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**Symbolic Computation of Exact Solutions
of Nonlinear Partial Differential Equations
Using Direct Methods**

by

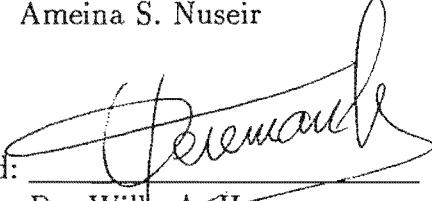
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Abstract

In this study, two methods for finding solitary-wave and soliton-solutions are presented and applied to a variety of nonlinear partial differential equations (PDEs). The solution techniques are: the truncated Painlevé expansion method, and a simplified version of Hirota's method. The truncated Painlevé expansion method is used to find the closed forms of solitary-wave solutions of several nonlinear evolution equations. The simplified version of Hirota's method, developed in the thesis, is used to explicitly construct one-, two-, and three-soliton solutions of completely integrable PDEs. The simplified version has the advantage over the original version of Hirota's method in that the bilinear forms of the PDEs are no longer needed. Exact solutions of PDEs can be constructed systematically by solving a perturbation scheme on the computer, using any symbolic manipulation program.

Using the simplified version of Hirota's method, the explicit form of the soliton solutions of the Kaup-Kupershmidt equation is found, in addition to multi-soliton solutions for various other PDEs. Finally, the relation between the admissibility of infinitely many conservation laws, the complete integrability and the existence of soliton solutions of PDEs is investigated.

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IN THE NAME OF ALLAH, THE MOST BENEFICENT, THE MOST MERCIFUL

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Dedication

**TO MY PARENTS, HUSBAND, SON,
BROTHERS AND SISTERS**

Chapter 1

Introduction

Nonlinear evolution and wave equations are partial differential equations (PDEs) involving first- or second-order derivatives with respect to time. Such equations have been intensively studied for the past decades [1, 2], and several new methods to solve nonlinear PDEs, either numerically or analytically, are now available.

When the dependent variable u in the PDE corresponds to a physical quantity (such as the surface height of a water wave, the magnitude of an electromagnetic wave, etc.), it is important to study the propagation or aggregation properties of u . This motivates the study of methods to analytically solve evolution or wave equations via symbolic methods. The goal is to find exact traveling wave solutions. If these solutions do not change their form during propagation, they are called *solitary waves*. Solitary waves that preserve their shape upon collision are called *solitons* [3]. Solitary-waves and solitons arise due to a critical balance between dispersion and nonlinearity.

Due to the complexity of the mathematics involved in finding exact solutions for these PDEs, the use of algorithmic techniques that can be implemented in the

symbolic language of computer algebra systems becomes a necessity. Several computer algebra packages now exist to aid in the study of nonlinear PDEs [4, 5, 6]. For example, Painlevé analysis offers an algorithm for testing whether or not a PDE is a good candidate to be completely integrable. In addition, the Painlevé method allows one to construct solitary wave solutions in explicit form. A more powerful technique is Hirota's bilinear method [7] which allows one to find N-soliton solutions of large classes of completely integrable PDEs [8].

The story of the first observation of solitary waves is worth telling. In 1834, while riding horseback beside the narrow Union canal near Edinburgh in Scotland, J. Scott Russell noticed that a bow wave, rolling away from a large barge, travelled as a huge heap of water for quite a long distance before finally dispersing into smaller ripples. In order to study this intriguing phenomenon, Russell did extensive experiments in a large water tank. Further investigations of solitary waves were done by Airy, Stokes, Boussinesq, and Rayleigh in an attempt to understand the mechanism behind this remarkable phenomenon [9]. The latter two scientists derived approximate models to describe solitary waves. In order to obtain his result, Boussinesq derived a one-dimensional nonlinear wave equation which now bears his name.

The issue was finally resolved (in 1895) by two Dutchmen, Korteweg and de Vries, when they derived a nonlinear evolution equation governing long, one-dimensional surface gravity waves (with small amplitude) propagating in shallow

water:

$$\frac{\partial \eta}{\partial \tau} = \frac{3}{2} \sqrt{\frac{g}{h}} \frac{\partial}{\partial \xi} \left(\frac{1}{2} \eta^2 + \frac{2}{3} \alpha \eta + \frac{1}{3} \sigma \frac{\partial^2 \eta}{\partial \xi^2} \right), \quad (1.1)$$

$$\sigma = \frac{1}{3} h^3 - \frac{Th}{\rho g}, \quad (1.2)$$

where η is the surface elevation of the wave above the equilibrium level h , α is a small arbitrary constant related to the uniform motion of the liquid, g is the gravitational constant, T is the surface tension, and ρ is the density. The independent variables τ and ξ are scaled versions of the time and space coordinates. Equation (1.1), which is called the Korteweg-de Vries (KdV) equation, can be brought into a non-dimensional form via the change of variables

$$t = \frac{1}{2} \sqrt{\frac{g}{h\sigma}} \tau, \quad x = -\sigma^{-\frac{1}{2}} \xi, \quad u = \frac{1}{2} \eta + \frac{1}{3} \alpha. \quad (1.3)$$

After some algebra, one obtains

$$u_t + 6uu_x + u_{3x} = 0, \quad (1.4)$$

where subscripts denote partial derivatives, e.g. $u_{3x} = \frac{\partial^3 u}{\partial x^3}$.

Despite this early derivation of the KdV equation, it was not until 1960 that any new applications of the equation were discovered [9]. In 1960, while studying collision-free hydrodynamic waves, Gardner and Morikawa rediscovered the KdV

equation [10]. Amazingly, the KdV equation started to show up in a number of other physical contexts such as the study of stratified internal waves, ion-acoustic waves in plasma physics, lattice dynamics, and so on (further details can be found in Jeffrey and Kakutani [11], Scott *et al.* [12], Miura [13], Ablowitz and Segur [14], Lamb [15], Calogero and Degasperis [16], Dodd *et al.* [17], and Novikov *et al.* [18]).

Since the late 1960's, the study of the properties of solitons, and the search for solitonic equations and methods to solve them, has been an active and exciting area of research. In regards to formal mathematical approaches that are close to what I will use in this thesis, Sawada and Kotera [19], Rosales [20], Whitham [21], Wadati and Sawada [22, 23], and Hickernell [24] have all employed perturbation-type techniques. Their methods lead, after some tedious algebraic manipulations, to the N-soliton solutions of the KdV, modified KdV (mKdV), Burgers', and Boussinesq equations. Their iterative procedures are closely connected to Hirota's direct method [25] for finding multi-soliton solutions. Taflin [26] and Santini *et al.* [27] presented a rigorous mathematical analysis of the problem of direct linearization of nonlinear PDEs.

