

ON THE DIFFRACTION OF LIGHT BY ADJACENT PARALLEL ULTRASONIC WAVES.
A GENERAL THEORY

R. Mertens and W. Hereman

Inst. Theor. Mechanica, Rijksuniversiteit Gent, B-9000 Gent, Belgium

A beam of light normal to the direction of two adjacent parallel ultrasonic waves is diffracted by the medium containing the first ultrasonic wave. According to physical conditions, known exact or approximate theories can be used. Each of the diffracted light waves then interacts with the second ultrasonic wave. For this second diffraction also exact or approximate theories can be applied. The present method not only leads to a simplification of the calculations, but, since only one system of reference is used, it also yields exact boundary conditions. In the case where the second ultrasonic is an harmonic of the first one, several waves diffracted by the second medium may combine. Some special cases are discussed.

INTRODUCTION

Since the experiments on diffraction of light by two parallel adjacent ultrasonic waves by Pande et al¹ this phenomenon has been studied extensively, experimentally as well as theoretically. For a survey of the literature since 1960 we refer to the paper of Leroy et al². Theoretically two methods were developed. A first one considers the diffraction of the incident light beam (normal to the direction of sound propagation) by the first ultrasonic wave, followed by the diffraction of each diffracted light wave by the second ultrasonic field. This was worked out by Mertens³ in the case of the elementary Raman-Nath theory and by Leroy and Mertens⁴ using the N-th order approximation (NOA) method for $N = 1$. A disadvantage of the previous method was that for each wave diffracted by the first medium a different system of reference had to be introduced for the description of the diffraction in the second medium. A second method introduced by Leroy⁵ is based on a global description of the two ultrasonic fields, leading, however, to the integration of rather involved systems of differential equations and moreover masking the physics of the problem, e.g. the oblique incidence in the second medium and eventual Bragg reflections in the case where the second ultrasonic is an harmonic of the first one.

In the present paper we shall generalize the first method, using a modified version of the generalized Raman-Nath theory⁶ in order to study the interaction of the light with the second ultrasonic field. Hereby only one coordinate system is used instead of two in the original Raman-Nath theory⁷. This general method has been worked out in the following cases :

- two adjacent ultrasonic waves having incommensurable frequencies;
- two adjacent ultrasonic waves, the second one being the second harmonic of the

first one.

Two approximate methods have been considered :

- the diffraction of light by two adjacent ultrasonic waves, one being the N-th harmonic of the other, in the case where the elementary theory of Raman-Nath is valid;
- the diffraction of light by two adjacent ultrasonic waves, one being the second harmonic of the other, in the case where the NOA-method for only one diffraction order (Bragg case) may be used.

GENERAL THEORY OF THE DIFFRACTION OF LIGHT BY ADJACENT PARALLEL ULTRASONIC WAVES WITH INCOMMENSURABLE FREQUENCIES

We consider an incident beam of light, with frequency ν and wavelength λ in the medium, normal to the direction of propagation of two adjacent parallel ultrasonic waves, with respective frequencies ν_1^* and ν_2^* , wavelengths λ_1^* and λ_2^* and widths L_1 and L_2 (Fig. 1). We put the x-axis along the direction of propagation of the sound waves, the z-axis along the direction of propagation of the incident light. In $z = L_1$ the electric field of the light will be split up in several plane subwaves⁶ making angles $\varphi_n = -n\lambda/\lambda_1^*$ ($n \in \mathbb{Z}$) with the z-axis, having frequency shifts $\Delta\nu_n = -n\nu_1^*$ ($n \in \mathbb{Z}$) and with amplitudes $\phi_n(\nu_1)$, where

$$\nu_1 = \pi \epsilon_1 L_1 / \epsilon_r \lambda = \zeta_1 L_1 / z . \quad (1)$$

