DETERMINATION OF A POSITION USING APPROXIMATE DISTANCES AND TRILATERATION

by

William S. Murphy Jr.

July 4, 2007

A thesis submitted to the Faculty and the Board of Trustees of the Colorado School of Mines in partial fulfillment of the requirements for the degree of Master of Science (Mathematics).

Golden, Colorado

Date ____________________

Signed: ____________________
William S. Murphy Jr.

Approved: ____________________
Dr. Willy A. Hereman
Thesis Advisor

Golden, Colorado

Date ____________________

____
Dr. Ardel Boes
Professor and Head,
Department of Mathematical and
Computer Sciences

ABSTRACT

This thesis discusses and illustrates the mathematical solution of a positioning problem that was developed for Thunder Basin Coal Company (TBCC) in Wright, Wyoming. TBCC is designing and implementing a system to electronically determine the three dimensional position of equipment in an open pit mine on a real time basis. This system will use radio beacons to measure the approximate distances between the equipment in the mine, and known fixed positions on the rim of the mine. The mathematical solution methods presented in this thesis use these approximate distances, and the coordinates of the known fixed beacon positions, to calculate the position of the equipment. The surveying term used to describe this class of problems is trilateration.

TBCC is developing this system because alternative methods of determining positions in a timely, accurate, and cost effective manner are not currently available. Traditional manual surveying techniques are too labor intensive and too slow. The Global Positioning System [Remondi 1991] currently does not provide elevations that are accurate enough for TBCC's applications.

The first solution proposed in this thesis is to treat the unknown position as the point of intersection of the surfaces of several spheres, whose centers are known fixed positions. This approach is not feasible for TBCC's problem because it leads to a nonlinear equation of a high degree. Alternatively, by linearizing the equations and converting the problem into one of finding the intersection of several planes, more useful solution techniques can be established and are developed in this thesis.

The nonlinear least squares technique provides the most accurate results of all methods proposed in this thesis. It calculates the exact position when exact distances are used; and a reasonably accurate position when used with approximate distances. The accuracy of the calculated position is degraded when the elevation of the unknown position is above the elevation of the lowest beacon position, and when the unknown position is located outside the perimeter of the beacon positions.

TBCC is procuring the distance measuring radio beacons that will be installed on the rim of the mine. They are also procuring touch screen computer terminals to mount in various pieces of equipment. These terminals will provide two way communication with the computer that calculates their position through the use of radio modems. TBCC will use this electronic surveying system for a variety of mining applications, including positioning of bulldozers, drills, etc.

CONTENTS

LIST OF FIGURES

LIST OF TABLES

Acknowledgements

I would like to express my appreciation to Dr. Willy Hereman for the time he invested in my education. Dr. Hereman's knowledge, guidance, and academic advice was invaluable in the preparation of this thesis.

I would also like to express my appreciation to Dr. Robert E. D. Woolsey for providing me with a solid applications oriented foundation in operations research. Dr. Woolsey's operations research program prepared me for my next position as an instructor in the Systems Engineering Department at the United States Military Academy.

Thanks are due Dr. Steven Pruess. His mathematical expertise and valuable comments contributed materially to the preparation of this thesis.

Recognition is due Mr. Fred Darland, and Mr. Jerry Durgin of Thunder Basin Coal Company. They proposed the positioning problem, and are spearheading the implementation of the system.

I would like to thank the members of the "Guild" for their support and assistance during the past two years.

I dedicate this thesis to my wife Brandee. She was a constant source of support, encouragement, and understanding.

BIBLIOGRAPHY

[Symbolics 1988] Computer Aided Mathematics Group. 1988. *MACSYMA Reference Manual Version 13.* Cambridge: Symbolics, Inc.

[DeLoach 1963] DeLoach, B.A. 1963. *Error Analysis of a Trilateration Network With and Without the Laplace Condition.* Ohio State University: Masters Thesis.

[Grossman 1969] Grossman, W. 1969. *Grundzüge der Ausgleichungsrechnung nach der Methode der Kleinsten Quadrate nebst Anwendung in der Geodesy.* Berlin: Springer Verlag.

[Kahaner, Moler, and Nash 1989] Kahaner, D., C. Moler, and S. Nash. 1989. *Numerical Methods and Software.* Englewood Cliffs, NJ: Prentice Hall.

[Lawson, and Hanson 1974] Lawson, C.L., and R.I. Hanson 1974. *Solving Least Squares Problems.* Englewood Cliffs, NJ: Prentice Hall.

[Manning 1979] Manning, P.E. 1979. *Tests of an ATCRBS Based Trilateration Sensor at Logan International Airport.* Campridge, MA: U.S. Department of Transportation Federal Aviation Administration. FAA-RD-79-115 (November).

[McKeown 1975] McKeown, J.J. 1975. Specialized Versus General-Purpose Algorithms for Minimizing Functions that are Sums of Squared Terms. *Mathematical Programming* **9**: 57-68.

[Merchant 1975] Merchant, D.C. 1981. *Passive Trilateration Positioning Procedure Error Analysis.* Griffis Air Force Base, NY: Rome Air Development Center. RADC-TR-81-225 (August).

[Mikhail 1976] Mikhail E.M. 1976. *Observations and Least Squares.* New York: IEP-Dun-Donnelley, Harper and Row.

[Moler, Little, Bangert, and Kleiman 1989] Moler, C., J. Little, S. Bangert, and S. Kleiman. 1989. *PC-MATLAB for MS-DOS Personal Computers.* Sherborn, MA: The MathWorks, Inc.

[Noble and Daniel 1988] Noble, B. and J.W. Daniel. 1988. *Applied Linear Algebra.* Englewood Cliffs: Prentice-Hall.

[Remondi 1991] Remondi, B.W. 1991. Artificial Constellations, The Global Positioning System. *The Military Engineer* 84 (545): 31-36.

1. INTRODUCTION

1.1 Purpose

Thunder Basin Coal Company (TBCC) is developing a system to electronically determine the position of equipment in an open pit mine on a real time basis. Their proposed system will use radio beacons to measure the approximate distances between the equipment in the mine, and known fixed positions on the rim of the mine Figure 1.1. The electronics firm, contracted by TBCC to develop and produce the radio beacons, argued that a mathematical solution to this three dimensional positioning problem does not exist. The electronics firm proposed that TBCC contract with them to develop and produce an oscillating, rotating laser which could be used in conjunction with the distance measuring radio beacons to determine the elevation of the equipment in the mine. TBCC contacted the Department of Mathematical and Computer Sciences at the Colorado School of Mines in December 1990, to determine if a mathematical solution to this three dimensional positioning problem exists; and if so, whether or not a programmable fast algorithm could be designed. This thesis is a presentation of the mathematical solution of this positioning problem.

1.2 Methodology

Conventional surveying methods, commonly referred to as *triangulation*, use measured angles to calculate the coordinates of an unknown position. Since the angles in this positioning problem are not known, an alternate solution technique known as *trilateration* must be used. Trilateration is the term used to describe the class of positioning problems which involve the use of measured distances from known positions to an unknown point. Since the distances used in this application are not exact, the solution techniques presented in this thesis use an iterative trilateration procedure to calculate the best approximation to the exact solution.

Trilateration gained more importance in surveying due to the development of electronic equipment that can measure distances accurately [DeLoach 1963]. These electronic measurements can be made with radar, lasers, radio signals, etc.

Numerous issues had to be resolved in order to determine if a practical mathematical solution to this trilateration problem exists. For instance, proposed solution techniques are unacceptable if they fail to calculate the exact position when used with exact distances. TBCC required that a position be accurate within a tolerance of five feet when used with approximate distances that are accurate within one-half foot. The evaluation of these solution techniques must be completed with actual data provided by TBCC. The effects of errors in the distance measurements on the accuracy of the calculated position had to be investigated to determine if the beacon manufacturer's proposed accuracy of one-half foot was acceptable. The theoretical minimum number of beacons that are needed to calculate a unique solution, and the number of these

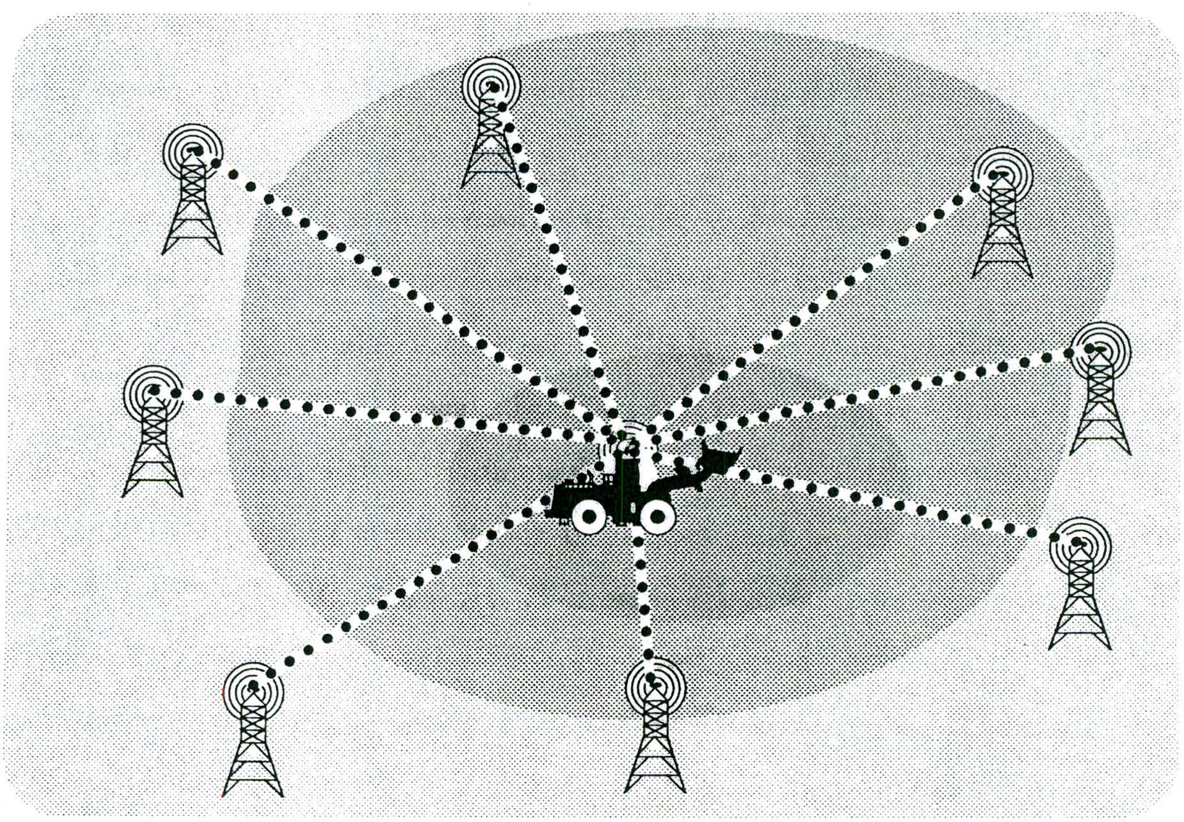

Fig. 1.1: Illustration of the Positioning Problem

beacons that are needed in practice had to be determined. A calibration program had to be developed to study the effects of various beacon placement patterns on the accuracy of the calculated position.

The algorithms presented in this thesis were first written in the MACSYMA [Symbolics 1988] symbolic manipulation computer program, and then rewritten in the C programming language. Comparison of the results from the two independent programs helped determine programming and roundoff errors. The C programming language was selected because it is fast and accurate, widely available, and was requested by TBCC.

1.2.1 Mathematical Notation and Data Collection

The mathematical notation and symbols used in this thesis are given in Figure 1.2. Descriptions of these symbols, and the corresponding data collection techniques are as follows.

Surveyors use conventional surveying techniques to determine the easting, northing, and elevation of each beacon. These coordinates, which are labeled x_i, y_i, and z_i for the ith beacon, are manually typed into a computer data file. TBCC arbitrarily selected a point in western Wyoming as the origin of their coordinate system. The following coordinates, which were supplied by TBCC, were used to test the various solution methods presented in this thesis.

$$\begin{aligned}
B_1(x_1, y_1, z_1) &= B_1(475060, 1096300, 4670)\\
B_2(x_2, y_2, z_2) &= B_2(481500, 1094900, 4694)\\
B_3(x_3, y_3, z_3) &= B_3(482230, 1088430, 4831)\\
B_4(x_4, y_4, z_4) &= B_4(478050, 1087810, 4775)\\
B_5(x_5, y_5, z_5) &= B_5(471430, 1088580, 4752)\\
B_6(x_6, y_6, z_6) &= B_6(468720, 1091240, 4803)\\
B_7(x_7, y_7, z_7) &= B_7(467400, 1093980, 4705)\\
B_8(x_8, y_8, z_8) &= B_8(468730, 1097340, 4747)
\end{aligned} \tag{1.1}$$

The coordinates of the mobile receiver/transmitter which is mounted on the equipment in the mine are denoted x, y, and z. Determining these unknown coordinates is the topic of this thesis.

Each beacon uses signal timing data to measure the approximate distance between itself and the mobile receiver/transmitter that is mounted on the equipment in the mine. The approximate distance between the ith beacon and the equipment in the mine is denoted r_i. The beacons electronically input these distances into a computer data base. When the exact distances are used in the development of the solution techniques, they are labeled $\hat{r}_i$ instead of r_i.

The distance data used to test various solution techniques presented in this thesis was obtained in the following manner. First, the coordinates of the equipment in the mine are arbitrarily selected. The exact radii, $\hat{r}_i$, are then calculated by using the coordinates of the beacons (1.1), the coordinates of the equipment in the mine, and the following formula

$$\hat{r}_i = \sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2} \qquad (i = 1, 2, ..., n), \tag{1.2}$$

where i denotes the beacon number, and n is the total number of beacons. The r_i are obtained by adding errors to each of these $\hat{r}_i$. The errors which are used in this thesis for testing purposes are shown in Table 1.1.