Finally, one can add yet another method to the arsenal of techniques for finding solitary-wave solutions: the truncated Painlevé expansion method. This analysis is named after Painlevé, who investigated the pole behavior of solutions to classify integrable ordinary differential equations (ODEs) [28]. Ablowitz, Ramani and Segur

[29], and McLeod and Olver [30] showed how useful Painlevé analysis can be to investigate the integrability of nonlinear ODEs. Later Weiss, Tabor and Carnevale [31] extended the technique to nonlinear PDEs and systems of nonlinear PDEs.

A PDE is said to have the Painlevé property when its solutions (in the complex plane) are single-valued about a movable, singularity manifold [32]. Ablowitz, Ramani, and Segur developed an algorithm that determines whether or not an ODE has the Painlevé property. The formulation of the Painlevé test for PDEs is due to Weiss, Tabor, and Carnevale. They showed that the test can be directly applied to PDEs without any need for reductions to ODEs. One should emphasize that the Painlevé property is a sufficient condition for the integrability of PDEs. Counter-examples, such as the Dym-Kruskal equation [33], show that it is not a necessary condition.

Since its introduction, Painlevé analysis has been heavily used by mathematicians for finding exact solutions for nonlinear PDEs. A search in the literature revealed that the Painlevé analysis has been used to find exact solutions for the two-dimensional Burgers' equation [34], the KdV-Burgers equation [35], the shallow water wave equation [36], the Kolmogorov-Petrovsky-Piskunov equation [37], and the Fisher equation [38]. On the other hand, Shivamoggi used Painlevé analysis to establish the integrability of the Zakharov-Kuznetsov equation [39].

Quite often, solutions obtained via Painlevé analysis can also be obtained via other techniques; in particular, symmetry group methods and the Hirota method.

Symmetry groups have been used in several different applications in the context of nonlinear differential equations. For example, one can derive new, more interesting solutions of differential equations by applying the symmetry group of a differential equation to known (often trivial) solutions. At the same time, group theoretical techniques are used to reduce the total number of dependent and independent variables of a PDE. For example, a PDE with one dependent and two independent variables can then be reduced to an ODE.

In the early stages of this research project, I spent considerable time trying to find equations for which the so-called nonclassical symmetry method produces reductions other than those obtained by the classical method due to Sophus Lie [40]. The goal was to obtain exact solutions via group theoretical methods, based on the reductions of PDEs to ODEs. Equations such as the Fisher equation with convection, the Kuramoto-Sivashinsky equation, and a general class of 5th-order evolution equations, $u_t + \alpha u u_{3x} + \beta u_x u_{2x} + \gamma u^2 u_x + u_{5x} = 0$, are among the many examples I tried to solve via symmetry reductions.

Since the search for new solutions via group analysis did not lead to many new results, I looked for related and much more direct methods to construct solitary and soliton solutions. The connections between symmetry techniques and two direct methods (one based on Painlevé analysis, the other on Hirota's bilinear transform) [41, 42, 43] encouraged me study these methods, refine them, and apply them to

several nonlinear PDEs. Initially, I concentrated my efforts on finding exact solutions for nonlinear PDEs using the truncated Painlevé expansion method. To broaden the scope of applicability of this method, I later designed a simplified version of Hirota's method and successfully applied it to a class of fifth-order PDEs.

In Chapter two, the truncated Painlevé expansion method is introduced and applied for finding solitary wave solutions of integrable and non-integrable PDEs. These PDEs include the Kuramoto-Sivashinsky equation, and the Fisher equation with convection, for which solitary wave solutions were constructed. For a non-linear reaction-diffusion equation, I obtained two solutions: one of them is finite as $x \rightarrow \infty$, and the other is finite as $x \rightarrow -\infty$. With a tanh-method, Malfliet [44] was able to find the solution that remains bounded when $x \rightarrow \infty$. The truncated Painlevé method covers both cases. Another example is the two-dimensional KdV-Burgers equation, for which a solitary wave solution is constructed. For a class of nonlinear dissipative-dispersive equation, I got three solutions, two are new, the remaining one was derived previously by Malfliet and Hereman [45]. I also solved an evolution equation due to Calogero [46]. Again, a new solitary wave solution was obtained. Finally, I studied a general class of evolution equations of type $u_t = (u^n)_{xx} + f(u)$. I investigated under which conditions the truncated Painlevé method can be applied. Exact solutions have been constructed for some specific equations in that class.

In Chapter three, a simplified version of Hirota's method is presented. This

method, which is developed in this thesis, has the advantage that it eliminates one major requirement of the original Hirota method. Indeed, in the simplified version, the knowledge of the bilinear form of the PDE is no longer needed. The job of finding soliton solutions is then easier and more efficient. Furthermore, without bilinear forms, the Hirota method is algorithmic, and a computer can be used in every step of the calculation of the solution. The simplified version of the Hirota method, although more elaborate than the truncated Painlevé expansion method, has a larger scope. With Hirota's technique multi-soliton solutions can be constructed, whereas the truncated Painlevé expansion method only leads to simple solitary wave solutions.

As a matter of introduction of the method, I used it to reconstruct already known soliton solutions for familiar integrable and non-integrable equations. Among these equations are the ubiquitous KdV equation for which the N-soliton solution is rederived, and the non-integrable Fisher equation with convection for which a solitary wave solution is reconstructed. Note that the original Hirota method could not be applied to this case because that equation does not have a bilinear form.

In Chapter Four, I apply the two methods presented in the previous two chapters to a general class of 5th-order evolution equations, $u_t + \alpha u u_{3x} + \beta u_x u_{2x} + \gamma u^2 u_x + u_{5x} = 0$. The well-known Sawada-Kotera (SK), Lax, Kaup-Kuperschmidt (KK), and Ito equations are all members of this class. In the first Section of Chapter four, the truncated Painlevé expansion method is used to find exact solutions for this class. As

expected, some constraints for the parameters α, β and γ in the general class must be satisfied in order to get solitary wave solutions. The solution I finally obtained turned out to solve the SK and Lax equations, but not the KK and Ito equations. After making small changes in the transformations, I applied the truncated Painlevé expansion method directly to the KK and Ito equations, for which I obtained solitary wave solutions.

In Section two of Chapter four, I substitute a two-soliton solution of KdV-type into the general class to test for specific equations for which this form of solution might be suitable. I found that only the SK and Lax equations have this type of solution. In Section three, the simplified version of Hirota's method has been applied to subcases of the general class. It turned out that this method is successful for finding a three-soliton solution for the SK, Lax, and the KK equations. Using the simplified version of Hirota's method, I was able to construct the two-soliton and three-soliton solutions for the Kaup-Kuperschmidt equation which, to my knowledge, have never been derived before. Although the Kaup-Kuperschmidt equation is known to be completely integrable [47], the computation of soliton solutions is very complicated and lengthy. Without a symbolic manipulation program the computations could not be done. A main advantage of the simplified version of Hirota's method is that it is algorithmic and that the knowledge of the bilinear representation of the equation is superfluous. Therefore, the method can be applied to equations for which the bilinear

form is not known. The drawback of the method is that it involves a large amount of tedious algebra and calculus. In nearly all cases the use of a computer algebra package will be necessary to carry out the lengthy but straightforward computations.

