0}$, $n = 0,\pm1,\pm2,\ldots,\pm N$.
Projecting a solution

$$\phi_n = A_n \exp(\tfrac{1}{2}is\zeta) \quad , \quad n = 0,\pm1,\ldots,\pm N \ , \tag{4}$$

the integration of system (3) is then reduced to an eigenvalue problem with matrix equation

$$(\mathbf{M}-s\mathbf{I}).\mathbf{A} = \mathbf{O} \ , \tag{5}$$

The Hermitian (2N+1) by (2N+1) matrix $\mathbf{M}$ has diagonal elements $M_{jj} = \rho(N-j+1)(N-j+1-\beta)$, $(j=1,2,\ldots,2N+1)$, subdiagonal elements $M_{p,p-1}=-i$ $(p=2,3,\ldots,2N+1)$, superdiagonal elements $M_{q,q+1}=i$ $(q=1,2,\ldots,2N)$, the remaining elements all being zero; $\mathbf{I}$ is the 2N+1 by 2N+1 unit matrix and $\mathbf{A}^T=(A_{-N},\ldots,A_{-1},A_0,A_1,\ldots,A_N)$. For a nonzero vector solution $\mathbf{A}$ the eigenvalues s must be the 2N+1 obviously real roots of the characteristic equation

$$\det(\mathbf{M}-s\mathbf{I}) = 0 \ . \tag{7}$$

Next, the eigenvector $\mathbf{A}^{(k)}$, $\mathbf{A}^{(k)T}=(A_{-N}^{(k)},\ldots,A_{-1}^{(k)},A_{0}^{(k)},A_{1}^{(k)},\ldots,A_{N}^{(k)})$ associated with the eigenvalue $s_k$ $(k=1,2,\ldots,2N+1)$ can be determined from the linear homogeneous system (5). The general solution of the linear system (3) may then be written as

$$\phi_n = \sum_{k=1}^{2N+1} C_k A_n^{(k)} \exp(\tfrac{1}{2} i s_k \zeta) \quad , \quad n=0,\pm1,\ldots,\pm N \ . \tag{8}$$

Regarding the boundary conditions (2), the 2N+1 real constants $C_k$ follow from

$$\sum_{k=1}^{2N+1} C_k A_n^{(k)} = \delta_{n0} \quad , \quad n=0,\pm1,\ldots,\pm N \ . \tag{9}$$

Finally, one can calculate the intensities, at z=L, yielding

$$I_n(v) = |\phi_n(v)|^2 = \delta_{n0} - 4 \sum_{\substack{k,\ell=1 \\ k<\ell}}^{2N+1} C_k C_\ell A_n^{(k)} \overline{A_n^{(\ell)}} \sin^2(s_k - s_\ell)\frac{v}{4} \ ,$$
$$n=0,\pm1,\ldots,\pm N \ . \tag{10}$$

The characteristic equation (7) of degree 2N+1 in s, can be solved analytically only for N=1; explicit expressions were obtained by Nagabhushana Rao[7]; otherwise the problem has to be treated numerically in the following steps : (i) determine the eigenvalues and eigenvectors of matrix $\mathbf{M}$; (ii) solve the linear system (9) for $C_k$; (iii) substitute these results into (10).

## NUMERICAL RESULTS AND DISCUSSION

(i) In Fig. 1 we compare $I_0$ and $I_{-1}$ versus $\varphi$ obtained from the NOA method (for N=7) with the experimental results of Mayer[8] for Q=4.28. The fitting of both curves is very good. A similar result is obtained for Q=8.36. The purpose of Mayer's experiments was to show the deviation from RN's elementary theory[14], confirmed here theoretically. Similar results were also obtained by Leroy and Blomme[15] by a so-called MN-OA method[16].