ϵ_r being the relative permittivity of the undisturbed medium and ϵ_1 its maximum variation. The amplitudes $\phi_n(\zeta_1)$ are obtained as follows⁶ : find the solution $\Psi^{(1)}(\xi_1, \zeta_1)$ of the partial differential equation

$$2 \frac{\partial \Psi^{(1)}}{\partial \zeta_1} - 2i \cos 2\xi_1 \Psi^{(1)} = -\frac{1}{4} i \rho_1 \frac{\partial^2 \Psi^{(1)}}{\partial \xi_1^2} \quad (0 \leq \zeta_1 \leq \nu_1) , \quad (2a)$$

$$\text{with the boundary condition} \quad \Psi^{(1)}(\xi_1, 0) = 1 , \quad (2b)$$

$$\text{and the periodicity condition} \quad \Psi^{(1)}(\xi_1 + \pi, \zeta_1) = \Psi^{(1)}(\xi_1, \zeta_1) , \quad (2c)$$

$$\text{where} \quad \rho_1 = 2\epsilon_r \lambda^2 / \epsilon_1 \lambda_1^{*2} , \quad (3)$$

$$\text{and} \quad \xi_1 = \frac{1}{2}(k_1^* x - \omega_1^* t + \frac{3\pi}{2}) , \quad \zeta_1 = \pi \epsilon_1 z / \epsilon_r \lambda . \quad (4)$$

Developing $\Psi^{(1)}(\xi_1, \zeta_1)$ in a Fourier series

$$\Psi^{(1)}(\xi_1, \zeta_1) = \sum_{n=-\infty}^{\infty} \phi_n(\zeta_1) i^n \exp(2in\xi_1) \quad (5)$$

gives the amplitudes. This method is called the "modified generating function method". The equation (2a) with the conditions (2b,c) may be solved exactly (arbitrary values

of ρ_1) or approximately for $\rho_1 = 0$ (elementary Raman-Nath theory), $\rho_1 \ll 1$, and $\rho_1 \gg 1$ (Bragg case).

Let us now consider the interaction of the n -th partial light wave with the second ultrasonic wave in the same frame of reference Oxz . In earlier theories⁴ an infinite set of reference systems, the new z -axis making angles φ_n with the original z -axis, had to be introduced. In $z = L_1 + L_2$ the electric field of the partial wave will consist of new subwaves, making angles $\theta_{n'} = \varphi_n - n'\lambda/\lambda_2^* = -\lambda(n/\lambda_1^* + n'/\lambda_2^*)$ with the z -axis, having frequency shifts $\Delta\nu_{nn'} = -(n\nu_1^* + n'\nu_2^*)$ and with amplitudes

$$\phi_{nn'} = \phi_n(v_1)\phi_{n'}(v_2;\varphi_n)\exp(-in'\delta), \quad (6)$$

$$\text{where } v_2 = \pi\varepsilon_2 L_2 / \varepsilon_r \lambda \cos\varphi_n = \zeta_2 L_2 / z \approx \pi\varepsilon_2 L_2 / \varepsilon_r \lambda, \quad (7)$$

δ being the phase difference of the two ultrasonic waves. The function $\phi_{n'}(\zeta_2;\varphi_n)$, where $v_1\varepsilon_2/\varepsilon_1 \leq \zeta_2 \leq v_2$, may be obtained in the following way⁶: solve the PDE for $\Psi^{(2)}(\xi_2, \zeta_2; \varphi_n)$,

$$2\frac{\partial\Psi^{(2)}}{\partial\zeta_2} - 2i\cos 2\xi_2 \Psi^{(2)} = -\frac{1}{4}i\rho_2 \frac{\partial^2\Psi^{(2)}}{\partial\xi_2^2} - a_2 \sin\varphi_n \frac{\partial\Psi^{(2)}}{\partial\xi_2}, \quad (8a)$$

$$\text{with boundary condition } \Psi^{(2)}(\xi_2, v_1\varepsilon_2/\varepsilon_1; \varphi_n) = 1, \quad (8b)$$

$$\text{and periodicity condition } \Psi^{(2)}(\xi_2 + \pi, \zeta_2; \varphi_n) = \Psi^{(2)}(\xi_2, \zeta_2; \varphi_n), \quad (8c)$$

$$\text{where } \rho_2 = 2\varepsilon_r \lambda^2 / \varepsilon_2 \lambda_2^{*2}, \quad a_2 = 2\varepsilon_r \lambda / \varepsilon_2 \lambda_2^*, \quad (9a,b)$$

$$\xi_2 = \frac{1}{2}(k_2^* x - \omega_2^* t + \frac{3\pi}{2} - \delta), \quad \zeta_2 = \pi\varepsilon_2 z / \varepsilon_r \lambda \cos\varphi_n \approx \pi\varepsilon_2 z / \varepsilon_r \lambda. \quad (10)$$

ε_2 being the maximum variation of relative permittivity in the second ultrasonic field. We remark here that for convenience the boundary condition (8b) has been normalized. Developing $\Psi^{(2)}(\xi_2, \zeta_2; \varphi_n)$ in a Fourier series,

$$\Psi^{(2)}(\xi_2, \zeta_2; \varphi_n) = \sum_{n'=-\infty}^{\infty} \phi_{n'}(\zeta_2; \varphi_n) i^{n'} \exp(2in'\xi_2), \quad (11)$$

leads to the unknown factor $\phi_{n'}(v_2; \varphi_n)$ in (6). Returning to the original variables x and t in (11) leads via (10) to the exponential factor $\exp(-in'\delta)$. Eq. (8a) with the conditions (8b,c) may also be solved, either exactly or approximately, according to the physical parameters of the second US field.