It is known that if a nonlinear PDE admits multiple-soliton solutions, it is solvable by the Inverse Scattering Transform, and it possesses an infinite number of conservation laws [9]. The one-to-one correspondence between the infinitesimal transformations (symmetries) and the conservation laws was reported and proved by Wadati [48]. In Chapter five I use these connections to confirm that the special cases in the general class of 5th-order evolution equations, for which I obtained soliton solutions before, are indeed completely integrable. In particular, I studied the conservation laws of that class of equations to find out which special equations could be solitonic, that means equations which admit soliton solutions.

The calculation of a set of conservation laws for the Lax, the Sawada-Kotera, the Kaup-Kuperschmidt, and Ito equations, which all belong to this class of 5th-order equations, was carried out explicitly with a MATHEMATICA program developed by Hereman and Göktas [49]. Based on the computation of a sufficiently large number of conservation laws, one gets a good indication that, with the exception of the Ito equation, these special equations are indeed completely integrable. The computation of their exact two- and three-soliton solutions in the previous chapter confirmed that conjecture. For these well-known examples, it was therefore explicitly verified that

a PDE having an N-soliton solution has a large (in principle infinite) number of conservation laws.

Chapter 2

Truncated Painlevé Expansion

In this chapter the truncated Painlevé expansion method will be described and applied to some nonlinear PDEs to find their closed-form solutions. The examples presented reflect how easy and fast this method can be, especially when a computer algebra package is used to perform the actual calculations. In addition to simplicity, this method is very effective for solving integrable nonlinear PDEs that admit soliton solutions, as well as some of the non-integrable ones with solitary wave solutions.

The truncated Painlevé expansion is a powerful method for finding exact solutions of nonlinear PDEs, provided these equations have the Painlevé property. A PDE has the Painlevé property when the solutions of the PDE are ‘single-valued’ about a singularity manifold, and have no worse singularities than movable poles. More precisely, the singularity manifold is given by $g(z_1, \dots, z_n) = 0$, where g is an analytic function of (z_1, \dots, z_n) . I will assume that a solution $u = u(z_1, \dots, z_n)$ of the

PDE can be represented in a Laurent series in g :

$$u(z_1, \dots, z_n) = g^\alpha(z_1, \dots, z_n) \sum_{k=0}^{\infty} u_k(z_1, \dots, z_n) g^k(z_1, \dots, z_n), \quad (2.1)$$

where α is an integer, $g = g(z_1, \dots, z_n)$, and $u_k = u_k(z_1, \dots, z_n)$ are analytic functions of (z_1, \dots, z_n) in the neighborhood of the manifold $g(z_1, \dots, z_n) = 0$. Substitution of (2.1) into the PDE, and balancing the various power terms in g allows the determination of the possible value(s) of α , and also defines the recursions relations for the coefficients u_k , $k = 0, 1, 2, \dots$ [50].

A special truncated Painlevé expansion is obtained by cutting the series (2.1) at the constant level term in g . The terms retained in the truncated expansion will then define a transformation of the dependent variable. This transformation will allow one to homogenize the equations, once the equation has been “homogenized” it can be readily solved as I will illustrate in the next section.

2.1 The Method

The truncated Painlevé expansion method has been used to find exact solutions for various nonlinear PDEs. It is a fairly systematic method that can be programmed using any symbolic computer package, such as MACSYMA, MATHEMATICA, MAPLE, and REDUCE. Using the Painlevé expansion method, I have found exact solutions for many nonlinear PDEs. Let me focus on single PDEs, with

dependent variable u and independent variables x (for space), and t (for time). In solving the candidate equation, I first write $u(x, t)$ as a Laurent series in the complex plane,

$$u(x, t) = g^\alpha(x, t) \sum_{k=0}^{\infty} u_k(x, t) g^k(x, t), \quad (2.2)$$

where $g(x, t)$ is the non-characteristic manifold for the poles, and α is the negative integer which gives the degree of the most singular terms. Second, by substituting the series into the equation and requiring that the most singular terms vanish, one obtains the values for α and $u_0(x, t)$. If the next most singular terms are required to vanish, one will obtain the expressions for $u_1(x, t)$, $u_2(x, t)$, etc. After that, the series will be truncated at the constant level term. The truncated series will define a transformation of the dependent variable, which turns out to be crucial in the process of determining exact closed-form solutions. To illustrate the truncated Painlevé expansion method, the well known Korteweg-de Vries (KdV) equation [51] will be used;

$$u_t + buu_x + u_{xxx} = 0, \quad (2.3)$$

where b is any real constant. In order to find the leading order α , let

$$u = g^\alpha(x, t)u_0(x, t). \quad (2.4)$$

Substituting (2.4) into (2.3) yields

$$\begin{aligned}
 & bg^{2\alpha} u_0 u_{0x} + \alpha b g^{2\alpha-1} g_x u_0^2 + g^\alpha [u_{03x} + u_{0t}] \\
 & + g^{\alpha-1} [3\alpha g_x u_{0xx} + 3\alpha g_{xx} u_{0x} + \alpha g_{3x} u_0 + \alpha g_t u_0] \\
 & + g^{\alpha-2} [3\alpha^2 g_x^2 u_{0x} - 3\alpha g_x^2 u_{0x} + 3\alpha^2 g_x g_{xx} u_0 - 3\alpha g_x g_{xx} u_0] \\
 & + g^{\alpha-3} [\alpha^3 g_x^3 u_0 - 3\alpha^2 g_x^3 u_0 + 2\alpha g_x^3 u_0] = 0. \tag{2.5}
 \end{aligned}$$

From the above equation, you can see that the most singular powers of g are $2\alpha - 1$, and $\alpha - 3$; therefore, by equating these powers, one gets $\alpha = -2$. Hence, the most singular terms, i.e. the terms in g^{-5} in (2.5):

$$(-2bg_x u_0^2 - 24g_x^3 u_0)g^{-5}, \tag{2.6}$$

will vanish if

$$u_0 = -\frac{12g_x^2}{b}. \tag{2.7}$$

Next, to find u_1 let

$$u = g^{-2}(x, t) [u_0(x, t) + g(x, t)u_1(x, t)] \tag{2.8}$$

i.e.