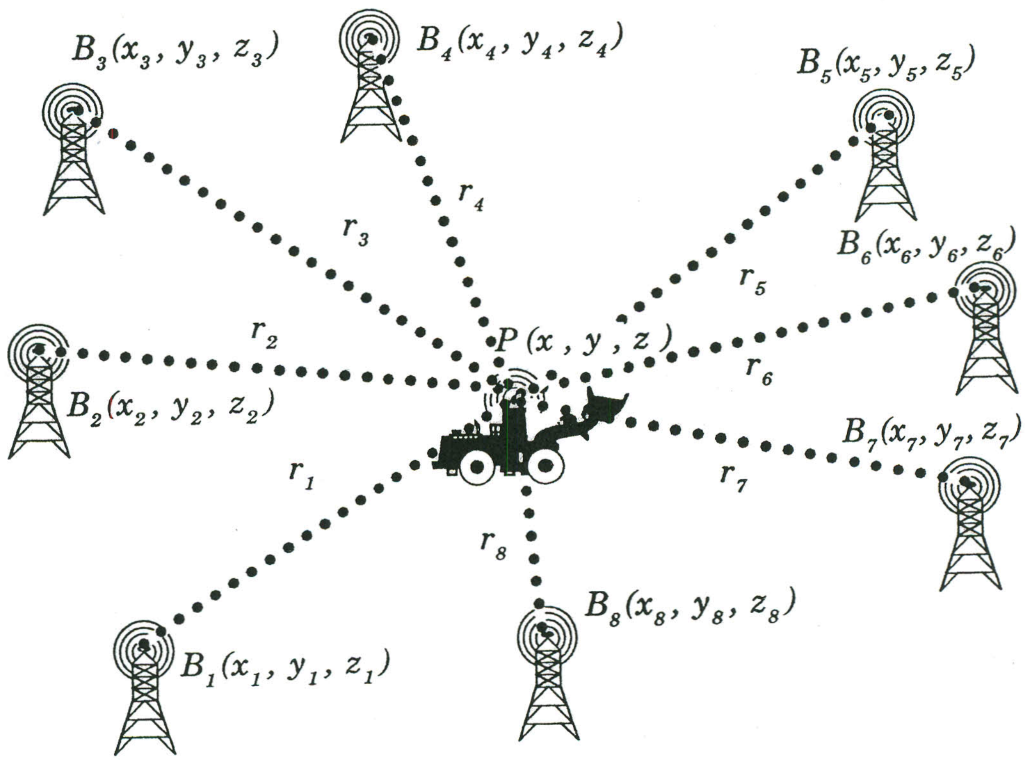

Fig. 1.2: Illustration of Mathematical Symbols

Tab. 1.1: Errors Induced on the Distances

Distance	Error
r_1	-0.457890
r_2	0.173050
r_3	0.316931
r_4	-0.191205
r_5	0.468339
r_6	0.141141
r_7	0.328659
r_8	-0.390460

These distances are then used to solve the positioning problem for the coordinates (x, y, z) of the equipment in the mine. The accuracy of the various solution methods presented in this thesis is evaluated by comparing the coordinates which were used to calculate the $\hat{r}_i$, with the coordinates calculated with the proposed methods.
The reader should realize that r_i does **not** refer to the distance between the beacon B_i and the arbitrarily positioned origin O of the reference system (x, y, z). Referring to Sections 2.2.1 and 2.2.2, $r_i \neq \| \overrightarrow{R_i} \| = \| \overrightarrow{OB_i} \|$; however $r_i = \| \overrightarrow{PB_i} \|$.

1.2.2 Solution Techniques

The obvious approach in solving this positioning problem is to treat the coordinates of the equipment in the mine (x, y, z) as the point of intersection of several spheres, whose centers are the locations of the n beacons (x_i, y_i, z_i) for $i = 1, 2, ..., n$. The exact distances between the beacons, and the equipment in the mine, r_i, are the radii of the individual spheres. The equation for any of these spheres is

$$(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2 = {r_i}^2. \tag{1.3}$$

The point of intersection of the surfaces of n of these spheres is obtained by letting $i = 1, 2, ..., n$, and solving the resulting n nonlinear equations simultaneously to eliminate two coordinates. This solution technique is not feasible because it produces a nonlinear equation of high degree. Furthermore, since the equations are quadratic, many cases for the signs would have to be considered.
Linearizing the system of equations will reduce the degree, and convert the problem into one of finding the point of intersection of several planes. When the exact distances from four beacons are available, the solution of the system of equations is completely determined. There are three equations, three unknowns, and exactly one solution. Consequently, the theoretical minimum number of beacons is four. When approximate distances are used, the position which is calculated by the direct solution of the linear equations is not acceptable. The sophistication, needed when working with approximate distances, is dealt with in the linear least squares, weighted linear least squares, and nonlinear least squares solution techniques.

The most effective approach presented in this thesis is the nonlinear least squares method. Table 1.2 summarizes the errors produced by the various calculation methods tested in this thesis when exact distances are used. Table 1.3 is a similar summary of these errors when approximate distances are used. The information in these two tables is discussed in detail throughout this thesis. The nonlinear least squares method calculates the locations within the required 5.0 foot tolerance for both exact and approximate distances. The accuracy of the calculated position is degraded when the elevation of the equipment in the mine is located above the elevation of the lowest beacon; and when the equipment is located outside the perimeter of the beacons.

Tab. 1.2: Summary of Errors Calculated by Various Solution Techniques using Exact Distances

Test Coordinates		Results from Various Solution Methods				
		Linearized Equations	Linear Least Squares	Arithmetic Average	Weighted Average	Nonlinear Least Squares
x	480000	0.000	0.000			0.000
y	1093000	0.000	0.000			0.000
z	4668	0.562	-0.313	-0.261	-0.112	0.014
x	480000	0.000	0.000			0.000
y	1093000	0.000	0.000			0.000
z	4525	0.562	-1.000	-0.019	-0.005	-0.010
x	480000	0.000	0.000			0.000
y	1095500	0.000	0.000			0.000
z	4525	0.219	-0.039	0.017	-0.003	0.000

1.2.3 Applications

The usefulness of the mathematical solution of the trilateration positioning problem presented in this thesis is not restricted to mining applications. This solution could be used to improve the accuracy of existing trilateration applications, or it could be implemented in new electronic positioning systems. Possible applications of this solution include precision farming; underwater positioning; navigational aids for ships, aircraft, or automobiles; determining the position of aircraft, rockets, missiles, and satellites; and any of the existing trilateration systems listed below.

Various positioning systems which apply trilateration principles are currently in use worldwide. A brief search in the literature led to the following examples. There are numerous military applications of trilateration. One such application is the Global Positioning System, which was developed by the Department of Defense. This system uses a constellation of satellites which transmit radio signal timing data to mobile receivers on earth. These receivers calculate unknown locations by using the satellites known orbital ephemeris, the radio signal timing data, and a combination of the Doppler principle and trilateration. The satellites produce two signals, the Standard Positioning Service which is available for civilian use, and the more accurate Precise Positioning Service which is coded and restricted to military use. Accurate elevations

Tab. 1.3: Summary of Errors Calculated by Various Solution Techniques using Approximate Distances

Test Coordinates		Results from Various Solution Methods: Linearized Equations	Linear Least Squares	Arithmetic Average	Weighted Average	Nonlinear Least Squares
x	480000	-0.469	0.219			-0.062
y	1093000	-0.125	0.875			0.125
z	4668	-10.750	13.437	-8.279	-19.725	-4.101
x	480000	-0.469	0.250			-0.063
y	1093000	-0.125	0.875			0.125
z	4525	-11.000	13.375	5.259	2.234	-1.514
x	480000	-0.375	0.219			-0.062
y	1095500	-0.500	0.875			0.375
z	4525	-35.813	14.039	15.275	10.312	1.271

are not available from this system on a real time basis [Remondi 1991]. Logan International Airport uses a radar based trilateration system for locating and identifying aircraft and other transponder equipped vehicles on the surface of the airport [Manning 1979]. The Geodetic Survey of Canada uses a trilateration system that involves the use of an aircraft flying between ground stations. In this application the distances between the aircraft and the ground stations are computed to calculate a position on the surface of the earth [DeLoach 1963]. The Rome Air Development Center at Griffiss Air Force Base uses a ground based system to determine the position of aircraft in flight [Merchant 1975].

2. LINEARIZATION

The procedures which were used to linearize and solve the equations (1.3) for the intersection of several spheres are based on geometry, linear algebra, and analysis. This linearization process reduces the degree, and converts the problem into one of finding the point of intersection of several planes. The solution of the system of linearized equations is completely determined when the exact distances from **four** beacons are known.

2.1 Development of the Linear System

The following mathematical notation was introduced in Section 1.2.2.
The constraints are the equations of the spheres with radii r_i,

$$(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2 = {r_i}^2 \qquad (i = 1, 2, ..., n). \tag{2.1}$$

The j^{th} constraint is used as a *linearizing* tool. Adding and subtracting x_j, y_j and z_j in (2.1) gives

$$(x - x_j + x_j - x_i)^2 + (y - y_j + y_j - y_i)^2 + (z - z_j + z_j - z_i)^2 = {r_i}^2 \tag{2.2}$$

with $(i = 1, 2, ..., j - 1, j + 1, ..., n)$.
Expanding and regrouping the terms, leads to

$$\begin{aligned} & (x - x_j)(x_i - x_j) + (y - y_j)(y_i - y_j) + (z - z_j)(z_i - z_j) \\ = \ & \frac{1}{2}[(x - x_j)^2 + (y - y_j)^2 + (z - z_j)^2 \\ & -{r_i}^2 + (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2] \\ = \ & \frac{1}{2}[{r_j}^2 - {r_i}^2 + d_{ij}^2] = b_{ij}, \end{aligned} \tag{2.3}$$

where

$$d_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2} \tag{2.4}$$

is the distance between beacons B_i and Bj.
Since it does not matter which constraint is used as a linearizing tool, arbitrarily select the first constraint ($j = 1$). This is analogous to selecting the first beacon. Since $i = 2, 3, ..., n$, this leads to a linear system of $(n - 1)$ equations in 3 unknowns:

$$\begin{aligned} & (x - x_1)(x_2 - x_1) + (y - y_1)(y_2 - y_1) + (z - z_1)(z_2 - z_1) \\ = \ & \frac{1}{2}[{r_1}^2 - {r_2}^2 + d_{21}^2] = b_{21} \qquad (2.5) \\ & (x - x_1)(x_3 - x_1) + (y - y_1)(y_3 - y_1) + (z - z_1)(z_3 - z_1) \end{aligned}$$

$$= \frac{1}{2}[{r_1}^2 - {r_3}^2 + d_{31}^2] = b_{31} \tag{2.6}$$

$$\vdots$$

$$(x - x_1)(x_n - x_1) + (y - y_1)(y_n - y_1) + (z - z_1)(z_n - z_1)$$
$$= \frac{1}{2}[{r_1}^2 - {r_n}^2 + d_{n1}^2] = b_{n1}. \tag{2.7}$$

This linear system is easily written in matrix form

$$\mathbf{A}\vec{x} = \vec{b}, \tag{2.8}$$

with

$$\mathbf{A} = \begin{pmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ \vdots & \vdots & \vdots \\ x_n - x_1 & y_n - y_1 & z_n - z_1 \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x - x_1 \\ y - y_1 \\ z - z_1 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} b_{21} \\ b_{31} \\ \vdots \\ b_{n1} \end{pmatrix}. \tag{2.9}$$

The linear system (2.8) has $(n-1)$ equations in three unknowns. Therefore, theoretically only four beacons ($n = 4$) are needed to determine the unique position of a piece of equipment in the mine; provided no more than two beacons are co-linear.

2.2 Geometrical Interpretation of the Linear System

In this section we digress by showing two alternative geometrical techniques to derive the linear system (2.8). The two alternative methods involve the use of analytic geometry, and trigonometry. In turn, they provide a nice geometrical interpretation of the equations in (2.8).

2.2.1 Analytic Geometry

The following analysis uses analytic geometry to show that each of the equations in the linear system (2.8) represents a plane.
Select equation (2.5), the first equation of the system (2.8), for analysis:

$$(x - x_1)(x_2 - x_1) + (y - y_1)(y_2 - y_1) + (z - z_1)(z_2 - z_1)$$
$$= \frac{1}{2}[{r_1}^2 - {r_2}^2 + d_{21}^2] = b_{21}. \tag{2.10}$$

In Figure 2.1, the points $B_1\,(x_1, y_1, z_1)$ and $B_2\,(x_2, y_2, z_2)$ refer to two beacon locations and point $P\,(x, y, z)$ represents the unknown point. The point O represents the (arbitrary) origin of the cartesian coordinate system. Further denote the plane PB_1B_2 by α.
Straightforward algebra allows us to rewrite (2.10) in the form

$$(x - x_0)(x_2 - x_1) + (y - y_0)(y_2 - y_1) + (z - z_0)(z_2 - z_1) = 0, \tag{2.11}$$

with

$$x_0 = x_1 + \frac{b_{12}}{d_{12}^2}(x_2 - x_1) = \frac{1}{2d_{12}^2}[({r_2}^2 - {r_1}^2)(x_1 - x_2) + d_{12}^2(x_1 + x_2)],$$

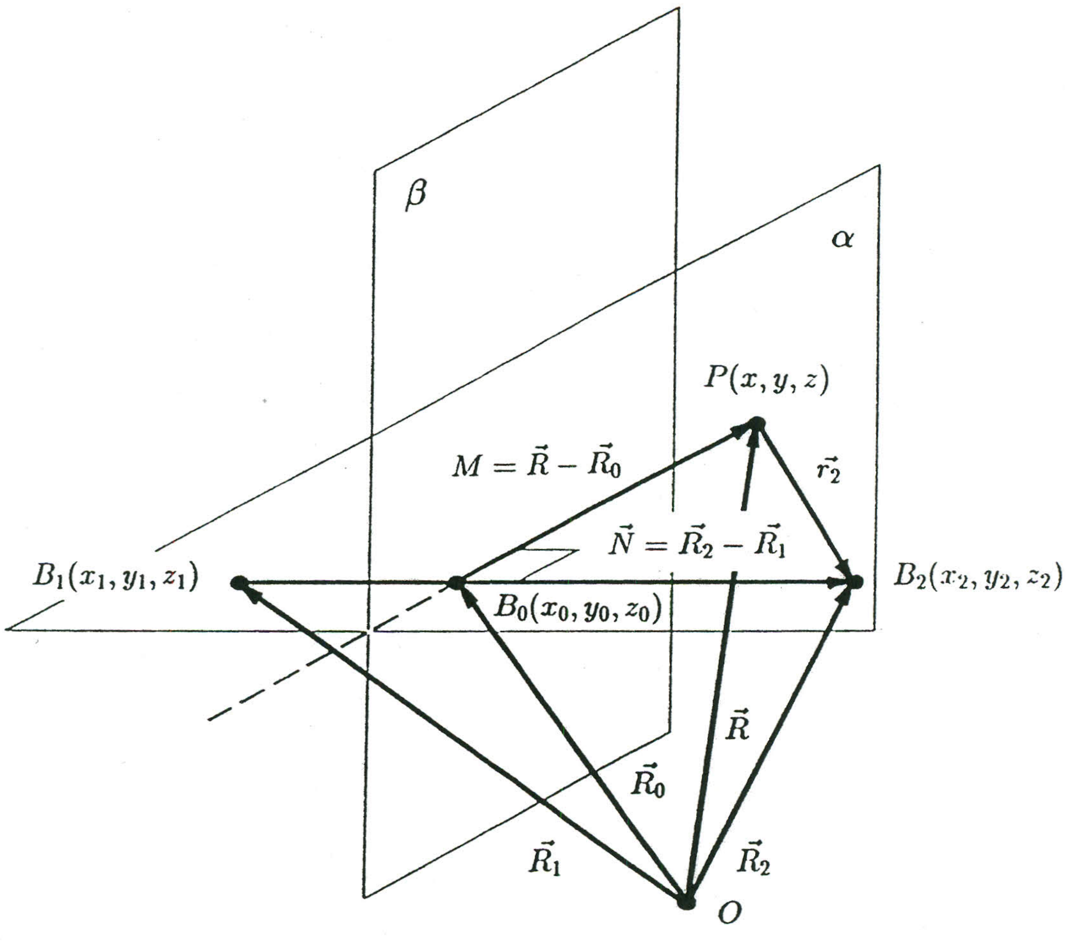

Fig. 2.1: Analytic Geometrical Interpretation

$$y_0 = y_1 + \frac{b_{12}}{d_{12}^2}(y_2 - y_1) = \frac{1}{2d_{12}^2}[({r_2}^2 - {r_1}^2)(y_1 - y_2) + d_{12}^2(y_1 + y_2)], \tag{2.12}$$
$$z_0 = z_1 + \frac{b_{12}}{d_{12}^2}(z_2 - z_1) = \frac{1}{2d_{12}^2}[({r_2}^2 - {r_1}^2)(z_1 - z_2) + d_{12}^2(z_1 + z_2)].$$

Using analytic geometry, it is obvious that (2.11) is the normal form of the plane β containing the points $P(x, y, z)$ and $B_0\,(x_0, y_0, z_0)$, with normal vector $\vec{N} = \vec{R_2} - \vec{R_1}$. Vector $\vec{N}$ has components $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$. Vector $\vec{M}$ is defined as $\vec{M} = \overrightarrow{B_0P} = \vec{R} - \vec{R_0}$, with components $(x - x_0, y - y_0, z - z_0)$. Equation (2.11) is obtained by expressing that vector $\vec{M}$ is *orthogonal* to vector $\vec{N}$. In mathematical notation this is $\vec{M} \cdot \vec{N} = 0$. The planes α and β are thus orthogonal. They intersect along the line carrying the vector $\vec{M} = \overrightarrow{B_0P}$.
A plane β is obtained for each of the equations in the linear system (2.8). The coordinates of the point of intersection of these β-planes is the location $P(x, y, z)$ of the equipment in the mine. Three of these β-planes are needed to uniquely determine this position.