Finally, the intensities of order (n, n') are given by

$$I_{nn'} = \phi_n(v_1)\overline{\phi_n(v_1)}\phi_{n'}(v_2;\varphi_n)\overline{\phi_{n'}(v_2;\varphi_n)}, \text{ where the bar signifies complex conjugation.}$$

Let the second USW be the N -th harmonic of the first one, i.e. $v_2^* = Nv_1^*$ and $\lambda_2^* = \lambda_1^*/N$. The angle of incidence of the n -th subwave in $z=L_1$ is then given by

$$\varphi_n = -n\lambda/\lambda_1^* = -(2n/N)(\lambda/2\lambda_2^*) . \quad (12)$$

If $p = -2n/N$, with $p \in \mathbb{Z}_0$, i.e. $2n$ is a multiple of N , then φ_n is the Bragg angle of order p denoted by $\varphi_{BR}^{(p)}$. So, in general, not all subwaves emerging from the first sound field give rise to Bragg incidence, unless $N=2$. In the latter case, where the second USW is the second harmonic of the first one, all angles of incidence for the second US field are Bragg angles, leading to more simple expressions for the amplitudes of the diffracted light waves. So, e.g. in the exact solution, the amplitudes are expressed with the Fourier coefficients of the periodic Mathieu functions of integral order⁸.

Investigating those subwaves coming out from the plane $z = L_1 + L_2$ in a particular direction defined by the angle $\theta_{2n} = -2n(\lambda/\lambda_1^*)$ and with the same frequency (cf. Fig. 2), we obtain respectively for the amplitudes of even and odd order

$$\phi_{2n} = \sum_{r=-\infty}^{\infty} \phi_{2r}(v_1) \phi_{n-r}(v_2; \varphi_{2r}) \exp[-i(n-r)\delta] , \quad (13a)$$

$$\phi_{2n+1} = \sum_{r=-\infty}^{\infty} \phi_{2r+1}(v_1) \phi_{n-r}(v_2; \varphi_{2r}) \exp[-i(n-r)\delta] . \quad (13b)$$

SPECIAL CASES

(a) Fundamental and N -th harmonic. $\rho_1 \approx 0$, $\rho_2 \approx 0$

In both US fields the physical conditions for the application of Raman-Nath's elementary theory are fulfilled⁹ leading to the amplitude of order s ,

$$\begin{aligned} \phi_s = \sum_{n=-\infty}^{\infty} J_{s-nN}(v_1) J_n \left[\frac{2 \sin(\frac{1}{2} a_2 v_2 \sin \varphi_{s-nN})}{a_2 \sin \varphi_{s-nN}} \right] \\ \times \exp[-\frac{1}{2} i a_2 v_2 \sin \varphi_{s-nN}] \exp(-in\delta) . \end{aligned} \quad (14)$$

This expression corresponds with the formulae obtained by Mertens³. If the condition $a_2 v_2 |\sin \varphi_{s-nN}| \ll 1$ is fulfilled (14) reduces to Leroy's Eq. (10) of Ref. 5.

(b) Fundamental and second harmonic. Bragg case

If $\rho_1 \gg 1$ and ϵ_1 and ϵ_2 are of the same order of greatness, we have also $\rho_2 \gg 1$. In this case and for normal incidence, in the first US field, we use the formulae ob-

tained by Nagendra Nath¹⁰ :

$$\phi_0(v_1) = \frac{1}{2} \left(1 - \frac{\rho_1}{\sqrt{\rho_1^2 + 8}} \right) \exp\left(i \frac{\rho_1 + \sqrt{\rho_1^2 + 8}}{4} v_1\right) + \frac{1}{2} \left(1 + \frac{\rho_1}{\sqrt{\rho_1^2 + 8}} \right) \exp\left(i \frac{\rho_1 - \sqrt{\rho_1^2 + 8}}{4} v_1\right) \quad (15a)$$

$$\phi_{\pm 1}(v_1) = \mp \frac{i}{\sqrt{\rho_1^2 + 8}} \left[\exp\left(i \frac{\rho_1 + \sqrt{\rho_1^2 + 8}}{4} v_1\right) - \exp\left(i \frac{\rho_1 - \sqrt{\rho_1^2 + 8}}{4} v_1\right) \right] \quad (15b)$$