$$u = \frac{u_1}{g} - \frac{12g_x^2}{bg^2}, \quad (2.9)$$

and substitute (2.9) for u into (2.5), and compute, again, the coefficient of the most singular term (g^{-4} in this case), one finds

$$30bg_x^3u_1 - 360g_x^3g_{xx}. \quad (2.10)$$

To eliminate this term, take

$$u_1 = \frac{12g_{xx}}{b}. \quad (2.11)$$

The next step is to calculate u_2 , for which

$$\begin{aligned} u &= g^{-2}(x, t) [u_0(x, t) + g(x, t)u_1(x, t) + g^2(x, t)u_2(x, t)] \\ &= \frac{u_0(x, t)}{g^2(x, t)} + \frac{u_1(x, t)}{g(x, t)} + u_2(x, t), \end{aligned} \quad (2.12)$$

is substituted into the given PDE. It turns out that u_2 has to satisfy the original equation,

$$(u_2)_t + bu_2(u_2)_x + (u_2)_{3x} = 0. \quad (2.13)$$

As a consequence, an auto-Bäcklund transformation has been derived, i.e. (2.12), which links two solutions to the same equation, in terms of a so far unknown function

$g(x, t)$.

For other equations, the same procedure will be followed. Setting different power terms in $g(x, t)$ equal to zero will allow the finding of u_0, u_1, u_2 , etc. The series will always be truncated at the constant level term of g , and the coefficient of that constant level will be set to zero. For the KdV equation, set $u_2 = 0$ to obtain

$$u = \frac{12g_{xx}}{bg} - \frac{12g_x^2}{bg^2} = \frac{12}{b} \frac{\partial^2}{\partial x^2} \ln g. \quad (2.14)$$

To solve the original KdV equation, substitute (2.14) into (2.3), after integrating once with respect to x . This gives

$$\frac{12}{b} g^{-2} \left[-g_t g_x + 3g_{xx}^2 - 4g_x g_{3x} \right] + \frac{12}{b} g^{-1} [g_{tx} + g_{4x}] = 0. \quad (2.15)$$

Then take the individual coefficients of different powers of g , set them equal to zero, and solve each equation separately;

$$g_{tx} + g_{4x} = 0, \quad (2.16)$$

$$g_t g_x - 3g_{xx}^2 + 4g_x g_{3x} = 0. \quad (2.17)$$

Note that one of the equations in g is linear and the other is quadratic. To obtain

solitary wave solutions, I will assume the following form for g :

$$g = 1 + c \exp(kx - \omega t + \delta), \quad (2.18)$$

where c , k , ω , and δ are all real constants. Substituting g into (2.16) leads to the dispersion law

$$\omega = k^3. \quad (2.19)$$

Remarkably, under the constraint $\omega = k^3$, both equations are satisfied simultaneously. Since u is written in terms of g in (2.14), g is given in (2.18) and ω in (2.19), the solution will be

$$u = \frac{12ck^2 \exp(kx - k^3t + \delta)}{b[c \exp(kx - k^3t + \delta) + 1]^2}. \quad (2.20)$$

For $c = 1$, the solution can be written in a more familiar form as

$$u = \frac{3k^2}{b} \left[1 - \tanh^2\left(\frac{kx - k^3t + \delta}{2}\right) \right] = \frac{3k^2}{b} \left[\operatorname{sech}^2\left(\frac{kx - k^3t + \delta}{2}\right) \right] \quad (2.21)$$

This is the well-studied solitary wave solution of the KdV equation with phase velocity $v = \frac{\omega}{k} = k^2$ and width $\frac{3k^2}{b}$. A picture is given in Figure 1, for $k = \frac{3}{2}$, $\delta = 0$, and $b = 6$.

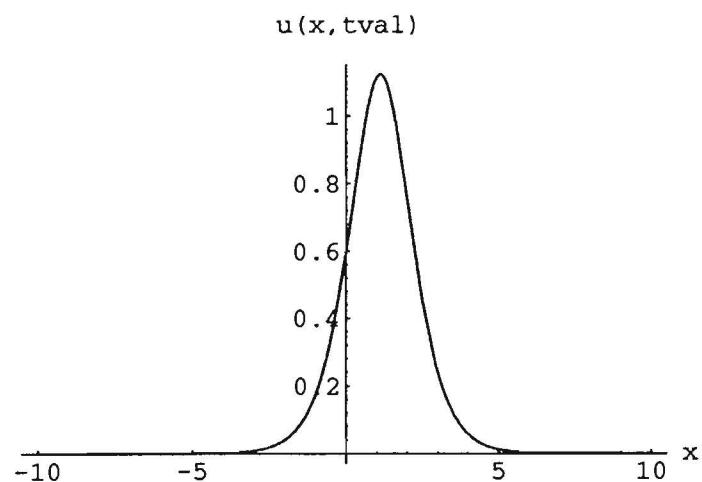


Figure 1a. The solitary-wave solution (2.21) to the KdV equation for $t\text{val}=0.5$.

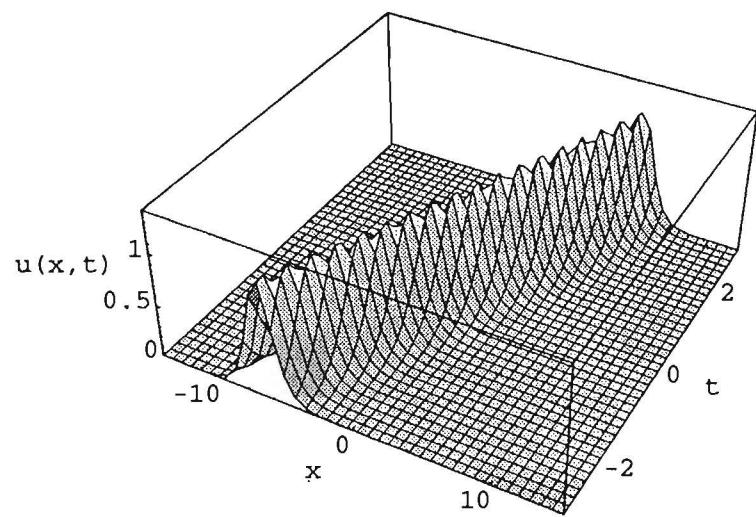


Figure 1b. The solitary-wave solution (2.21) to the KdV equation in 3-dimensions.