(ii) Further we compare $I_0$ and $I_{-1}$ versus $\beta$, calculated with the NOA method (N=7, although N=5 suffices for all values considered) with the experimental points obtained by Klein et al[9] for different combinations of Q's (0.57,2.25,3.75,6.28 and 9.3) and v's (2 and 3). In Figs. 2,3,4,5 we represent our numerical and Klein's experimental results respectively for Q=0.57 and v=2, Q=2.25 and v=3, Q=6.28 and v=2, Q=9.3 and v=3. There is excellent agreement with the measured values. Our curves fit the data even better than those calculated by Klein et al using a direct numerical integration of the RN equations. Furthermore the numerical integration used here results in accurate plots for even larger values of $|\beta|$. From the figures the following theoretical properties[17] are confirmed :

- Intensities $I_{-1}$ show an extremum at the first order Bragg angle $\varphi_{BR}^{(-1)}$ ($\beta$=1);
- Intensity distributions of $I_{-1}$ are symmetric about $\varphi_{BR}^{(-1)}$. From our computer data it is clear that for $Q \gg 1$ (we took Q=9.3), Bragg reflection of the light is a dominant effect, although, for certain values of $\beta$ the intensities of orders +2 or -2 are not negligible.

(iii) In addition to comparison with experiments we have calculated the intensities $I_0$, $I_{\pm 1}$, $I_{\pm 2}$ versus v, for Q=2.25 with $\beta$=1 and $\beta$=2 (Figs. 6 and 7), showing clearly the asymmetry of the spectrum. Up to v=10, N=5 largely suffices and even N=4 would have given excellent results.

(iv) Finally, we have computed $I_0$, $I_{\pm 1}$ versus $\beta$ for $Q \ll 1$, namely Q=0.1, with v=1, v=2 and v=3 using the NOA method (N=6). Compared with the formulae from the elementary RN theory[13] (expressed here in Q and $\beta$) :

$$I_{+n} = I_{-n} = J_n^2(v_\varphi) \ (n=0,1) \ , \ v_\varphi = v\sec\varphi\sin(Q\beta/4)/(Q\beta/4) \ , \tag{11}$$

these practically yield the same results (Fig. 8). In this case the symmetry of the diffraction pattern is also displayed.

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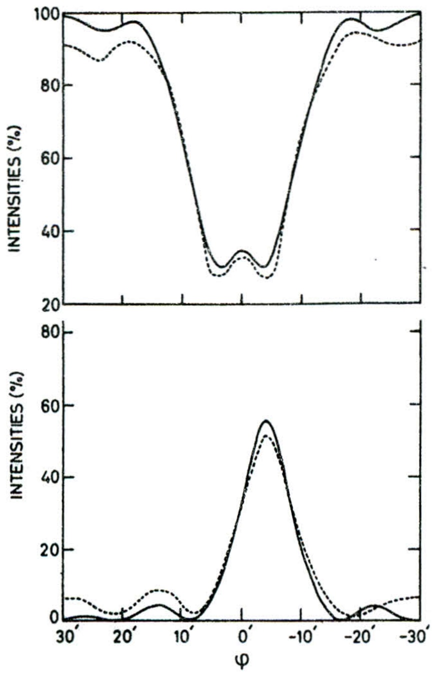
REFERENCES

1 Leroy O. and Claeys J.-M., Wave Motion, vol. 6 (1984), pp. 33-39.
2 R. Mertens, W. Hereman and J.-P. Ottoy, Proc. Ultrason. Intern. 85 (1985), pp. 422-428.
3 R. Mertens, J.-P. Ottoy and W. Hereman, Proc. 12th Intern. Congr. Acoust., Toronto, Canada, vol. 2 (1986) 2 pp.
4 R.A. Mertens, W. Hereman and J.-P. Ottoy, Acad. Anal. (Med. Kon. Acad. Wet. Belg.), to appear.
5 N.S. Nagendra Nath, Proc. Indian Acad. Sci., vol. 8A (1938), pp. 499-503.
6 R. Mertens, Proc. Indian Acad. Sci., vol. 55A (1962), pp. 63-98.
7 K. Nagabhushana Rao, Proc. Indian Acad. Sci., vol. 9A (1939), pp. 422-446.
8 W.G. Mayer, J. Acoust. Soc. Am., vol. 36 (1964), pp. 779-781.
9 W.R. Klein, C.B. Tipnis and E.A. Hiedemann, J. Acoust. Soc. Am., vol. 38 (1965), pp. 229-233.
10 W.R. Klein and B.D. Cook, IEEE Trans. Son. Ultrason., vol.SU-14(1967), pp.123-134.
11 C.V. Raman and N.S. Nagendra Nath, Proc. Indian Acad. Sci., vol. 2A (1935), pp. 406-412.
12 C.V. Raman and N.S. Nagendra Nath, Proc. Indian Acad. Sci., vol. 3A (1936), pp. 459-465.
13 R. Mertens and W. Hereman, Fortsch. Akust. (DAGA'80, München)(1980), pp. 563-566.
14 C.V. Raman and N.S. Nagendra Nath, Proc. Indian Acad. Sci., vol. 2A (1935), pp. 413-420.
15 O. Leroy and E. Blomme, personal communication.
16 E. Blomme and O. Leroy, Acustica, to be published.
17 G. Plancke-Schuyten and R. Mertens, Physica, vol. 62 (1972), pp. 600-613.