Since the zero order line coming out from the first US field has normal incidence with respect to the second one (15a) and (15b) apply, changing v_1 into v_2 , and ρ_1 into ρ_2 .

For Bragg angle incidence in the second US field we shall apply Phariseau's formulae¹¹ :

$$\text{- for a Bragg angle of order } +1 : \phi_0(v_2; \varphi_{BR}^{(1)}) = \cos \frac{v_2}{2}, \phi_1(v_2; \varphi_{BR}^{(1)}) = \sin \frac{v_2}{2}, \quad (16a, b)$$

$$\text{- for a Bragg angle of order } -1 : \phi_0(v_2; \varphi_{BR}^{(-1)}) = \cos \frac{v_2}{2}, \phi_{-1}(v_2; \varphi_{BR}^{(-1)}) = -\sin \frac{v_2}{2}. \quad (17a, b)$$

Considering the combinations of subwaves as given in Fig. 2, the following intensities are obtained up to terms in $1/\rho_1^2$ and $1/\rho_2^2$:

$$I_{-1, -1} \approx I_{1, 1} \approx 0 \quad (18a, b)$$

$$I_{0, -1} = |\phi_0(v_1) \phi_{-1}(v_2; 0)|^2 = |\phi_0(v_1) \phi_1(v_2; 0)|^2 = I_{0, 1} \approx \frac{4}{\rho_2^2 + 8} \sin^2 \frac{\sqrt{\rho_2^2 + 8}}{4} v_2 \quad (18c, d)$$

$$\begin{aligned} I_{\mp 1} &= |\phi_{\mp 1}(v_1) \phi_0(v_2; \varphi_{BR}^{(\pm 1)}) + \phi_{\pm 1}(v_1) \phi_{\mp 1}(v_2; \varphi_{BR}^{(\mp 1)}) e^{\pm i \delta}|^2 \\ &= I_{\mp 1}(v_1) (1 \pm \sin v_2 \cos \delta) \end{aligned} \quad (18e, f)$$

$$\text{with } I_{+1}(v_1) = I_{-1}(v_1) = \frac{4}{\rho_1^2 + 8} \sin^2 \frac{\sqrt{\rho_1^2 + 8}}{4} v_1 \quad (18g)$$

$$I_0 = |\phi_0(v_1) \phi_0(v_2; 0)|^2 \approx 1 - \frac{8}{\rho_1^2 + 8} \sin^2 \frac{\sqrt{\rho_1^2 + 8}}{4} v_1 - \frac{8}{\rho_2^2 + 8} \sin^2 \frac{\sqrt{\rho_2^2 + 8}}{4} v_2. \quad (18h)$$

Formula (18e, f) has been obtained by the global method of Leroy⁵ and has been verified experimentally by Sliwinski and collaborators^{2, 12}, and employed for measuring the ultrasonic pressure of the second harmonic.

ACKNOWLEDGEMENTS

One of the authors (R.M.) wishes to thank the National Fund for Scientific Research (Belgium) for research grants.