To further illustrate this method, Burgers' equation is introduced [2]:

$$u_t + buu_x - au_{2x} = 0, \quad (2.22)$$

where a , and b are real constants. To solve this equation, substitute (2.2) into (2.22), and require the most singular term to vanish. This leads to $\alpha = -1$, and

$$u_0 = -\frac{2ag_x}{b}. \quad (2.23)$$

Therefore, the transformation will be

$$u = -\frac{2ag_x}{bg} + u_1 = -\frac{2a}{b} \frac{\partial}{\partial x} \ln g + u_1, \quad (2.24)$$

where u_1 must satisfy (2.22). Proceeding to solve the equation in the same way as before, set $u_1 = 0$, and substitute (2.24) into (2.22) after integrating once with respect to x . This yields

$$ag_{xx} - g_t = 0. \quad (2.25)$$

Note that (2.25) is a linear equation. Now, solving this equation for

$$g = 1 + \exp(kx - \omega t + \delta) \quad (2.26)$$

produces the following solution

$$u = -\frac{ak}{b} \left[1 + \tanh\left(\frac{\theta}{2}\right) \right], \quad (2.27)$$

where $\theta = kx + ak^2t + \delta$. Figure 2, illustrates the solution for $k = \frac{3}{2}$, $\delta = 0$, $a = 4$, and $b = 2$.

2.2 Application

In this section some of the examples solved using the truncated Painlevé expansion method will be presented. For some of these examples new solutions have been found while for others, I have simply reconstructed previously known solutions. All the calculations in this section were performed using the computer algebra system MACSYMA.

Example 1: The Kuramoto-Sivashinsky Equation

Consider the Kuramoto-Sivashinsky (KS) equation [52]

$$u_t + uu_x + au_{xx} + bu_{4x} = 0. \quad (2.28)$$

This equation was independently established by Kuramoto in connection with reaction-diffusion systems, and by Sivashinsky to model flame propagation. To find the appropriate transformation for the dependent variable for the KS equation, first express

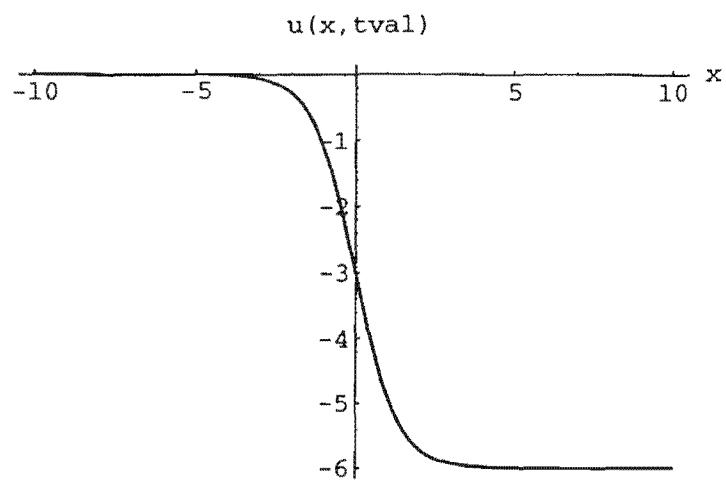


Figure 2a. The solution (2.27) to Burgers' equation for $tval=0$.

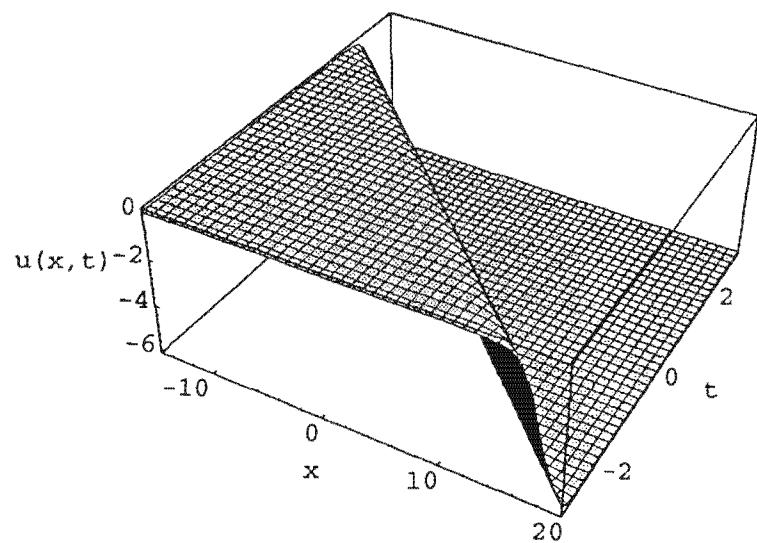


Figure 2b. The solution (2.27) to Burgers' equation in 3-dimensions.

$u(x, t)$ as a Laurent series in the complex plane

$$u(x, t) = g^\alpha(x, t) \sum_{k=0}^{\infty} u_k(x, t) g^k(x, t). \quad (2.29)$$

Second, by substituting (2.29) into (2.28), and requiring that the most singular terms vanish, one obtains $\alpha = -3$, and

$$u_0 = \frac{60}{19}(19bg_{3x} + ag_x). \quad (2.30)$$

If the next two most singular terms are required to vanish, one gets

$$u_1 = -180bg_xg_{2x}, \quad (2.31)$$

and

$$u_2 = 120bg_x^3. \quad (2.32)$$

Using the truncated Painlevé expansion, the transformation

$$u = \frac{u_0}{g^3} + \frac{u_1}{g^2} + \frac{u_2}{g} \quad (2.33)$$

emerges. Substituting the explicit forms for u_0, u_1 , and u_2 , (2.33) can be written as

$$u = \frac{\partial h(x, t)}{\partial x}, \quad (2.34)$$

where,

$$h = 60 \frac{bg_{2x}}{g} - 60 \frac{bg_x^2}{g^2} + \frac{60a}{19} \ln g. \quad (2.35)$$

Since u can be written as in (2.34), (2.28) can be integrated once with respect to x and replaced by

$$h_t + \frac{h_x^2}{2} + ah_{2x} + bh_{4x} = 0. \quad (2.36)$$

Furthermore, substituting h into (2.36) and deriving the system of equations for g (similar to (2.16)) by splitting the equation into different power terms in g , produces three equations. To solve these equations, again assume that

$$g = 1 + \exp(kx - \omega t + \delta), \quad (2.37)$$

and substitute g into each of these equations. Skipping the technical details, the following two cases have to be considered.

Case 1: $19bk^2 + a \neq 0$.