2.2.2 Trigonometry

The following analysis uses trigonometry to derive the linear system (2.8). Consider the triangle B_1B_2P in Figure 2.2, and let O be the arbitrary origin of the coordinate system. Recall that $\vec{r_1} = \overrightarrow{B_1P} = \vec{R} - \vec{R_1}$ has components $(x - x_1, y - y_1, z - z_1)$ and $\vec{r_2} = \overrightarrow{B_2P}$. Furthermore, $\vec{N} = \vec{R_2} - \vec{R_1} = \vec{r_2} - \vec{r_1}$ has components $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$.
Applying the cosine rule in the triangle B_1B_2P, and taking into account that $r_1 = \|\vec{r_1}\|, r_2 = \|\vec{r_2}\|$, and $d_{12} = \|\vec{N}\|$, one obtains

$$\begin{aligned} {r_2}^2 &= {r_1}^2 + d_{12}^2 - 2\,\vec{r_1} \cdot \vec{N} \\ &= {r_1}^2 + d_{12}^2 - 2[(x - x_1)(x_2 - x_1) + (y - y_1)(y_2 - y_1) + (z - z_1)(z_2 - z_1)], \end{aligned} \tag{2.13}$$

which is nothing else than equation (2.5), the first equation of the linear system (2.8).

2.3 Test Data

To illustrate the accuracy of the solution obtained from the three equations of the linear system (2.8), it was tested by calculating the location of three test points in separate trials. The individual trials used the exact and approximate distances from four arbitrarily selected beacons (B_1, B_2, B_3, and B_4). These distances are listed in Table 2.1. Notice that the errors, r_1 through r_4, are also given in Table 1.1. In practice, the linear system (2.8) was tested for 1000 points by using the calibration procedure in Chapter 6.
The position of the three test points in relation to the perimeter of the beacons is illustrated in Figure 2.3. The first test point, $P_1(480000, 1093000, 4668)$, is located inside the perimeter of the beacons, near the surface of the mine. Its elevation is two feet below the elevation of the lowest beacon. The second test point, $P_2(480000, 1093000, 4525)$, is located under point P_1, but at an elevation that is 143 feet below the elevation of the lowest beacon in the mine which is 600 feet deep. The third test point, $P_3(480000, 1095500, 4525)$, is located outside the perimeter of the beacons, at the same elevation as the second test point.

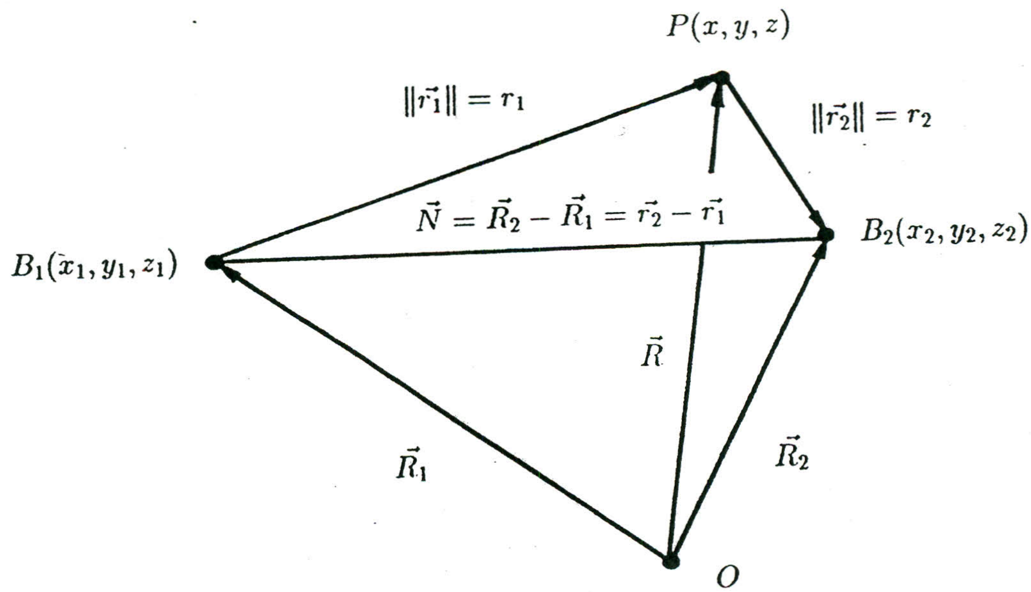

Fig. 2.2: Trigonometrical Interpretation

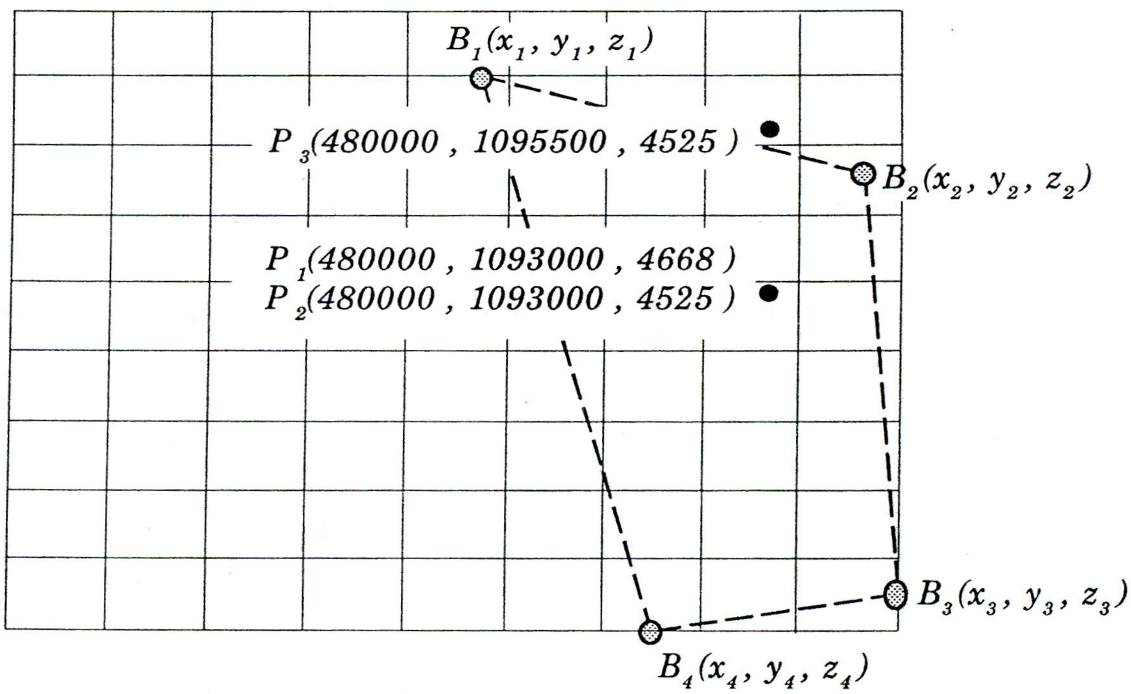

Fig. 2.3: Test Point Locations with Four Beacons

Using the four designated beacons with beacon B_1 as the linearizing tool,

$$\tilde{\mathbf{A}} = \begin{pmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ x_4 - x_1 & y_4 - y_1 & z_5 - z_1 \end{pmatrix} = \begin{pmatrix} 6440 & -1400 & 24 \\ 7170 & -7870 & 161 \\ 2990 & -8490 & 105 \end{pmatrix}. \tag{2.14}$$

Vector $\vec{b}$ is expressed as

$$\vec{b} = \begin{pmatrix} b_{21} \\ b_{31} \\ b_{41} \end{pmatrix}. \tag{2.15}$$

The $\vec{b}$'s which were calculated with (2.9) for the exact as well as the approximate distances associated with test point $P_1(480000, 1093000, 4668)$ are

$$\vec{b}_{exact} = \begin{pmatrix} 36433530 \\ 61390490 \\ 42787380 \end{pmatrix}, \qquad \vec{b}_{approximate} = \begin{pmatrix} 36430390 \\ 61386160 \\ 42785710 \end{pmatrix}. \tag{2.16}$$

The $\vec{b}$'s for test point $P_2(480000, 1093000, 4525)$ are

$$\vec{b}_{exact} = \begin{pmatrix} 36430100 \\ 61367450 \\ 42772350 \end{pmatrix}, \qquad \vec{b}_{approximate} = \begin{pmatrix} 36426980 \\ 61363150 \\ 42770710 \end{pmatrix}. \tag{2.17}$$

The $\vec{b}$'s for test point $P_3(480000, 1095500, 4525)$ are

$$\vec{b}_{exact} = \begin{pmatrix} 32930120 \\ 41692480 \\ 21547390 \end{pmatrix}, \qquad \vec{b}_{approximate} = \begin{pmatrix} 32927540 \\ 41687830 \\ 21546600 \end{pmatrix}. \tag{2.18}$$

Designate $\vec{B}_1$ as

$$\vec{B}_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 475060 \\ 1096300 \\ 4670 \end{pmatrix}. \tag{2.19}$$

Solve for the location of the equipment in the mine by using

$$\begin{aligned} \vec{R} &= \vec{x} + \vec{B}_1 \\ &= (\tilde{\mathbf{A}}^{-1})\vec{b} + \vec{B}_1. \end{aligned} \tag{2.20}$$

These calculated locations and their associated errors are listed in Table 2.2.

2.4 Analysis

The solution of the linear system $\tilde{\mathbf{A}}\vec{x} = \vec{b}$ is unacceptable because it does not determine the locations within a tolerance of five feet when used with approximate distances with errors of up to one-half foot.

The linear system (2.8) can be used with exact distances and four arbitrarily selected beacons to accurately calculate an unknown location. For the three test points, this solution technique calculated the x and y coordinates within a tolerance of 0.0 feet, and the z coordinate (elevation) within a tolerance of 0.562 feet.
The straightforward solution of any three equations of the linear system (2.8) will produce unacceptable results when approximate distances are used. It calculated the x and y coordinates within a 0.5 foot tolerance, and the z coordinate within a 35.813 foot tolerance for the three test points.
The depth of the test point in the mine does not have a major impact on the accuracy of the position calculated by this solution technique.
The results produced when the test point is located outside the perimeter of the beacons are much worse than the results obtained for the points located inside the perimeter.

2.5 *Robustness*

The robustness of the calculation depends on the condition number of the coefficient matrix. Let $\tilde{\mathbf{A}}$ be any 3×3 matrix selected from $\mathbf{A}$ as in (2.14). The robustness of the calculation depends on the condition number $c(\tilde{\mathbf{A}}) = \|\tilde{\mathbf{A}}\|\,\|\tilde{\mathbf{A}}^{-1}\|$ of the coefficient matrix $\tilde{\mathbf{A}}$. The equations should be "well-conditioned" in order to be able to determine all three components of $\vec{x}$ with high accuracy. The change $\delta\vec{x}$ of the solution $\vec{x}$ resulting from the changes of $\tilde{\mathbf{A}}$ and $\vec{b}$ is given by [Noble and Daniel 1988]

$$\frac{\|\delta\vec{x}\|}{\|\vec{x}\|} \leq M\, c(\tilde{\mathbf{A}})\Big(\frac{\|\delta\vec{b}\|}{\|\vec{b}\|} + \frac{\|\delta\tilde{\mathbf{A}}\|}{\|\tilde{\mathbf{A}}\|}\Big), \tag{2.21}$$

where $M = \frac{1}{1-\alpha}$, with $\alpha = \|(\delta\tilde{\mathbf{A}})\tilde{\mathbf{A}}^{-1}\| < 1$. Both the data $\vec{b}$ and the position of the beacons (reflected in $\tilde{\mathbf{A}}$) can be changed to optimize the calculation of the position $\vec{x}$ of the equipment in the mine.
For matrix $\tilde{\mathbf{A}}$ in (2.14), and using the 2-norm, we obtained with MATLAB [Moler, Little, Bangert, and Kleiman 1989],

$$c(\tilde{\mathbf{A}}) = \|\tilde{\mathbf{A}}\|\|\tilde{\mathbf{A}}^{-1}\| = (14538)(0.0252) = 366.3576. \tag{2.22}$$

Using point $P_2(480000, 1093000, 4525)$ and the $\vec{b}$'s in (2.17) we compute

$$\delta\vec{b} = \begin{pmatrix} 36426980 \\ 61363150 \\ 42770710 \end{pmatrix} - \begin{pmatrix} 36430100 \\ 61367450 \\ 42772350 \end{pmatrix} = \begin{pmatrix} -3120 \\ -4300 \\ -1640 \end{pmatrix}. \tag{2.23}$$

Hence,

$$\|\delta\vec{b}\| = 5560, \tag{2.24}$$

$$\|\vec{b}_{exact}\| = 83202000, \tag{2.25}$$

and

$$\|\vec{x}\| = \|\vec{R} - \vec{B}_1\| = \left\| \begin{pmatrix} 475060 \\ 1096300 \\ 4670 \end{pmatrix} - \begin{pmatrix} 480000 \\ 1093000 \\ 4525 \end{pmatrix} \right\| = \left\| \begin{pmatrix} -4940 \\ 3300 \\ 145 \end{pmatrix} \right\| = 5942.6. \tag{2.26}$$

With $\|\delta\tilde{\mathbf{A}}\| = 0$ and $\alpha = 0$ we have $M = 1$, and from (2.21) we get

$$\|\delta\vec{x}\| \leq (366.3576)(\frac{5560}{83202000})(5942.6) = 145.4865. \tag{2.27}$$

With this method the 2-norm of $\|\delta\vec{x}\| = \|\delta\vec{R} - \delta\vec{B}_1\| \leq 145.4865$, which means that

$$\sqrt{(\delta x)^2 + (\delta y)^2 + (\delta z)^2} \leq 145 \text{ feet}. \tag{2.28}$$

Since $\delta x \approx 0$ and $\delta y \approx 0$ we estimate $\delta z \approx 145$. We conclude that the z-coordinate could be as much off as 145 feet.