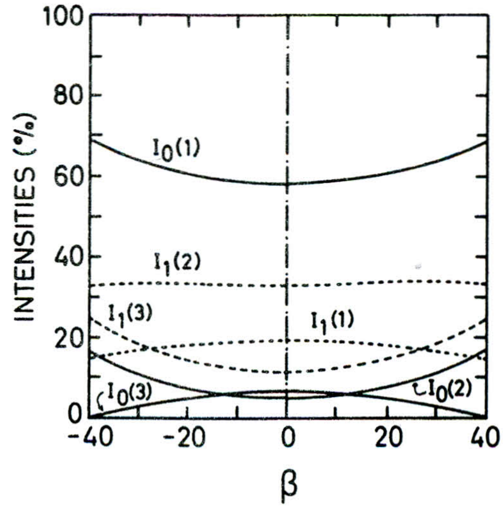
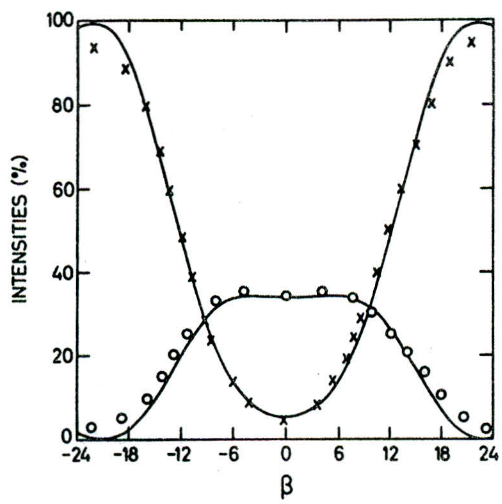




Fig. 1 $I_0$ (top) and $I_{-1}$ (bottom) versus $\varphi$. Experimental[8]:--;NOA method:—, for Q=4.28.

Fig. 8 $I_0$, $I_{-1}$, $I_{+1}$ versus $\beta$ for Q=0.1; v=1, v=2, v=3 from NOA method ($I_{-1} \cong I_{+1}$)



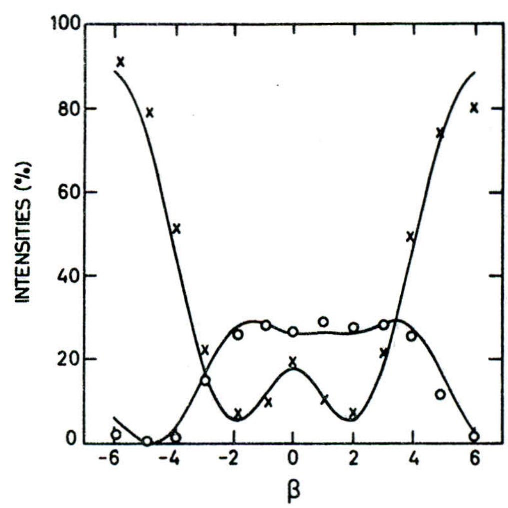




Fig. 2 $I_0$ and $I_{-1}$ versus $\beta$ for Q=0.57, v=2. NOA method:—. Experimental[9]:$I_0$:×, $I_{-1}$:o.