For this case, the dispersion law, relating ω and k , reads

$$\omega = \frac{19b^2k^6 + 20abk^4 + a^2k^2}{19bk^2 + a}. \quad (2.38)$$

Substituting the expression of ω into the other equations determines the value of k to be

$$k = \sqrt{\frac{11a}{19b}}. \quad (2.39)$$

Thus, after substitution for k , (2.38) becomes,

$$\omega = \frac{330a^2}{361b}. \quad (2.40)$$

Returning to u via (2.34) and (2.35), the solution is

$$u = \frac{15a}{19} \sqrt{\frac{11a}{19b}} \left[11 \tanh^3\left(\frac{kx - \omega t + \delta}{2}\right) - 9 \tanh\left(\frac{kx - \omega t + \delta}{2}\right) + 2 \right]. \quad (2.41)$$

Case 2: $19bk^2 + a = 0$. Then,

$$k = \sqrt{-\frac{a}{19b}}, \quad (2.42)$$

and

$$\omega = -\frac{30a^2}{361b}, \quad (2.43)$$

which leads to the solution

$$u = -\frac{15a\sqrt{-\frac{a}{b}}}{19\sqrt{19}} \left[\tanh\left(\frac{kx - \omega t + \delta}{2}\right) - 2 \right] \left[\tanh\left(\frac{kx - \omega t + \delta}{2}\right) + 1 \right]^2. \quad (2.44)$$

Boling and Xingde [52] discussed the structure of the traveling wave solution of this equation. They also studied the similarity solutions and the similarity transformations of the equation. Pictures of the solutions (2.41) and (2.44) are given in Figure 3 and Figure 4, respectively, for $a = 1$, $b = 1$, and $\delta = 0$.

Example 2: The Fisher Equation With Convection

Consider the Fisher equation with convection [53]

$$u_t + buu_x - u_{xx} - u(1 - u) = 0, \quad (2.45)$$

where b is any real constant. The transformation that follows from the truncated Painlevé expansion is

$$u = -\frac{2}{b} \frac{g_x}{g}. \quad (2.46)$$

By substituting it into the original equation and solving the resulting equations for g (as in the previous example) with

$$g = 1 + \exp(kx - \omega t + \delta), \quad (2.47)$$

one gets,

$$\omega = \frac{2k - bk^2}{b}, \quad \text{and} \quad \omega = -1 - k^2. \quad (2.48)$$

To satisfy both equations for ω one must have $k = -\frac{b}{2}$.

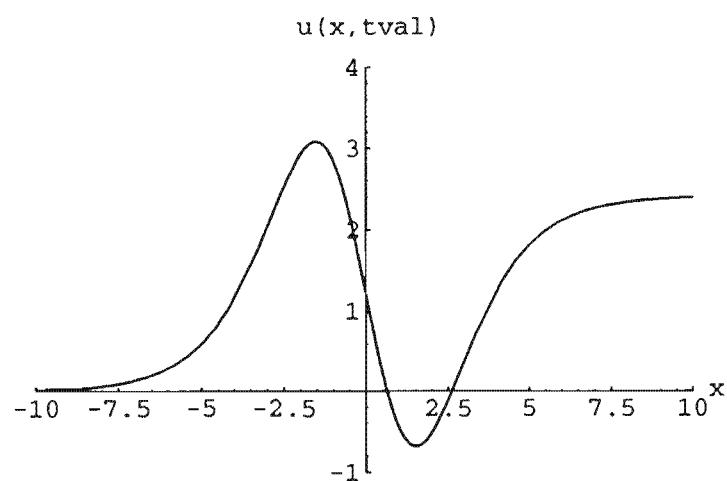


Figure 3a. The solution (2.41) to the KS equation for $tval=0$.

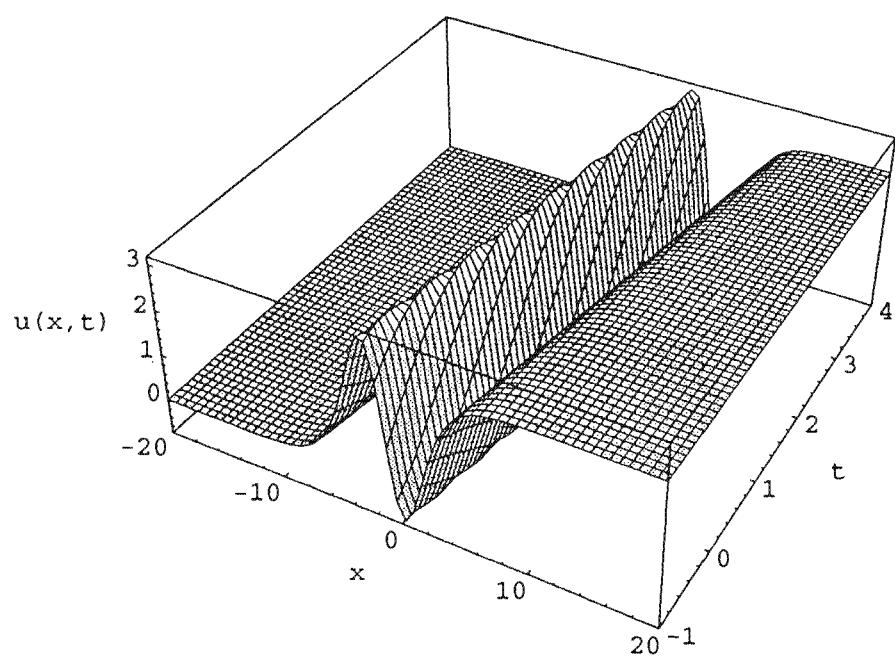


Figure 3b. The solution (2.41) to the KS equation in 3-dimensions.

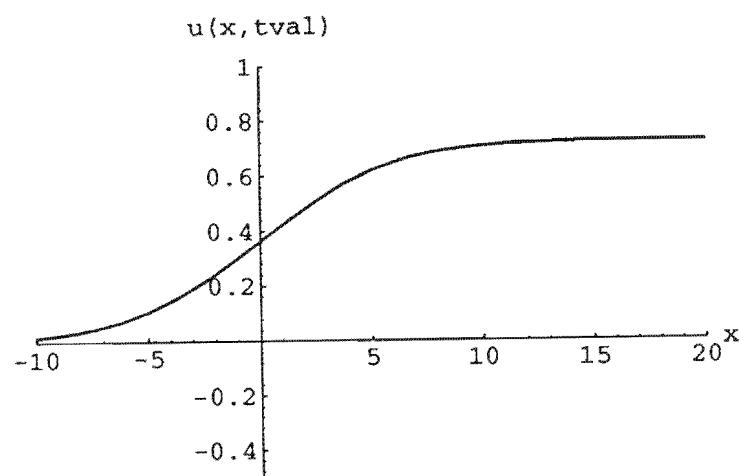


Figure 4a. The solution (2.44) to the KS equation for $tval=0$.