REFERENCES

- 1 Pande, A., Pancholy, M. and Parthasarathy, S. 'Simultaneous oscillations of a piezo-electric quartz at two frequencies'. J. Sci. Industr. Res., vol. 3 (1944), pp. 2-8.
- 2 Leroy, O., Sliwinski, A., Kwiek, P. and Markiewicz, A. 'Diffraction of light by adjacent fundamental and second or fourth harmonic ultrasound beams'. Ultrasonics, vol. 20 (1982), pp. 135-141.
- 3 Mertens, R. 'On the theory of the diffraction of light by two parallel ultrasonic waves, one being the n-th harmonic of the other'. Z. Physik, vol. 160 (1960), pp. 291-296.
- 4 Leroy, O. and Mertens, R. 'Diffraction of light by adjacent parallel ultrasonic waves with arbitrary frequencies (NOA-method)'. Acustica, vol. 26 (1972), pp. 96-102.
- 5 Leroy, O. 'Diffraction of light by two adjacent parallel ultrasonic beams'. Acustica, vol. 29 (1973), pp. 303-310.
- 6 Mertens, R. and Hereman, W. 'Diffraction of light by ultrasonic waves in the case of oblique incidence of the light. General theory and approximations'. Proc. 2nd Spring School on Acousto-optics, Gdansk (Wiezyca) (1983), in the press.
- 7 Raman, C.V. and Nagendra Nath, N.S. 'The diffraction of light by high frequency sound waves : Part V'. Proc. Indian Acad. Sci., vol. 3A (1936), pp. 459-465.
- 8 Plancke-Schuyten, G. and Mertens, R. 'The diffraction of light by progressive supersonic waves. Oblique incidence of the light II'. Physica, vol. 62 (1972), pp. 600-613.
- 9 Plancke-Schuyten, G., Mertens, R. and Leroy, O. 'The diffraction of light by progressive supersonic waves. Oblique incidence of the light I'. Physica, vol. 61 (1971), pp. 299-306.
- 10 Nagendra Nath, N.S. 'The diffraction of light by supersonic waves'. Proc. Indian Acad. Sci., vol. 8A (1938), pp. 499-503.
- 11 Phariseau, P. 'On the diffraction of light by progressive supersonic waves. Oblique incidence : Intensities in the neighbourhood of the Bragg angle'. Proc. Indian Acad. Sci., vol. 44 (1956), pp. 165-170.
- 12 Kwiek, P., Sliwinski, A. and Leroy, O. 'Measurements of the ultrasonic pressure of the second harmonic using ultrasonic-light-diffraction by adjacent beams'. Proc. Ultrasonics International 79, Graz, pp. 494-499.

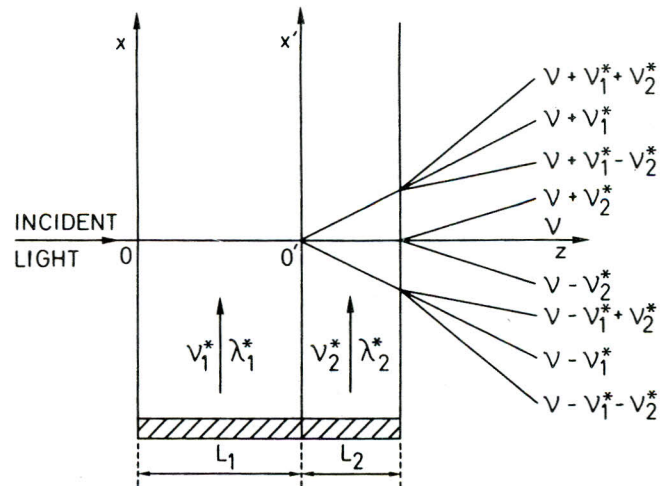


Fig. 1 Geometry in the case of two parallel adjacent US beams, with incommensurable frequencies

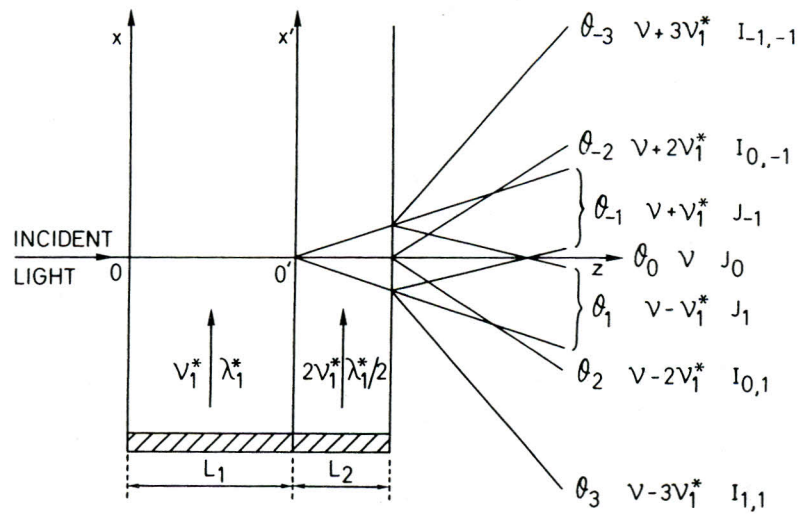


Fig. 2 Geometry of the special case $v_2^* = 2v_1^*$