Fig. 3 $I_0$ and $I_{-1}$ versus $\beta$ for Q=2.25, v=3. NOA method:—. Experimental[9]:$I_0$: ×, $I_{-1}$:o.

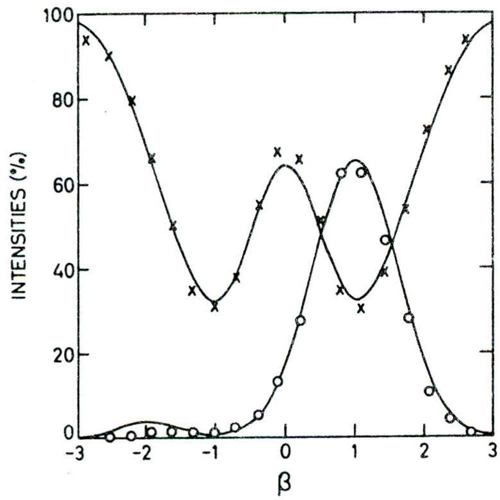



Fig. 4 $I_0$ and $I_{-1}$ versus β for Q=6.28, v=2.
NOA method:—. Experimental[9]: $I_0$:×, $I_{-1}$:o.

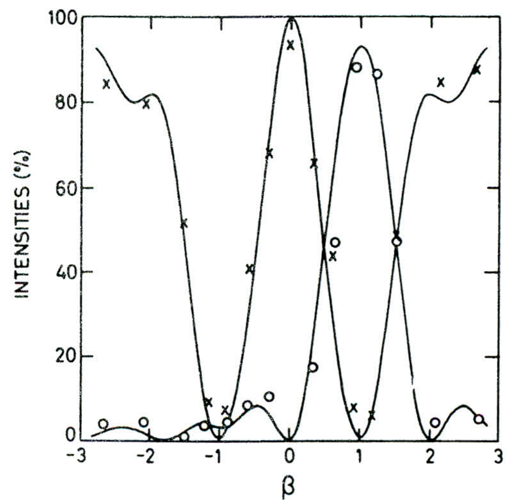



Fig. 5 $I_0$ and $I_{-1}$ versus β for Q=9.3, v=3.
NOA method:—. Experimental[9]: $I_0$: ×, $I_{-1}$:o.

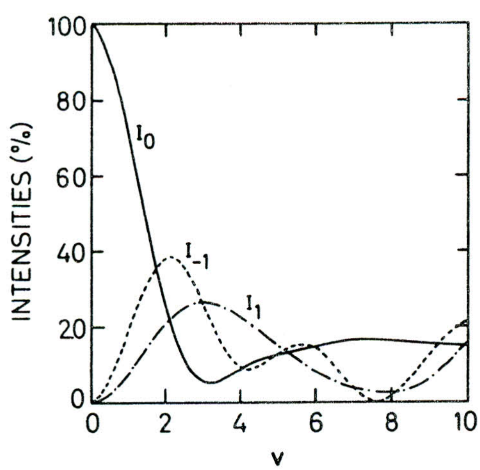



Fig. 6 $I_0$, $I_1$, $I_{-1}$ versus v for Q=2.25, β =2
from NOA method.

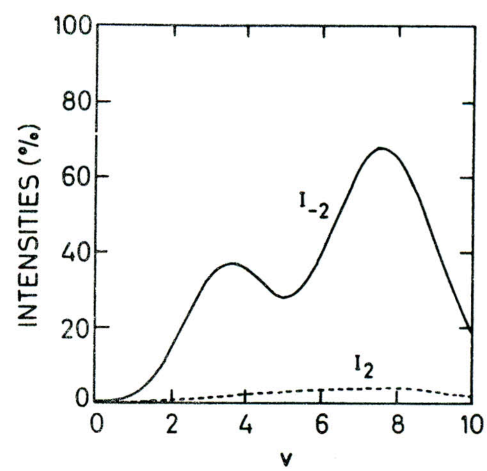



Fig. 7 $I_2$, $I_{-2}$ versus v for Q=2.25, β =2
from NOA method.