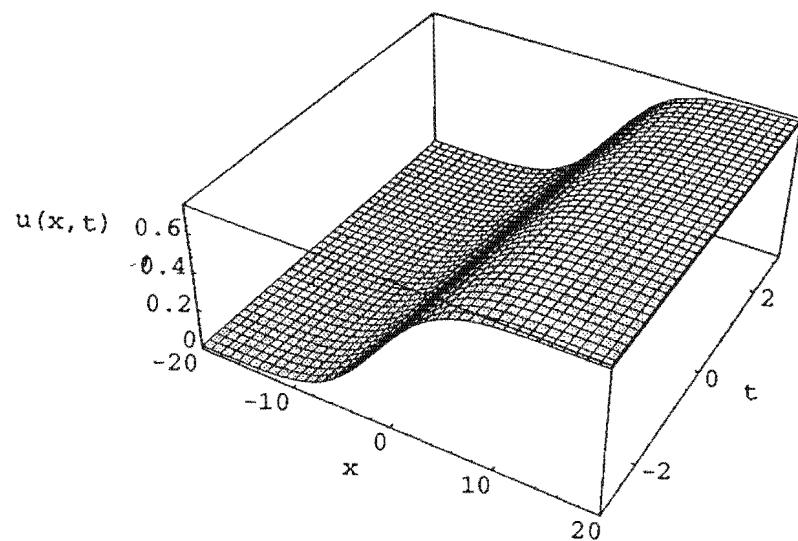


Figure 4b. The solution (2.44) to the KS equation in 3-dimensions.

Then,

$$\omega = -\frac{b^2}{4} - 1, \quad (2.49)$$

and, therefore, the solution is

$$u = -\frac{1}{2} \left[\tanh\left(\frac{kx - \omega t + \delta}{2}\right) + 1 \right]. \quad (2.50)$$

The same solution was derived by Malfliet via the tanh-method [54]. Figure 5, illustrates the solution (2.50) for $b = 1$, and $\delta = 0$.

Example 3: A Nonlinear Reaction-Diffusion Equation

The nonlinear reaction-diffusion equation [44, 54] can be written as

$$u_t = (u^2)_{xx} + u(1 - u). \quad (2.51)$$

In this example, the truncated Painlevé expansion gives $\alpha = 1$. However, by making the transformation $u = \frac{1}{v}$, and writing the equation in terms of v , one gets,

$$v^2 v_t + 6v_x^2 - 2vv_{xx} + v^2(v - 1) = 0. \quad (2.52)$$

The new equation passes the Painlevé test with $\alpha = -1$, and upon application

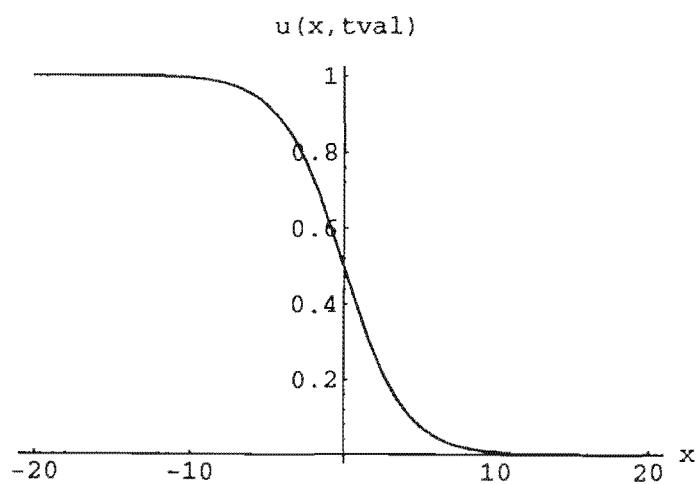


Figure 5a. The solution (2.50) to the Fisher equation with convection for $tval=0$.

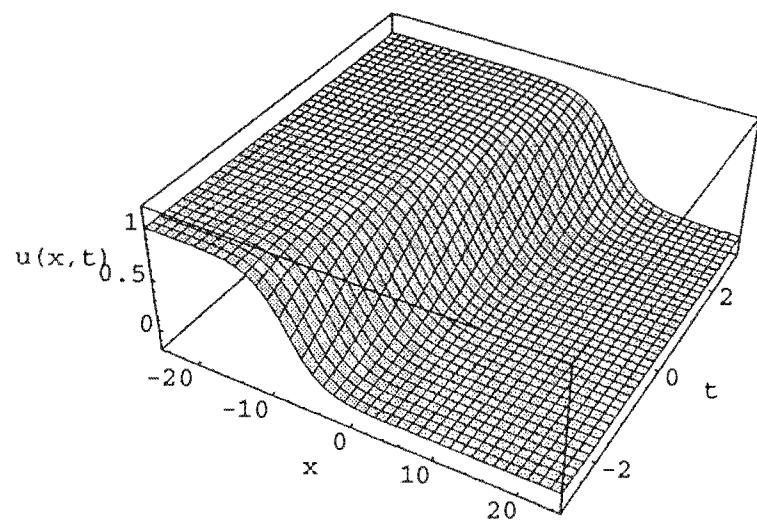


Figure 5b. The solution (2.50) to the Fisher equation with convection in 3-dimensions.

of the method described here, the transformation turns out to be

$$v = \frac{2g_x^2}{gg_t}. \quad (2.53)$$

Now substituting v into (2.52), splitting the resulting equation into different power terms in g , and solving these remaining PDEs in g , for

$$g = 1 + \exp(kx - \omega t + \delta), \quad (2.54)$$

one obtains

$$\omega = -\frac{1}{2} \quad \text{and} \quad k = \frac{\pm 1}{2}. \quad (2.55)$$

Thus, particular solutions for (2.51) are

$$u = \frac{2}{1 + \tanh(\frac{x+t+\delta_1}{4})}, \quad (2.56)$$

and

$$u = \frac{2}{1 + \tanh(\frac{t-x+\delta_2}{4})}. \quad (2.57)$$

Notice that, the solution in (2.56) is finite as $x \rightarrow \infty$, while the solution in (2.57) is finite as $x \rightarrow -\infty$. Using the tanh-method, Malfliet [44] got only the first solution where $k = \frac{1}{2}$. Figure 6, shows the solutions in (2.56) for $\delta = 0$. Note that (2.51) is invariant for the change $x \rightarrow -x$, which also transforms (2.56) to (2.57).

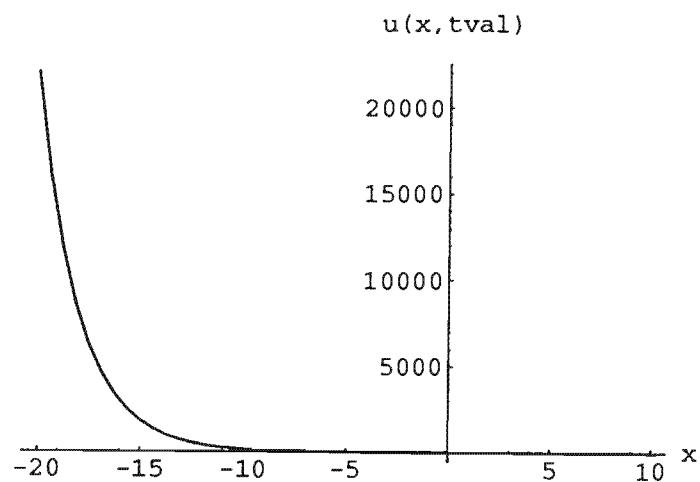


Figure 6a. The solution (2.56) to a nonlinear reaction-diffusion equation for $tval=0$.