Tab. 2.1: Distances from Four Beacons to the Equipment

Test Data	Beacons	Exact Distances	Approximate Distances	Distance Errors
Test Point Inside Mine Near the Top $P_1(480000, 1093000, 4668)$	B_1	5940.893	5940.382	-0.457890
	B_2	2420.883	2421.056	0.173050
	B_3	5087.666	5087.983	0.316931
	B_4	5545.271	5545.082	-0.191205
Test Point Inside Mine Near the Bottom $P_2(480000, 1093000, 4525)$	B_1	5942.607	5492.151	-0.457890
	B_2	2426.635	2426.809	0.173050
	B_3	5094.254	5094.570	0.316931
	B_4	5549.874	5549.683	-0.191205
Test Point Outside Mine Near the Bottom $P_3(480000, 1095500, 4525)$	B_1	5006.458	5005.998	-0.457890
	B_2	1624.635	1624.538	0.173050
	B_3	7419.664	7419.983	0.316931
	B_4	7937.320	7937.128	-0.191205

Tab. 2.2: Locations Calculated by Linearized Equations

		Exact Distances		Approximate Distances	
Test Data	Test Coordinates	Calculated Position	Errors	Calculated Position	Errors
Point Inside Mine	480000.0	480000.000	0.000	479999.531	-0.469
Near the Top	1093000.0	1093000.000	0.000	1092999.875	-0.125
$P_1(x, y, z)$	4668.0	4668.562	0.562	4657.250	-10.750
Point Inside Mine	480000.0	480000.000	0.000	479999.531	-0.469
Near the Bottom	1093000.0	1093000.000	0.000	1092999.875	-0.125
$P_2(x, y, z)$	4525.0	4525.562	0.562	4514.000	-11.000
Point Outside Mine	480000.0	480000.000	0.000	479999.625	-0.375
Near the Bottom	1095500.0	1095500.000	0.000	1095499.500	-0.500
$P_3(x, y, z)$	4525.0	4525.219	0.219	4489.187	-35.813

3. LINEAR LEAST SQUARES METHOD

In this chapter we will show that applying the linear least squares method to the linear system (2.8) is an unacceptable solution technique because it does not calculate the locations within a tolerance of five feet when used with approximate distances. The equipment locations obtained by the entire linear least squares method are generally more accurate than the locations obtained by solving four equations of the linear system (2.8) directly.

3.1 Development of the Linear Least Squares Method

In practice, the distances r_i are only approximate. Thus the problem requires the determination of $\vec{x}$ such that $\mathbf{A}\vec{x} \approx \vec{b}$. Minimizing the sum of the squares of the residuals,

$$S = \vec{r}^{\,T}\vec{r} = (\vec{b} - \mathbf{A}\vec{x})^T (\vec{b} - \mathbf{A}\vec{x}), \tag{3.1}$$

leads to the *normal* equation [Noble and Daniel 1988]

$$\mathbf{A}^T\mathbf{A}\vec{x} = \mathbf{A}^T\vec{b} \tag{3.2}$$

for $\vec{x}$.
There are several methods to solve (3.2) for $\vec{x}$. The condition number of $\mathbf{A}^T\mathbf{A}$ determines which method is best.
If $\mathbf{A}^T\mathbf{A}$ is non-singular and well-conditioned then

$$\vec{x} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{b} \tag{3.3}$$

is used.
If $\mathbf{A}^T\mathbf{A}$ is singular or badly conditioned then the normalized QR-decomposition [Noble and Daniel 1988] of $\mathbf{A}$ is generally used. In this method $\mathbf{A} = \mathbf{QR}$, where $\mathbf{Q}$ is an orthonormal matrix and $\mathbf{R}$ is upper-triangular matrix. The solution for $\vec{x}$ in the normalized QR-decomposition is then found from

$$\mathbf{R}\,\vec{x} = \mathbf{Q}^T\vec{b} \tag{3.4}$$

by back substitution when $\mathbf{A}$ is full rank.
It may happen that the matrix $\mathbf{A}^T\mathbf{A}$ is close to singular even when the original matrix $\mathbf{A}$ was not close to singular. For situations like that, $\mathbf{QR}$ decomposition may overcome the problem. If not, singular value decomposition (SVD) can be used to solve the least squares problem fairly accurately.

3.2 Singular Value Decomposition (SVD)

The optimal solution $\vec{x}_0$ is then given by $\vec{x}_0 = \mathbf{A}^+\vec{b}$. The pseudo-inverse [Noble and Daniel 1988] $\mathbf{A}^+ = \mathbf{V}\mathbf{\Sigma}^+\mathbf{U}^H$ involves the unitary matrices $\mathbf{U}, \mathbf{V}$ occurring in the SVD of $\mathbf{A}$, this is $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^H$. The matrix $\mathbf{\Sigma}^+$ is obtained from the "diagonal" matrix $\mathbf{\Sigma}$ as follows: The $p \times q$ matrix $\mathbf{\Sigma}$ has entries $< \mathbf{\Sigma} >_{ij} = 0$ if $i \neq j$ and $< \mathbf{\Sigma} >_{ij} = \sigma_i \geq 0$ for $1 \leq i \leq k$ and $k + 1 \leq i \leq \min\{p, q\}$. The numbers σ_i are called the *singular values*. The matrix $\mathbf{\Sigma}^+$ is then the $q \times p$ matrix whose nonzero entries are $< \mathbf{\Sigma}^+ >_{ii} = \frac{1}{\sigma_i}$, for $1 \leq i \leq k$.
To detect degeneracy of the matrix $\mathbf{A}$ one computes the ratio σ_1/σ_n, where σ_1 is the largest singular value and σ_n is the smallest singular value when $\mathbf{A}$ is full rank. The ratio σ_1/σ_n may be regarded as a condition number of the matrix $\mathbf{A}$. It is not the same condition number as in (2.21), but is usually about the same order of magnitude numerically.
The smallest singular value, σ_n, is the distance in the 2-norm from $\mathbf{A}$ to the nearest singular matrix. The fact that σ_1/σ_n is small may be considered as a condition of near-singularity of $\mathbf{A}$ [Kahaner, Moler, and Nash 1989][Lawson, and Hanson 1974].

3.3 Test Data

To investigate the accuracy of the solution produced by the linear least squares method, it was tested by calculating the location of three test points in separate trials. The individual trials used exact and approximate distances from eight beacons. The errors are the same as in Table 1.1. These distances are listed in Table 3.1. In practice, the linear least squares method was tested for 1000 points by using the calibration procedure in Chapter 6.
The position of the three test points in relation to the eight beacons is illustrated in Figure 3.1. The first test point, $P_1(480000, 1093000, 4668)$, is located inside the perimeter of the beacons, at an elevation that is two feet below the elevation of the lowest beacon. The second test point, $P_2(480000, 1093000, 4525)$, is located under point P_1, but at an elevation that is 143 feet below the elevation of the lowest beacon. The third test point, $P_3(480000, 1095500, 4525)$, is located outside the perimeter of the beacons, at the same elevation as the second test point.
Using the eight beacons, and beacon B_1 as the linearizing tool, we now have

$$\mathbf{A} = \begin{pmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ x_4 - x_1 & y_4 - y_1 & z_5 - z_1 \\ x_5 - x_1 & y_5 - y_1 & z_5 - z_1 \\ x_6 - x_1 & y_6 - y_1 & z_6 - z_1 \\ x_7 - x_1 & y_7 - y_1 & z_7 - z_1 \\ x_8 - x_1 & y_8 - y_1 & z_8 - z_1 \end{pmatrix} = \begin{pmatrix} 6440 & -1400 & 24 \\ 7170 & -7870 & 161 \\ 2990 & -8490 & 105 \\ -3630 & -7720 & 82 \\ -6340 & -5060 & 133 \\ -7660 & -2320 & 35 \\ -6330 & 1040 & 77 \end{pmatrix}. \tag{3.5}$$

Tab. 3.1: Distances from Eight Beacons to the Equipment

Test Data	Beacon	Exact Distances	Approximate Distances	Distance Errors
Point Inside Mine Near the Top $P_1(480000, 1093000, 4668)$	B_1	5940.893	5940.382	-0.457890
	B_2	2420.883	2421.056	0.173050
	B_3	5087.666	5087.983	0.316931
	B_4	5545.271	5545.082	-0.191205
	B_5	9643.044	9643.513	0.468339
	B_6	11417.270	11417.420	0.141141
	B_7	12638.110	12638.430	0.328659
	B_8	12077.030	12076.640	-0.39046
Point Inside Mine Near the Bottom $P_2(480000, 1093000, 4525)$	B_1	5942.607	5492.151	-0.457890
	B_2	2426.635	2426.809	0.173050
	B_3	5094.254	5094.570	0.316931
	B_4	5549.874	5549.683	-0.191205
	B_5	9645.353	9645.819	0.468339
	B_6	11419.870	11420.000	0.141141
	B_7	12639.330	12639.660	0.328659
	B_8	12078.820	12078.420	-0.39046
Point Outside Mine Near the Bottom $P_3(480000, 1095500, 4525)$	B_1	5006.458	5005.998	-0.457890
	B_2	1624.635	1624.538	0.173050
	B_3	7419.664	7419.983	0.316931
	B_4	7937.32	7937.128	-0.191205
	B_5	11047.380	11017.850	0.468339
	B_6	12060.820	12060.950	0.141141
	B_7	12692.620	12692.950	0.328659
	B_8	11421.370	11420.980	-0.39046

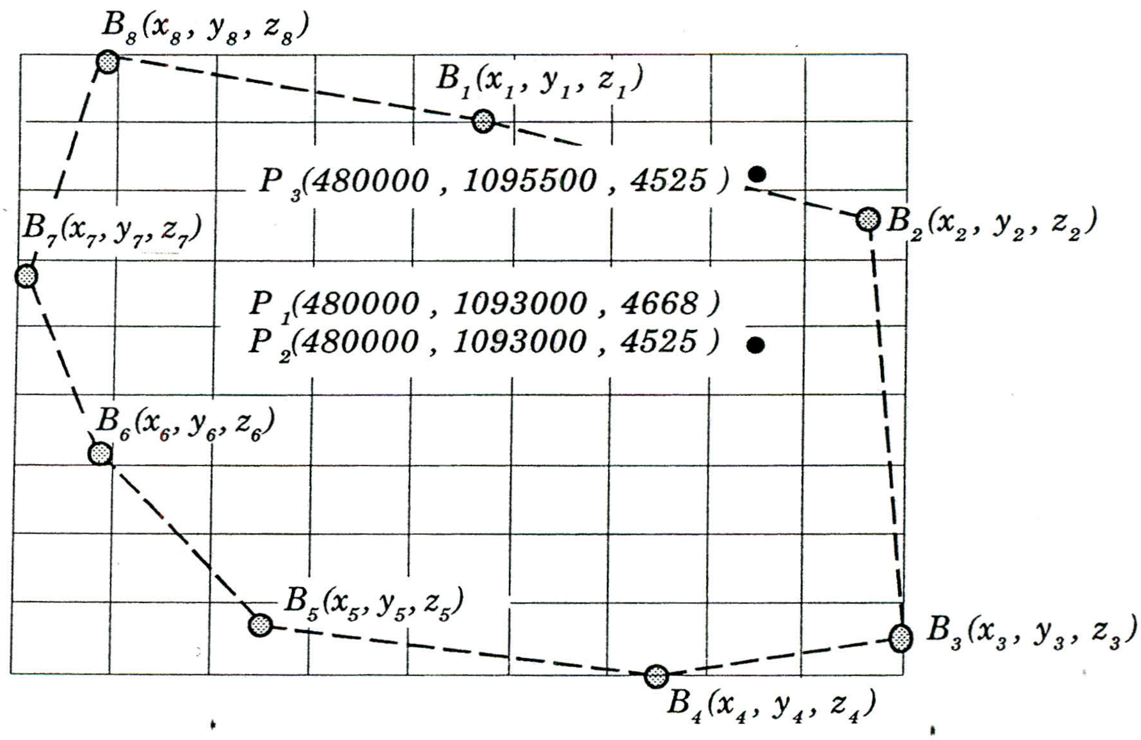

Fig. 3.1: Test Point Locations with Eight Beacons

Vector $\vec{b}$ is expressed as

$$\vec{b} = \begin{pmatrix} b_{21} \\ b_{31} \\ b_{41} \\ b_{51} \\ b_{61} \\ b_{71} \\ b_{81} \end{pmatrix}. \tag{3.6}$$

The $\vec{b}$'s which were calculated with the exact and approximate distances associated with test point $P_1(480000, 1093000, 4668)$ are

$$\vec{b}_{exact} = \begin{pmatrix} 36433530 \\ 61390490 \\ 42787380 \\ 7543642 \\ -14621910 \\ -30184530 \\ -34702420 \end{pmatrix}, \qquad \vec{b}_{approximate} = \begin{pmatrix} 36430390 \\ 61386160 \\ 42785710 \\ 7543692 \\ -14626250 \\ -30191370 \\ -34700410 \end{pmatrix}. \tag{3.7}$$

The $\vec{b}$'s for test point $P_2(480000, 1093000, 4525)$ are

$$\vec{b}_{exact} = \begin{pmatrix} 36430100 \\ 61367450 \\ 42772350 \\ 7531882 \\ -14640990 \\ -30189410 \\ -34713480 \end{pmatrix}, \qquad \vec{b}_{approximate} = \begin{pmatrix} 36426980 \\ 61363150 \\ 42770710 \\ 7524660 \\ -14645310 \\ -30196370 \\ -34711440 \end{pmatrix}. \tag{3.8}$$

The $\vec{b}$'s for test point $P_3(480000, 1095500, 4525)$ are

$$\vec{b}_{exact} = \begin{pmatrix} 32930120 \\ 41692480 \\ 21547390 \\ -11768090 \\ -27290940 \\ -35989400 \\ -32113340 \end{pmatrix}, \qquad \vec{b}_{approximate} = \begin{pmatrix} 32927540 \\ 41687830 \\ 21546600 \\ -11775480 \\ -27294910 \\ -35996000 \\ -32111160 \end{pmatrix}. \tag{3.9}$$

The $\mathbf{A}^T\mathbf{A}$ for $P_1(480000, 1093000, 4668)$ for both exact and approximate distances is

$$\mathbf{A}^T\mathbf{A} = \begin{pmatrix} 253939600 & -19537000 & -273510 \\ -19537000 & 227643000 & -3499260 \\ -273510 & -3499260 & 69089 \end{pmatrix}. \tag{3.10}$$

For point $P_2(480000, 1093000, 4525)$

$$\mathbf{A}^T\mathbf{A} = \begin{pmatrix} 253939600 & -19537000 & -273510 \\ -19537000 & 227643000 & -3499260 \\ -273510 & -3499260 & 69089 \end{pmatrix}, \tag{3.11}$$

Tab. 3.2: Locations Calculated by Linear Least Squares

Test Data	Test Coordinates	Exact Distances: Calculated Position	Exact Distances: Errors	Approximate Distances: Calculated Position	Approximate Distances: Errors
Point Inside Mine	480000	480000.000	0.000	480000.219	0.219
Near the Top	1093000	1093000.000	0.000	1093000.875	0.875
$P_1(x, y, z)$	4668	4667.687	-0.313	4681.437	13.437
Point Inside Mine	480000	480000.000	0.000	480000.250	0.250
Near the Bottom	1093000	1093000.000	0.000	1093000.875	0.875
$P_2(x, y, z)$	4525	4524.000	-1.000	4538.375	13.375
Point Outside Mine	480000	480000.000	0.000	480000.219	0.219
Near the Bottom	1095500	1095500.000	0.000	1095500.875	0.875
$P_3(x, y, z)$	4525	4524.961	-0.039	4539.039	14.039

and for $P_3(480000, 1095500, 4525)$

$$\mathbf{A}^T\mathbf{A} = \begin{pmatrix} 253939600 & -19537000 & -273510 \\ -19537000 & 227643000 & -3499260 \\ -273510 & -3499260 & 69089 \end{pmatrix}. \tag{3.12}$$

The $\mathbf{A}^T\vec{b}$ for $P_1(480000, 1093000, 4668)$, using exact and approximate distances respectively, are

$$\mathbf{A}^T\vec{b}_{exact} = \begin{pmatrix} 1318935000000 \\ -847727400000 \\ 10196270000 \end{pmatrix}, \qquad \mathbf{A}^T\vec{b}_{approximate} = \begin{pmatrix} 1318973000000 \\ -847578900000 \\ 10194060000 \end{pmatrix}. \tag{3.13}$$

For $P_2(480000, 1093000, 4525)$

$$\mathbf{A}^T\vec{b}_{exact} = \begin{pmatrix} 1318974000000 \\ -847226600000 \\ 10186370000 \end{pmatrix}, \qquad \mathbf{A}^T\vec{b}_{approximate} = \begin{pmatrix} 1319012000000 \\ -847078600000 \\ 10184180000 \end{pmatrix}, \tag{3.14}$$

and for $P_3(480000, 1095500, 4525)$

$$\mathbf{A}^T\vec{b}_{exact} = \begin{pmatrix} 1270131000000 \\ -278120000000 \\ 1438253000 \end{pmatrix}, \qquad \mathbf{A}^T\vec{b}_{approximate} = \begin{pmatrix} 1270167000000 \\ -277978400000 \\ 1436164000 \end{pmatrix}. \tag{3.15}$$

Since $\mathbf{A}^T\mathbf{A}$ is non-singular, we solve (3.3) for $\vec{x}$ to determine the location of the equipment in the mine. These calculated locations for the three test points are listed in Table 3.2.

3.4 Analysis

The linear least squares method as implemented here is an unacceptable solution technique because it does not calculate the locations within a tolerance of less than five feet.

The linear least squares solution in (3.3) can be used with exact distances and more than four beacons to accurately calculate an unknown location. For the three test points, this solution technique calculated the x and y coordinates within a tolerance of 0.0 feet, and the z coordinate (elevation) within a tolerance of 1.000 feet.
The linear least squares solution technique produces unacceptable results when approximate distances are used. The x and y coordinates are within a 0.875 foot tolerance, but the z coordinate is within only a 14.039 foot tolerance for the three test points.
The depth of the test point in the mine does not appear to have a major impact on the accuracy of the position calculated by this solution technique.
The results for a test point located outside the perimeter of the beacons are of the same order of magnitude as the results for the points located inside the perimeter.
The impact of the approximate distances on the accuracy of the calculated elevation is magnified by a condition which is inherent in open pit mining operations in Wyoming. The magnitudes of the x and y coordinates are substantially larger than the magnitude of the z coordinate because the coal is relatively close to the surface of the mine. This difference in scale, introduces inaccuracy in the calculated z coordinate (elevation).

3.5 Robustness

We use two different methods to analyze the robustness of the linear least squares method. The first technique involves singular value decomposition. The second technique uses symbolic manipulation. It allows us to predict theoretically the errors on the coordinates of the calculated location.

3.5.1 Application of Singular Value Decomposition

To analyze the inaccuracy that is introduced by the differences in magnitude of the coordinates we perform a singular value decomposition on matrix $\mathbf{A}$ in (3.5) using MATLAB [Moler, Little, Bangert, and Kleiman 1989].
Matrix $\mathbf{A}$ has singular values

$$\sigma_1 = 16259, \tag{3.16}$$

$$\sigma_2 = 14741, \tag{3.17}$$

and

$$\sigma_3 = 118. \tag{3.18}$$

Hence,

$$\sigma_1/\sigma_3 = 16259/118 = 137.788, \tag{3.19}$$

which confirms that the entries in the third column of A are about 100 times smaller than the entries in the first column of $\mathbf{A}$.
Furthermore,

$$\mathbf{A} = \mathbf{U\Sigma V^H} \tag{3.20}$$

with

$$\mathbf{U} = \begin{pmatrix} 0.3901 & -0.1216 & 0.1425 & 0.4442 & 0.4634 & 0.5609 & 0.2933 \\ 0.6169 & 0.2425 & 0.4632 & -0.0262 & -0.4278 & 0.0860 & -0.3937 \\ 0.4079 & 0.4129 & -0.1722 & -0.5263 & 0.1164 & -0.1285 & 0.5713 \\ 0.0263 & 0.5780 & -0.3926 & 0.6619 & -0.1840 & -0.1913 & 0.0495 \\ -0.1977 & 0.5053 & 0.3365 & -0.0662 & 0.6641 & -0.1501 & -0.3528 \\ -0.3487 & 0.3833 & -0.1568 & -0.2424 & -0.2092 & 0.7731 & -0.0822 \\ -0.3736 & 0.1398 & 0.6660 & 0.1537 & -0.2647 & -0.0681 & 0.5467 \end{pmatrix}, \tag{3.21}$$

$$\mathbf{V} = \begin{pmatrix} 0.8825 & -0.4703 & 0.0023 \\ -0.4702 & -0.8824 & 0.0156 \\ 0.0053 & 0.0148 & 0.9999 \end{pmatrix}, \tag{3.22}$$

$$\mathbf{\Sigma} = \begin{pmatrix} 16259 & 0 & 0 \\ 0 & 14741 & 0 \\ 0 & 0 & 118 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.23}$$

Let us continue with point $P_2(480000, 1093000, 4525)$ and $\vec{b}_{approximate}$ in (3.8). Since neither of the singular values is very close to zero, we compute the true pseudo-inverse of $\mathbf{A}$, namely

$$\mathbf{A}^+ = \mathbf{V}\mathbf{\Sigma}^+\mathbf{U}^\mathbf{H} \tag{3.24}$$

with

$$\mathbf{\Sigma}^+ = \begin{pmatrix} 0.00006150 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.00006784 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.0085 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.25}$$

We obtain

$$\mathbf{A}^+ = \frac{1}{100000} \begin{pmatrix} 2.505 & 2.574 & 0.896 & -1.701 & -2.685 & -3.115 & -2.474 \\ -0.400 & -3.236 & -3.652 & -3.536 & -2.453 & -1.286 & 0.244 \\ 0.001 & 0.045 & 0.055 & 0.059 & 0.044 & 0.027 & 0.002 \end{pmatrix}. \tag{3.26}$$

The optimal solution is then given by

$$\vec{x_0} = \mathbf{A}^+\vec{b}_{approximate} = \begin{pmatrix} 4940.2 \\ -32991.0 \\ -131.8 \end{pmatrix}, \tag{3.27}$$

hence,

$$\vec{R} = \vec{x_0} + \vec{B_1} = \begin{pmatrix} 480000 \\ 1093000 \\ 4500 \end{pmatrix}. \tag{3.28}$$

This result is not better than what could be obtained via (3.3).

Replacing $1/\sigma_3$ by 0 in $\mathbf{\Sigma}^+$ does not improve matters much. Indeed, with

$$\tilde{\mathbf{\Sigma}}^+ = \begin{pmatrix} 0.00006150 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.00006784 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.29}$$

one would obtain

$$\vec{R} = \begin{pmatrix} 480000 \\ 1093000 \\ 4700 \end{pmatrix}, \tag{3.30}$$

a result worse than what follows from (3.3).

3.5.2 Application of Symbolic Manipulation

To analyze the effects of the errors on the distance measurements we perform a symbolic calculation with MACSYMA [Symbolics 1988]. In this calculation we use the beacon locations in (1.1), and the point $P_2(480000, 1093000, 4525)$. Using these numerical values we solve for x, y, and z in terms of $err1$, $err2$, $err3$, $err4$, $err5$, $err6$, $err7$, and $err8$, where $err1$ represents the symbolic error on the distance associated with the first beacon, etc.
Using this method we get the theoretical errors on the coordinates of P :

$$\begin{aligned} \delta x = \; & -0.00000596589(err8)^2 - 0.144122(err8) - 0.00001708401(err7)^2 \\ & - 0.431861(err7) - 0.00001019150(err6)^2 - 0.232771(err6) \\ & - 0.00001228196(err5)^2 - 0.236928(err5) + 0.00000282727(err4)^2 \\ & + 0.031382(err4) + 0.00001732629(err3)^2 + 0.176529(err3) \\ & + 0.00001389623(err2)^2 + 0.067442(err2) + 0.00001147358(err1)^2 \\ & + 0.136366(err1), \end{aligned} \tag{3.31}$$

$$\begin{aligned} \delta y = \; & 0.00004504562(err8)^2 + 1.088196(err8) - 0.00001674421(err7)^2 \\ & - 0.423271(err7) + 0.00000987568(err6)^2 + 0.225558(err6) \\ & - 0.00004351283(err5)^2 - 0.839393(err5) - 0.00002958805(err4)^2 \\ & - 0.328420(err4) + 0.00001429731(err3)^2 + 0.145668(err3) \\ & + 0.00000737499(err2)^2 + 0.035793(err2) + 0.00001325149(err1)^2 \\ & + 0.157496(err1), \end{aligned} \tag{3.32}$$

and

$$\begin{aligned} \delta z = \; & 0.002815000(err8)^2 + 68.00692(err8) - 0.00066240(err7)^2 \\ & - 16.74474(err7) + 0.001422000(err6)^2 + 32.48654(err6) \\ & - 0.00165900(err5)^2 - 32.00419(err5) - 0.000727510(err4)^2 \\ & - 8.075191(err4) + 0.00195800(err3)^2 + 19.94801(err3) \\ & + 0.000602234(err2)^2 + 2.922807(err2) - 0.00374900(err1)^2 \\ & - 44.55374(err1) + 0.984375. \end{aligned} \tag{3.33}$$

To get an overall estimate of how accurate the distances need to be for this specific test point we perform the following calculation. Assume that

$$err1 = err2 = \cdots = err8 = \Delta. \tag{3.34}$$

Starting with the elevation, add the absolute values of the coefficients of the linear terms in $err1$ through $err8$ in (3.33). Similarly, add the absolute values of the coefficients of the quadratic terms in $err1$ through $err8$. Requiring an accuracy of five feet on the elevation of the equipment in the mine, leads to

$$(224.741)\Delta + (0.01359)\Delta^2 < 5 \text{ feet}. \tag{3.35}$$

Solving for Δ gives $\Delta \approx 0.022$ feet. This means that the error on the distances should be less than 0.022 feet, in order to calculate the elevation within a tolerance of five feet. Similar calculations for the x and y coordinates requires that the error on the distances should be less than 3.432 and 1.542 feet respectively.

If the errors in Table 1.1 are used, we can evaluate (3.31), (3.32), and (3.33). Substituting the errors for $err1$ through $err8$, gives the results in Table 3.3. This table also lists the results for the same data using the linear least squares method directly. The errors generated by both methods are equal, as expected.

Tab. 3.3: Comparison of the Locations Calculated Directly using Linear Least Squares, and the Locations Calculated by Substituting Known Errors into the Symbolic Linear Least Squares Solution

		Symbolic Equations		Linear Least Squares	
Test Data	Test Coordinates	Calculated Position	Errors	Calculated Position	Errors
Point Inside Mine	480000	480000.250	0.250	480000.250	0.250
Near the Bottom	1093001	1093001.875	0.875	1093000.875	0.875
$P_2(x, y, z)$	4525	4538.375	13.375	4538.375	13.375

4. AVERAGING AND WEIGHTING TECHNIQUES

In the previous chapters we obtained good results for the x and y coordinates. Returning to the equations of the spheres, we could calculate the corresponding values of z and use appropriate averaging. The z coordinates calculated from the weighted average and arithmetic average are not within the required 5.0 foot tolerance when these techniques are used with approximate distances. The weighting and averaging methods applied to the z coordinate generally provide a more accurate result for the elevation than the z coordinate obtained from the linear least squares method alone. These techniques reduce the impact of the differences in magnitude between the x and y coordinates, and the z coordinate. Their effectiveness is limited when the elevation of the equipment is too close to or above the elevation of the more or less common plane of the beacons, and when the equipment is located outside the perimeter of the beacons.

4.1 *Development of Averaging and Weighting Techniques*

Averaging and weighting techniques can be applied to the z-coordinate to obtain a better guess for the elevation of the piece of equipment.

4.1.1 *Arithmetic Average*

One way to obtain a better approximation of the value for the z-coordinate is to first calculate x and y from equation (3.3), and then solve each of the given constraints in (2.1) for z,

$$z \approx z_i \pm \sqrt{r_i^2 - (x - x_i)^2 - (y - y_i)^2}, \tag{4.1}$$

where $i = 1, 2, ..., n$. This gives n values for z which are denoted by $z_{\{k\}}$ $(k = 1, 2, ..., n)$.
If we restrict this solution method to applications where the elevation of the equipment in the mine is lower than *all* of the beacons, then only the $-$ sign in (4.1) is relevant. Since the $+$ sign was eliminated from (4.1), an arithmetic average can be calculated for the n values of $z_{\{k\}}$. In practice, we will only take the average of the *real* values of $z_{\{k\}}$. This arithmetic average is the value that best approximates z in a least squares sense.

4.1.2 *Weighted Average*

A weighted average of the $z_{\{k\}}$ values provides a better solution than the arithmetic average [Grossman 1969].
To illustrate the weighted average, consider the error f_1 between the measured distance r_1 and the theoretical distance $\hat{r_1}$ from the bulldozer to the first beacon , i.e.

$$f_1 = \hat{r_1} - r_1 = \sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2} - r_1. \tag{4.2}$$

Let $(\tilde{x}, \tilde{y}, \tilde{z})$ denote a 'good' guess for (x, y, x) and let $(\delta x, \delta y, \delta z)$ denote the errors on these coordinates. Taylor expansion produces

$$\begin{aligned} f_1(x,y,z) &= f_1(\tilde{x}+\delta x, \tilde{y}+\delta y, \tilde{z}+\delta z) \\ &= f_1(\tilde{x},\tilde{y},\tilde{z}) + \left(\frac{\partial f_1}{\partial x}\right)_{x=\tilde{x},y=\tilde{y},z=\tilde{z}} \delta x + \left(\frac{\partial f_1}{\partial y}\right)_{x=\tilde{x},y=\tilde{y},z=\tilde{z}} \delta y \\ &+ \left(\frac{\partial f_1}{\partial z}\right)_{x=\tilde{x},y=\tilde{y},z=\tilde{z}} \delta z + ... \end{aligned} \tag{4.3}$$

Taking into account the explicit form of f_1 as in (4.2), one obtains

$$\begin{aligned} f_1(x,y,z) &= f_1(\tilde{x},\tilde{y},\tilde{z}) + \frac{\tilde{x}-x_1}{f_1(\tilde{x},\tilde{y},\tilde{z})+r_1}\delta x \\ &+ \frac{\tilde{y}-y_1}{f_1(\tilde{x},\tilde{y},\tilde{z})+r_1}\delta y + \frac{\tilde{z}-z_1}{f_1(\tilde{x},\tilde{y},\tilde{z})+r_1}\delta z + ..., \end{aligned} \tag{4.4}$$

where

$$f_1(\tilde{x},\tilde{y},\tilde{z}) = \sqrt{(\tilde{x}-x_1)^2 + (\tilde{y}-y_1)^2 + (\tilde{z}-z_1)^2} - r_1. \tag{4.5}$$

This result reveals that the weight factors should be proportional to the reciprocals of the distances. Therefore, a good approximation will be

$$z = \frac{\sum_{k=1}^{s} \frac{1}{r_k} z_{\{k\}}}{\sum_{k=1}^{s} \frac{1}{r_k}}, \tag{4.6}$$

where the sum is only taken over those values of k for which the corresponding $z_{\{k\}}$ from equation (4.1) is real.

4.2 Test Data

To perform a suitable test we calculate the x and y coordinates for the three test points given in Table 3.1 using the linear least squares method. Keeping the x and y coordinates fixed we then use arithmetic and weighted averaging to calculate the values for the elevation z. A comparison of the values of the z coordinates obtained by the linear least squares, by arithmetic averaging, and by weighted averaging is found in Table 4.1.

4.3 Analysis

The arithmetic average and weighted average give unacceptable results because they do not determine the locations within a tolerance of 5.0 feet when used with approximate distances. The z coordinate obtained from the weighted average is generally more accurate than the z coordinates obtained from the arithmetic average, and the linear least squares techniques. The z obtained from the arithmetic average is generally more accurate than the z obtained from the linear least squares.

Tab. 4.1: Locations Calculated by Averaging Techniques

Test Data	Method	Exact Distances Calculated z	Exact Distances Errors	Approximate Distances Calculated z	Approximate Distances Errors
$P_1(x,y,z)$	Linear Least Squares	4667.678	-0.313	4681.437	13.437
$z = 4668.0$	Arithmetic Average	4667.739	-0.261	4659.721	-8.279
	Weighted Average	4667.888	-0.112	4648.275	-19.725
$P_2(x,y,z)$	Linear Least Squares	4524.000	-1.000	4538.375	13.375
$z = 4525.0$	Arithmetic Average	4524.981	-0.019	4530.259	5.259
	Weighted Average	4524.995	-0.005	4527.234	2.234
$P_2(x,y,z)$	Linear Least Squares	4524.961	-0.039	4539.039	14.039
$z = 4525.0$	Arithmetic Average	4525.017	0.017	4540.275	15.275
	Weighted Average	4525.003	-0.003	4535.312	10.312

The weighted average (3.3), and the arithmetic average can both be used with exact distances to accurately calculate an unknown location. For the three test points, these solution techniques calculated the z coordinate within a tolerance of 0.112 feet, and 0.261 feet respectively.

The depth of the test point in the mine has a major impact on the accuracy of the position calculated by these solution techniques. The accuracy of the calculated z decreases as the elevation of the equipment in the mine approaches the elevation of the more or less common plane of the beacons. This adverse impact affects the weighted average more than the arithmetic average.

When approximate distances were used, the weighted average leads to a z coordinate within a tolerance of 19.725 feet for the three test points. This 19.725 value was obtained from a point that has an elevation that is within 2.0 feet of the elevation of the lowest beacon. The tolerance for the two points near the bottom of the mine was 10.312 feet.

When approximate distances were used, the arithmetic average calculated the z coordinate within a 15.275 foot tolerance for the three test points. Its tolerance for the two points near the bottom of the mine was also 15.275 feet.

As a point moves farther outside the perimeter, the accuracy of the calculated z decreases for both the weighted, and the arithmetic averages.

If the two geometrically 'bad' cases are eliminated, the accuracy of the calculated z is within the required 5.0 foot tolerance for both the weighted average, and the arithmetic average. These methods produce acceptable results for points that are located inside the mine, at an elevation that is not close to or above the elevation of the lowest beacon.

4.4 *Robustness*

The discussion of the robustness in Section 3.5 applies to the arithmetic and weighted average solution methods because they are modifications of the linear least squares solution method. The accuracy of the elevations calculated by the two averaging methods could by analyzed theoretically. However, such an analysis would be of little use. Simple tests as in Section 4.3 reveal that the z coordinates are still not accurate enough. To drastically improve the accuracy of the calculated z, we invented a new method which is discussed in Chapter 5.

5. NONLINEAR LEAST SQUARES METHOD

The nonlinear least squares method developed in this chapter is acceptable for use by TBCC. The accuracy of the z coordinate calculated from approximate distances is within a tolerance of 5.0 feet. This accuracy is attainable if the equipment is inside the perimeter of the beacons at an elevation that is not close to or above the elevation of the lowest beacon.

5.1 *Development of the Nonlinear Least Squares Method*

The sum of the squares of the errors on the distances is minimized in this least squares method. Recall that r_i denotes the *approximate* distance between the equipment in the mine, and the i^{th} beacon; and that $\hat{r}_i$ stands for the *exact* distance, i.e.

$$(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2 = \hat{r}_i^{\,2}. \tag{5.1}$$

To minimize the sum of the squares of the errors on the distances, one must minimize the function

$$F(x, y, z) = \sum_{i=1}^{n} (\hat{r}_i - r_i)^2 = \sum_{i=1}^{n} f_i(x, y, z)^2, \tag{5.2}$$

with

$$f_i(x, y, z) = \hat{r}_i - r_i = \sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2} - r_i. \tag{5.3}$$

Minimizing the sum of the square errors is a fairly common problem in applied mathematics for which various algorithms are available [McKeown 1975]. Numerous different approaches can be taken, from simple to very complicated [Mikhail 1976]. The Newton iteration was selected from among those available to find the 'optimal' solution $P(x, y, z)$.

A 'good' initial guess for $(\tilde{x}, \tilde{y}, \tilde{z})$ is obtained from the linear least squares method. A 'better guess' for z can be obtained by solving the constraint equations and using the weighted averaging described in Section 4.1.2.

The only case considered is the case for which $F_{\min} > 0$ and therefore $n > 3$. Differentiating (5.2) with respect to x yields

$$\frac{\partial F}{\partial x} = 2\sum_{i=1}^{n} f_i \frac{\partial f_i}{\partial x}. \tag{5.4}$$

The formulae for the partials with respect to y and z are similar. Introducing the vectors $\vec{f}, \vec{g}$ and the Jacobian matrix $\mathbf{J}$, leads to

$$\vec{g} = 2\mathbf{J}^T \vec{f}, \tag{5.5}$$

where

$$\mathbf{J} = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_n}{\partial x} & \frac{\partial f_n}{\partial y} & \frac{\partial f_n}{\partial z} \end{pmatrix}, \quad \vec{f} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}, \quad \vec{g} = \begin{pmatrix} \frac{\partial F}{\partial x} \\ \frac{\partial F}{\partial y} \\ \frac{\partial F}{\partial z} \end{pmatrix}. \tag{5.6}$$

Using the vector $\vec{R}$

$$\vec{R} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{5.7}$$

Newton iteration gives

$$\vec{R}_{\{k+1\}} = \vec{R}_{\{k\}} - (\mathbf{J}^T_{\{k\}}\mathbf{J}_{\{k\}})^{-1}\mathbf{J}^T_{\{k\}}\vec{f}_{\{k\}}, \tag{5.8}$$

where $\vec{R_{\{k\}}}$ denotes the kth approximate solution. The subscript $\{k\}$ in $\mathbf{J}$ and $\vec{f}$ means that these quantities are evaluated at $\vec{R_{\{k\}}}$. Obviously $\vec{R}_{\{1\}} = (\tilde{x}, \tilde{y}, \tilde{z})^T$.
Using the explicit form of the function $f_i(x, y, z)$ leads to

$$\mathbf{J}^T\mathbf{J} = \begin{pmatrix} \sum_{i=1}^{n} \frac{(x-x_i)^2}{(f_i+r_i)^2} & \sum_{i=1}^{n} \frac{(x-x_i)(y-y_i)}{(f_i+r_i)^2} & \sum_{i=1}^{n} \frac{(x-x_i)(z-z_i)}{(f_i+r_i)^2} \\ \sum_{i=1}^{n} \frac{(x-x_i)(y-y_i)}{(f_i+r_i)^2} & \sum_{i=1}^{n} \frac{(y-y_i)^2}{(f_i+r_i)^2} & \sum_{i=1}^{n} \frac{(y-y_i)(z-z_i)}{(f_i+r_i)^2} \\ \sum_{i=1}^{n} \frac{(x-x_i)(z-z_i)}{(f_i+r_i)^2} & \sum_{i=1}^{n} \frac{(y-y_i)(z-z_i)}{(f_i+r_i)^2} & \sum_{i=1}^{n} \frac{(z-z_i)^2}{(f_i+r_i)^2} \end{pmatrix}, \tag{5.9}$$

and

$$\mathbf{J}^T\vec{f} = \begin{pmatrix} \sum_{i=1}^{n} \frac{(x-x_i)f_i}{(f_i+r_i)} \\ \sum_{i=1}^{n} \frac{(y-y_i)f_i}{(f_i+r_i)} \\ \sum_{i=1}^{n} \frac{(z-z_i)f_i}{(f_i+r_i)} \end{pmatrix}. \tag{5.10}$$

In practice this type of iteration works fast, in particular when the matrix $\mathbf{J}^T\mathbf{J}$ is augmented by a diagonal matrix which effectively biases the search direction towards that of *steepest decent.* Levenberg and Marquardt [Lawson, and Hanson 1974] developed this improvement. As the solution is approached such modifications can be expected to have a decreasing effect.

5.2 *Test Data*

The x, y, and z coordinates of the linear least squares solution are used as the initial values of x, y, and z in (5.7), (5.8), (5.9), and (5.10).
At the start, and after every iteration the norm squared of the vector in (5.10) is calculated to determine if a nonlinear least squares iteration will improve the solution any further. If the

Tab. 5.1: Locations Calculated by Nonlinear Least Squares

Description		Initial Guess	1st Iteration	2nd Iteration	3rd Iteration	Error
$P_1(x,y,z)$	x	480000.000	480000.000	480000.000	480000.000	0.000
Exact	y	1093000.000	1093000.000	1093000.000	1093000.000	0.000
Distances	z	4667.687	4667.741	4668.012	4668.014	0.014
Norm Squared		0.000247	0.000169	0.000196	0.000109	
$P_1(x,y,z)$	x	480000.219	479999.969	479999.938	479999.938	-0.062
Approximate	y	1093000.875	1093000.375	1093000.125	1093000.125	0.125
Distances	z	4681.437	4678.745	4662.531	4663.899	-4.101
Norm Squared		3.614748	0.018587	0.002435	0.005657	
$P_2(x,y,z)$	x	480000.000	480000.000	480000.000	480000.000	0.000
Exact	y	1093000.000	1093000.000	1093000.000	1093000.000	0.000
Distances	z	4524.000	4524.442	4524.988	4524.990	-0.010
Norm Squared		0.002632	0.000661	0.000025	0.0000289	
$P_2(x,y,z)$	x	480000.250	480000.000	479999.938	479999.938	-0.062
Approximate	y	1093000.875	1093000.250	1093000.125	1093000.125	0.125
Distances	z	4538.375	4531.425	4523.331	4523.486	-1.514
Norm Squared		3.221449	0.018470	0.027271	0.029720	
$P_3(x,y,z)$	x	480000.000	480000.000	480000.000		0.000
Exact	y	1095500.000	1095500.000	1095500.000		0.000
Distances	z	4524.961	4524.973	4525.000		0.000
Norm Squared		0.000030	0.000013	0.000002		
$P_3(x,y,z)$	x	480000.219	479999.907	479999.938	479999.938	-0.062
Approximate	y	1095500.875	1095500.625	1095500.375	1095500.375	0.375
Distances	z	4539.039	4534.588	4526.040	4526.271	1.271
Norm Squared		2.952818	0.083713	0.020888	0.014418	

norm squared of (5.10) with $x = \tilde{x}$, $y = \tilde{y}$, and $z = \tilde{z}$ is less than or equal to the .00001, the initial guess $(\tilde{x}, \tilde{y}, \tilde{z})$ is accepted as the 'best' solution. Otherwise, the nonlinear least squares iterative procedure is implemented. The norm squared of (5.10) of each subsequent solution is tested to determine if an additional iteration is required to achieve the 'best' solution. There are exceptional cases when the norm squared cycles through several different values, and does not drop below a level of 0.00001. To avoid this situation a maximum of one hundred iterations are calculated in practice. Three of the iterations for each the three test points are listed in Table 5.1.

5.3 *Analysis*

The nonlinear least squares method gives the most accurate results of all methods developed and examined in this thesis, when approximate distances are involved in the calculations.

The nonlinear least squares solution procedure calculates the coordinates of the equipment within the required tolerance of 5.0 feet for both exact and approximate distances. The use of this method should be restricted to situations where the equipment in the mine is inside the perimeter of the beacons, and below the more or less common plane of the beacons. This method will provide results if these constraints are violated. The accuracy of the solution decreases as the elevation of the equipment increases, and as the equipment moves farther outside the perimeter of the beacons.

When exact distances were used, the nonlinear least squares solution technique calculated the x and y coordinates within a tolerance of 0.0 feet, and the z coordinate within a tolerance of 0.01 feet for the three test points.

When approximate distances were used, the calculated x and y coordinates were within a tolerance of 0.375 feet, and the z coordinate within a tolerance of 4.101 feet for the three test points.

The accuracy of the z coordinate, calculated with approximate distances, increases as the constraints are imposed. The z tolerance for the two points near the bottom of the mine was 1.514 feet.

5.4 *Robustness*

The complete theoretical analysis of the nonlinear least squares method and the underlying iteration process are beyond the scope of this thesis.

Further research at the PhD level must determine what the range is for the initial guess so that the iteration process converges (contraction principle); if the point of convergence is unique; and what the constraints are on the matrices and the columns for the iteration process to converge.

A comprehensive search of the literature for other trilateration methods must be performed to address some of the previous questions. Furthermore, a comparison of the nonlinear least squares method with other existing methods must be done.

6. CALIBRATION

A computerized calibration procedure was used to test the accuracy of the various solution techniques presented in this thesis. This standardized calibration procedure makes it possible to compare the effectiveness of different solution techniques. If the same beacon placement pattern is used with different solution procedures, the results of this calibration procedure can be compared to determine which one produces the most accurate results.

The effects of various beacon placement patterns on the accuracy of the positioning solution can also be tested with this calibration procedure. This capability will allow TBCC to test the effectiveness of proposed beacon placement patterns, without having to physically install beacons in the mine. Use of this calibration procedure will help to ensure that the position which is calculated by their electronic positioning system is within the stated tolerance range.

6.1 Procedure

The first step in the computerized calibration procedure is to establish a rectangular solid which encompasses the mine. This rectangular solid is formed in the following manner. The easting, northing, and elevation of the proposed beacon locations is manually input into the calibration program. These coordinates are then electronically sorted to determine the maximum and minimum northing and easting, and the minimum beacon elevation of the beacons. These extreme values are then used to establish a rectangular solid test area. The sides of this rectangular solid are formed from planes that intersect the beacons which contain the extreme northing and easting values. These planes have north-south, and east-west orientations. The top of the solid is formed from a horizontal plane at a specified elevation. This plane is below the beacon which has the minimum elevation. In this thesis, the distance between the horizontal plane and the lowest beacon is two feet. The bottom of the solid is formed from the horizontal plane at the elevation that is considered to be the bottom of the mine. This specified elevation is manually input into the computer.

The next step in the calibration procedure is to superimpose a three dimensional grid on the rectangular solid. This grid is formed by passing planes through the rectangular solid at evenly spaced intervals. These planes have north-south, east-west, and horizontal orientations, which are parallel to the boundaries of the rectangular solid. When three of these planes intersect, they intersect at a point. Each of these points of intersection is used to test the accuracy of a specific solution procedure for the corresponding beacon placement pattern. A representative rectangular solid with a superimposed grid is shown Figure 6.1.

After the grid of test points is established, the various solution procedures are tested in the following manner. The exact distances between the test points and the beacons are calculated. Uniform errors that are within the specified tolerance are then added to these distances. The position of each point is then calculated using the approximate distances. The coordinates of the

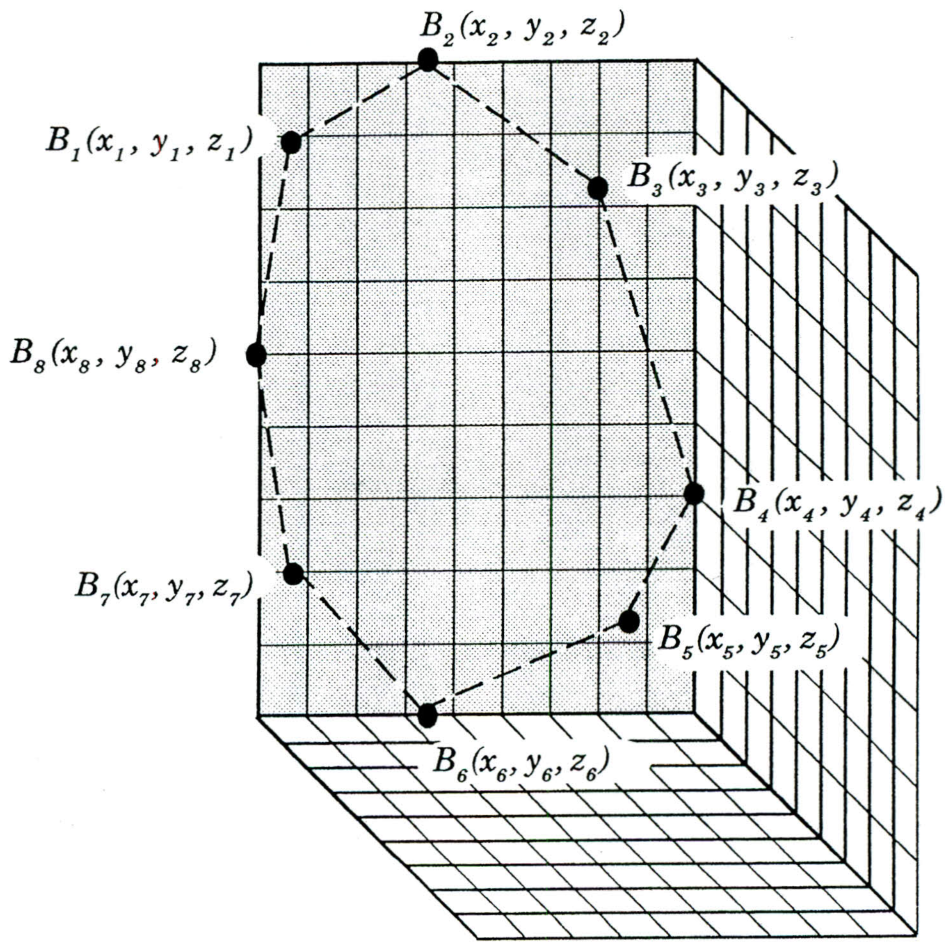

Fig. 6.1: Rectangular Solid Formed by the Calibration Procedure

original exact location of each test point, and the coordinates of the location which is calculated from the approximate distances are then compared. Ideally, these coordinates should equal each other. The solution procedure is acceptable if the differences between these coordinates is within the specified tolerance of 5.0 feet. The MACSYMA version of this calibration program is in Appendix .1.

6.2 Test Data

The number of locations out of 1000 that are not within the 5.0 foot tolerance for the various solution methods discussed in this thesis are listed in Table 6.1.

Tab. 6.1: Summary of Locations Out of Tolerance Calculated with Various Solution Techniques

	Number of Calculated Locations Out or 1000 That Are Not Within a Tolerance of 5.0 Feet	
Method	Exact Distances	Approximate Distances
Linearized Equations *	0	919
Linear Least Squares	0	856
Arithmetic Average	0	225
Weighted Average	0	142
Nonlinear Least Squares	0	81

* This is a smaller test area than the other cases.

Sample output from the C version of the calibration program is as follows:

```
At Point 1,1,1
Easting   =   467399.711870  Correct =   467400.0  Off = -0.288130
Northing  =  1087809.923138  Correct =  1087810.0  Off = -0.076862
Elevation =     4683.241594  Correct =     4668.0  Off = 15.241594
Out of Tolerance
```

The values labeled 'Correct' represent the actual coordinates of the test point. The 'Off' values represent the 'error.' These values are the differences between the actual coordinates and the calculated coordinates. Point$(1, 1, 1)$ refers to the location of a point that is based on a record keeping system which refers to a point in the test grid. The coordinates of the record keeping system are defined as Point(easting, northing, elevation). The starting point of the record keeping system is south-west corner of the test grid, at the highest tested elevation. Thus, Point$(1, 1, 1)$ is in the south-west corner of the mine. Point$(10, 10, 10)$ refers to the point in the north-east corner of the test grid, at the lowest elevation tested.

The 81 points that are not within the required tolerance for the nonlinear least squares method because the z coordinates were off more than 5.0 feet. These points are plotted in Figures 6.2, 6.3, 6.4, 6.5, 6.6, and 6.7. Each of these figures corresponds to an elevation level in the mine where the z coordinate was off more than five feet. The perimeter of the beacons is also plotted at each elevation.

We conclude from these figures, that the accuracy of the calculated position decreases as the equipment location moves farther outside the perimeter of the beacons; and as the elevation of the equipment in the mine increases. The worst results are near the top of the mine. In practice the beacons could be installed on poles, or on the highest points on the rim of the mine.

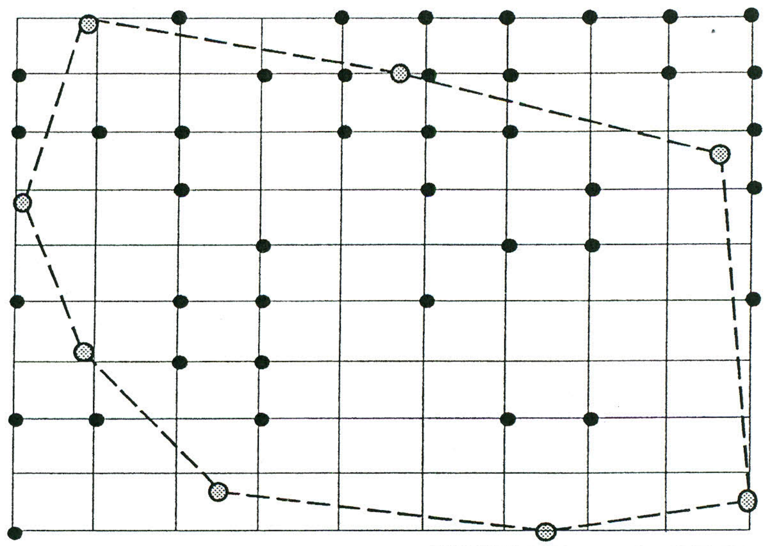

Fig. 6.2: Locations of Calculated Positions at Elevation 4668 (2 Feet Below Lowest Beacon) that are Not Within a Tolerance of 5.0 Feet

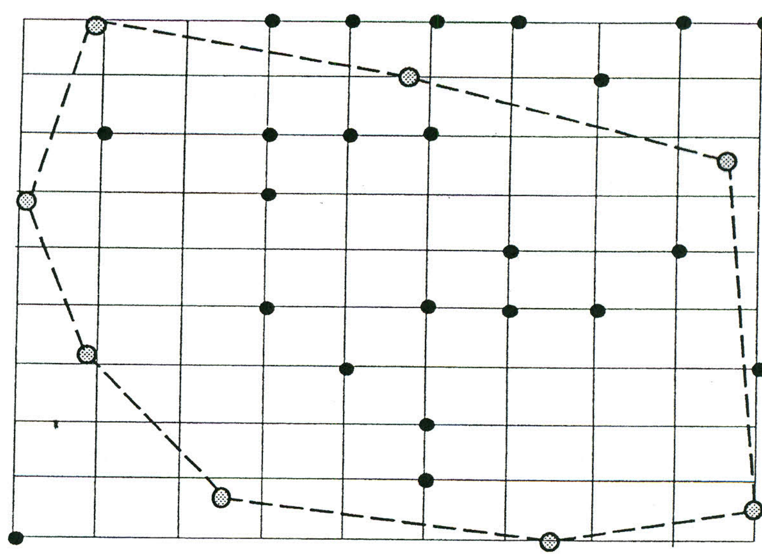

Fig. 6.3: Locations of Calculated Positions at Elevation 4601.3 (68.7 Feet Below Lowest Beacon) that are Not Within a Tolerance of 5.0 Feet

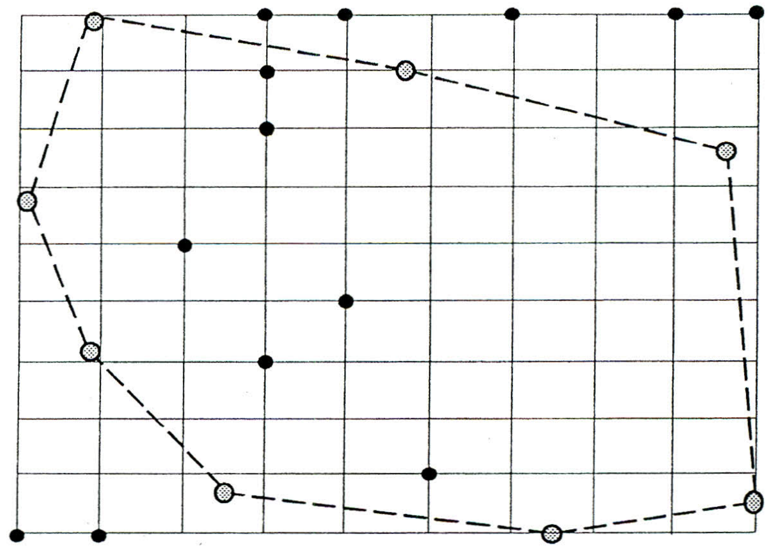

Fig. 6.4: Locations of Calculated Positions at Elevation 4534.6 (135.4 Feet Below Lowest Beacon) that are Not Within a Tolerance of 5.0 Feet

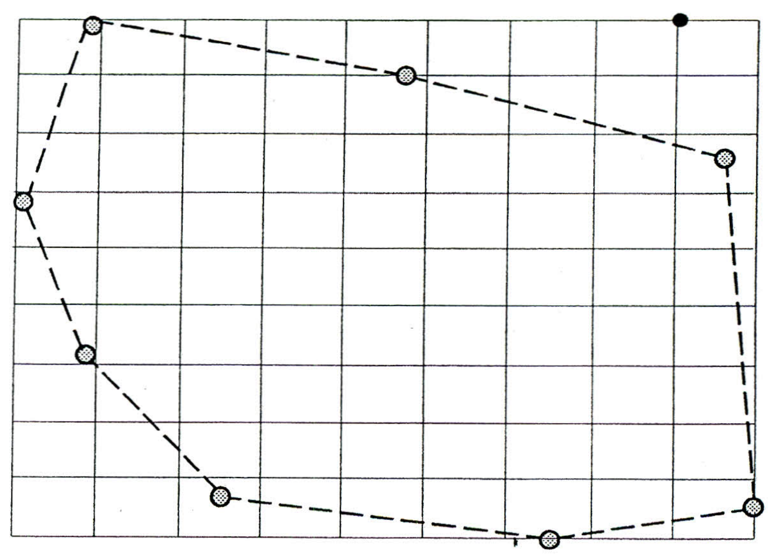

Fig. 6.5: Locations of Calculated Positions at Elevation 4468 (200 Feet Below Lowest Beacon) that are Not Within a Tolerance of 5.0 Feet

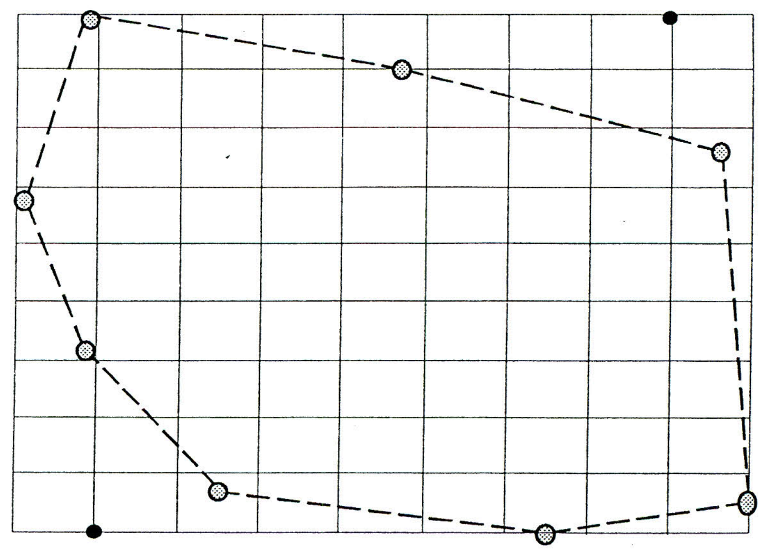

Fig. 6.6: Locations of Calculated Positions at Elevation 4401.3 (268.7 Feet Below Lowest Beacon) that are Not Within a Tolerance of 5.0 Feet

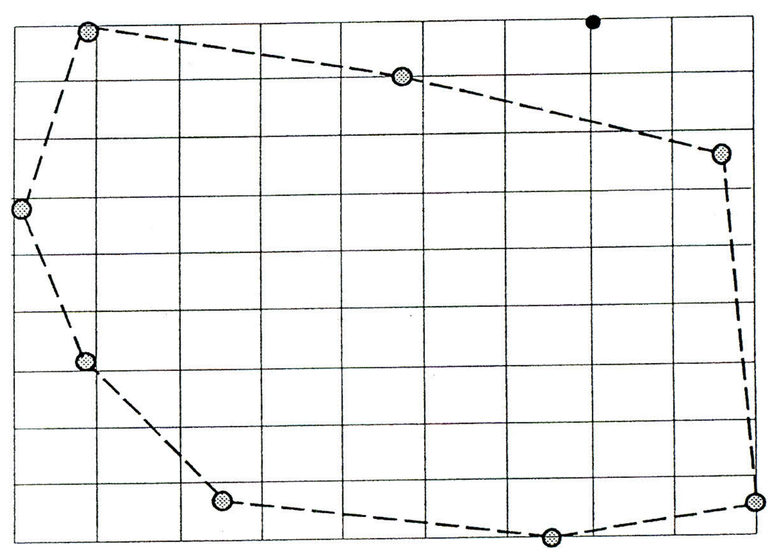

Fig. 6.7: Locations of Calculated Positions at Elevation 4268 (400 Feet Below Lowest Beacon) that are Not Within a Tolerance of 5.0 Feet

7. CONCLUSION

The mathematical solution of the trilateration positioning problem posed by Thunder Basin Coal Company, was presented in this thesis. This solution will be used in mining applications to determine the position of a piece of equipment in the mine in three dimensional space using trilateration with approximate distances.

The nonlinear least squares method gives the most accurate results of all methods developed and examined in this thesis, when approximate distances were used in the calculations. The nonlinear least squares solution procedure calculates the exact position when exact distances are known, and reasonably accurate answers when inaccurate distances are known. Although this method is restricted to applications where the elevation of the unknown position is below the elevation of the lowest beacon, this method will provide results if this constraint is violated. The accuracy of the solution is degraded when the elevation of the unknown position is close to or above the elevation of the lowest known position, and when the unknown position is outside the perimeter of the beacons.

Symbolics Computer Aided Mathematics Group. 1988. *MACSYMA Reference Manual Version 13.* Cambridge: Symbolics, Inc.

DeLoach, B.A. 1963. *Error Analysis of a Trilateration Network With and Without the Laplace Condition.* Ohio State University: Masters Thesis.

Grossman, W. 1969. *Grundzüge der Ausgleichungsrechnung nach der Methode der Kleinsten Quadrate nebst Anwendung in der Geodesy.* Berlin: Springer Verlag.

Kahaner, D., C. Moler, and S. Nash. 1989. *Numerical Methods and Software.* Englewood Cliffs, NJ: Prentice Hall.

Lawson, C.L., and R.I. Hanson 1974. *Solving Least Squares Problems.* Englewood Cliffs, NJ: Prentice Hall.

Manning, P.E. 1979. *Tests of an ATCRBS Based Trilateration Sensor at Logan International Airport.* Campridge, MA: U.S. Department of Transportation Federal Aviation Administration. FAA-RD-79-115 (November).

McKeown, J.J. 1975. Specialized Versus General-Purpose Algorithms for Minimizing Functions that are Sums of Squared Terms. *Mathematical Programming* **9**: 57-68.

Merchant, D.C. 1981. *Passive Trilateration Positioning Procedure Error Analysis.* Griffis Air Force Base, NY: Rome Air Development Center. RADC-TR-81-225 (August).

Mikhail E.M. 1976. *Observations and Least Squares.* New York: IEP-Dun-Donnelley, Harper and Row.

Moler, C., J. Little, S. Bangert, and S. Kleiman. 1989. *PC-MATLAB for MS-DOS Personal Computers.* Sherborn, MA: The MathWorks, Inc.

Noble, B. and J.W. Daniel. 1988. *Applied Linear Algebra.* Englewood Cliffs: Prentice-Hall.

Remondi, B.W. 1991. Artificial Constellations, The Global Positioning System. *The Military Engineer* 84 (545): 31-36.

APPENDIX

CALIBRATION PROGRAM IN MACSYMA

```
/* Save the output in the following file                          */
writefile("nout100.out")$

/* MACSYMA command                                                */
ratprint:false$

/* The beacon locations are specified here.                       */
/* These beacons were specified as test data by TBCC              */

/* Beacon 1                                                       */
x[1]:   475060.0$       /* X Coordinate of Beacon 1               */
y[1]:  1096300.0$       /* Y Coordinate of Beacon 1               */
z[1]:     4670.0$       /* Z Coordinate of Beacon 1               */

/* Beacon 2                                                       */
x[2]:   481500.0$
y[2]:  1094900.0$
z[2]:     4694.0$

/* BEACON 3                                                       */
x[3]:   482230.0$
y[3]:  1088430.0$
z[3]:     4831.0$

/* BEACON 4                                                       */
x[4]:   478050.0$
y[4]:  1087810.0$
z[4]:     4775.0$

/* BEACON 5                                                       */
x[5]:   471430.0$
y[5]:  1088580.0$
z[5]:     4752.0$

/* BEACON 6                                                       */
x[6]:   468720.0$
y[6]:  1091240.0$
z[6]:     4803.0$

/* BEACON 7                                                       */
x[7]:   467400.0$
y[7]:  1093980.0$
z[7]:     4705.0$
```

```
/* BEACON 8                                                          */
x[8]:   468730.0$
y[8]:  1097340.0$
z[8]:     4747.0$

/* n is the Number of Beacons.                                       */
n:8$

/* Relative coordinates are being calculated.                        */
for i: 2 thru n do(
   k[i,1]: ev(x[i]-x[1],numer),
   k[i,2]: ev(y[i]-y[1],numer),
   k[i,3]: ev(z[i]-z[1],numer),
   sqd[i,1]: ev(k[i,1]^2+k[i,2]^2+k[i,3]^2),numer)$

/* These are the starting coordinates.  The start point is at the    */
/* south-west corner of test grid, 2 feet below the lowest beacon.   */
xstart:  475060$
ystart: 1087810$
zstart:    4668$


/* (np)^3 is the number of test points.                              */
/* With np=10 we will have 1000=10*10*10 test points.                */
np:10$

/*  This routine fills an array with exact beacon locations;         */
/*  calculates the exact distances to "n" beacons; adds distance     */
/*  errors; calculates locations; and calculates location errors.    */

for i:1 thru np do(
   for j:1 thru np do(
      for k:1 thru np do(

 print("For Test Point [",i,",",j,",",k,"]"),

 /* The test point is being specified here.                     */
 /* This procedure tests the 1000 point in sequence.            */
 xbulact[i,j,k]: ev(xstart + (i-1) * (797/(np-1)),numer),
 ybulact[i,j,k]: ev(ystart + (j-1) * (8000/(np-1)),numer),
 zbulact[i,j,k]: ev(zstart - (k-1) * (600/(np-1)),numer),

 /* The exact distances are being calculated.                   */
 for m:1 thru n do(
```

```
   r[i,j,k,m]: ev(sqrt((x[m]-xbulact[i,j,k])^2
      +(y[m]-ybulact[i,j,k])^2+(z[m]-zbulact[i,j,k])^2),numer)),

/* Uniform errors are added to the exact distances.         */
err[1]: -0.457890,
err[2]:  0.173050,
err[3]:  0.316931,
err[4]: -0.191205,
err[5]:  0.468339,
err[6]:  0.141141,
err[7]:  0.328659,
err[8]: -0.390460,
for m:1 thru n do(
   r[i,j,k,m]:r[i,j,k,m]+err[m]),


/* LINEAR LEAST SQUARES */

/* A and b are being calculated here.                        */
for m: 2 thru n do(
   b[m,1]: ev((1/2)*(r[i,j,k,1]^2-r[i,j,k,m]^2+sqd[m,1]),numer)),

a: ev(matrix([k[2,1],k[2,2],k[2,3]],[k[3,1],k[3,2],k[3,3]],
    [k[4,1],k[4,2],k[4,3]],[k[5,1],k[5,2],k[5,3]],
    [k[6,1],k[6,2],k[6,3]],[k[7,1],k[7,2],k[7,3]],
    [k[8,1],k[8,2],k[8,3]]),numer),

b: ev(matrix([b[2,1]],[b[3,1]],[b[4,1]],[b[5,1]],[b[6,1]],
    [b[7,1]],[b[8,1]]),numer),

/* The transpose of A times A  (traa); and                   */
/* the transpose of A times B  (trab) are calculated         */
tra:transpose(a),
traa: tra.a,
trab: tra.b,

/* The solution of the linear least squares                  */
/* from the relative positions is calculated.                */
solx : ev(((traa)^^(-1)).trab,numer),

/* Results obtained with the induced random errors are       */
/* obtained by adding the coordinates of the first beacon    */
xbul: ev(solx[1,1]+x[1],numer),
ybul: ev(solx[2,1]+y[1],numer),
zbul: ev(solx[3,1]+z[1],numer),
```

```
 /* The location errors are calculated by subtracting the    */
 /* test coordinates from the calculated coordinates.        */
 /* The calculated location and the error are printed        */
 print("x linear lsq =",xbul,"xerr",xbul-xbulact[i,j,k]),
 print("y linear lsq =",ybul,"yerr",ybul-ybulact[i,j,k]),
 print("z linear lsq =",zbul,"zerr",zbul-zbulact[i,j,k]),


/* ARITHMETIC AVERAGE */
/* WEIGHTED AVERAGE   */


 /* azbul, the arithmetic average of the z is calculated.   */
 /* wzbul, the weighted average of the z, using a weight    */
 /* factor of f = 1/r, is also calculated.                  */
 /* These calculations of z are completed for each beacon   */
 /* for the case of the negative sign after x and y are     */
 /* determined and fixed. The imaginary parts are not used. */

 acounter:0,
 zbulsum:0,
 wcounter:0,
 wzbulsum:0,

 for m:1 thru n do(
    zbul[m]:ev( z[m]-sqrt( (r[i,j,k,m])^2
    -(y[m]-ybul)^2-(x[m]-xbul)^2 ),numer),

    if imagpart(rectform(zbul[m]))=0 then (
       acounter:acounter+1,
       zbulsum:zbulsum+zbul[m],
       f: ev(1/r[i,j,k,m],numer),
       wcounter:wcounter+f,
       wzbul[m]:zbul[m]*f,
       wzbulsum:wzbulsum+wzbul[m]) ),

 azbul:ev(zbulsum/acounter,numer),
 print("The arithmetic average of real z's is ",azbul),
 print("The error is ",azbul-zbulact[i,j,k]),
 asolxx:matrix([xbul],[ybul],[azbul]),
 print("The arithmetic average solution is",asolxx),

 wzbul:ev(wzbulsum/wcounter,numer),
 print("The weighted average of real z's is ",wzbul),
```

```
 print("The error is ",wzbul-zbulact[i,j,k]),
 wsolxx:matrix([xbul],[ybul],[wzbul]),
 print("The weighted average solution is",wsolxx),


/* NONLINEAR LEAST SQUARES */

 /* Start the iterations with the solution of              */
 /* the linear least squares method.                       */
 solxx:matrix([xbul],[ybul],[zbul]),

 itercounter:0,

 /* Specify the maximum number of iterations.              */
 numiter:100,

 for p:1 while itercounter<numiter do(
    xit:solxx[1,1],
    yit:solxx[2,1],
    zit:solxx[3,1],

    for m:1 thru n do(
    dit[m]:sqrt( (xit-x[m])^2+(yit-y[m])^2+(zit-z[m])^2 ),
    sqdit[m]:(xit-x[m])^2+(yit-y[m])^2+(zit-z[m])^2 ),

    for m:1 thru n do(
       v[i,j,k,m]:dit[m]-r[i,j,k,m] ),

bb1:ev(sum(v[i,j,k,m]*(xit-x[m])/dit[m],m,1,n),numer),
bb2:ev(sum(v[i,j,k,m]*(yit-y[m])/dit[m],m,1,n),numer),
bb3:ev(sum(v[i,j,k,m]*(zit-z[m])/dit[m],m,1,n),numer),

    /* Test for deciding if iterations are complete          */
    normbbsq:ev(bb1^2+bb2^2+bb3^2,numer),
    print("This is the norm squared of column bb",normbbsq),
    if normbbsq > 0.00001 then (

       bb:matrix([bb1],[bb2],[bb3]),
       aa11:ev(sum((xit-x[i])^2/sqdit[i],i,1,n),numer),
       aa12:ev(sum(((xit-x[i])*(yit-y[i]))/sqdit[i],i,1,n),numer),
       aa13:ev(sum(((xit-x[i])*(zit-z[i]))/sqdit[i],i,1,n),numer),
       aa23:ev(sum(((zit-z[i])*(yit-y[i]))/sqdit[i],i,1,n),numer),
       aa22:ev(sum((yit-y[i])^2/sqdit[i],i,1,n),numer),
       aa33:ev(sum((zit-z[i])^2/sqdit[i],i,1,n),numer),
       aa:matrix([aa11,aa12,aa13],[aa12,aa22,aa23],
```

```
 [aa13,aa23,aa33]),

      invaa: invert(aa),
      solxx:ev(solxx-invaa.bb,numer),

      print("Nonlinear least squares solution after the",
      p,"-th iteration: ", solxx),

      print("x error ",solxx[1]-xbulact[i,j,k]),
      print("y erorr ",solxx[2]-ybulact[i,j,k]),
      print("z error ",solxx[3]-zbulact[i,j,k]),

      itercounter:itercounter+1)

   else (itercounter:numiter))

/* The iteration is complete for that test point.          */
     )
   )
)$
```