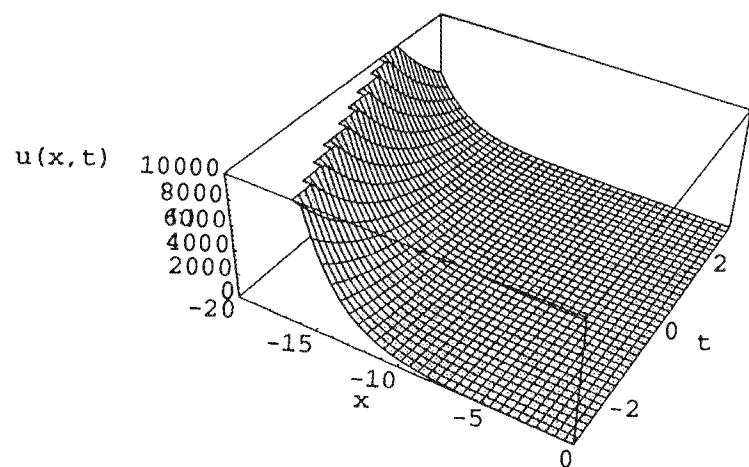


Figure 6b. The solution (2.56) to a nonlinear reaction-diffusion equation in 3-dimensions .

Example 4: The Two Dimensional KdV-Burgers Equation

Consider the two dimensional KdV-Burgers equation [55]

$$(u_t + uu_x - mu_{xx} + nu_{3x})_x + su_{yy} = 0, \quad (2.58)$$

where m, n , and s are any real constants. The truncated Painlevé expansion gives

$\alpha = -2$, and the transformation

$$u = \frac{\partial}{\partial x} \left(\frac{12ng_x}{g} - \frac{12m \ln g}{5} \right). \quad (2.59)$$

So, I can integrate the equation once with respect to x before substituting the transformation into (2.58). The new equation is then

$$h_{tx} + h_x h_{xx} - mh_{3x} + nh_{4x} + sh_{yy} = 0, \quad (2.60)$$

where

$$h = \frac{12ng_x}{g} - \frac{12m \ln(g)}{5}. \quad (2.61)$$

To solve this equation, assume

$$g = 1 + c \exp(kx + ly - \omega t + \delta), \quad (2.62)$$

and substitute it into (2.61) and (2.60). Just as in the previous examples, one ends up with two cases:

Case 1: With

$$k = -\frac{m}{5n} \quad \text{and} \quad \omega = \frac{-625l^2n^3s - 6m^4}{125n^3}, \quad (2.63)$$

the solution is

$$u = \frac{12m^2c \exp(kx + ly - \omega t + \delta)[2 + c \exp(kx + ly - \omega t + \delta)]}{25n [c \exp(kx + ly - \omega t + \delta) + 1]^2}, \quad (2.64)$$

or

$$u = \frac{12cm^2(1 + \tanh(\frac{\theta_1}{2})) \left[c \tanh(\frac{\theta_1}{2}) - 2 \tanh(\frac{\theta_1}{2}) + c + 2 \right]}{25n \left[c \tanh(\frac{\theta_1}{2}) - \tanh(\frac{\theta_1}{2}) + c + 1 \right]^2} \quad (2.65)$$

where

$$\theta_1 = kx + ly - \omega t + \delta, \quad (2.66)$$

with k and ω as in (2.63). Figure 7 shows a picture of the solution in (2.65) for $m = -3, n = -1, s = 2, l = 1, c = 2$, and $\delta = 0$

Case 2: With

$$k = \frac{m}{5n} \quad \text{and} \quad \omega = \frac{625l^2n^3s - 6m^4}{125n^3}, \quad (2.67)$$

the solution is

$$u = -\frac{12m^2c^2 \exp[2(kx + ly - \omega t + \delta)]}{25n [c \exp(kx + ly - \omega t + \delta) + 1]^2}, \quad (2.68)$$

which can be written as

$$u = -\frac{12c^2m^2 \left[1 + \tanh\left(\frac{\theta_2}{2}\right) \right]^2}{25n \left[c \tanh\left(\frac{\theta_2}{2}\right) - \tanh\left(\frac{\theta_2}{2}\right) + c + 1 \right]^2}, \quad (2.69)$$

where

$$\theta_2 = kx + ly - \omega t + \delta, \quad (2.70)$$

with k and ω as in (2.67). Figure 8, shows a picture of the solution in (2.69) for $m = \frac{1}{2}$, $n = -1$, $s = 2$, $l = 1$, $c = 1$, and $\delta = 0$.

It is worth mentioning that Zhibin and Mingliang [55] got the same solution

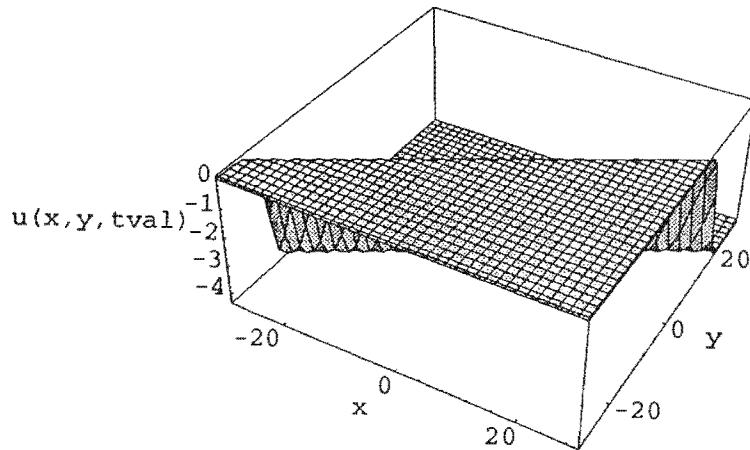


Figure 7. The solution (2.65) to the two-dimensional KdV-Burgers equation for $tval=0$.

(using MATHEMATICA), by expressing the solution as a sum of shock-wave solutions of the two-dimensional Burgers' equation and the solitary wave solutions of the two-dimensional KdV equation. This study shows that such an assumption is not needed. Application of the truncated Painlevé expansion method leads automatically and directly to the form they assumed *a priori*.

Example 5: A Nonlinear Dissipative-Dispersive Equation

Consider the nonlinear dissipative-dispersive equation

$$u_t + uu_x + bu_{3x} - a(u_t + muu_x)_x = 0, \quad (2.71)$$

where a, b and m are real constants. This equation was derived by Kakutani and

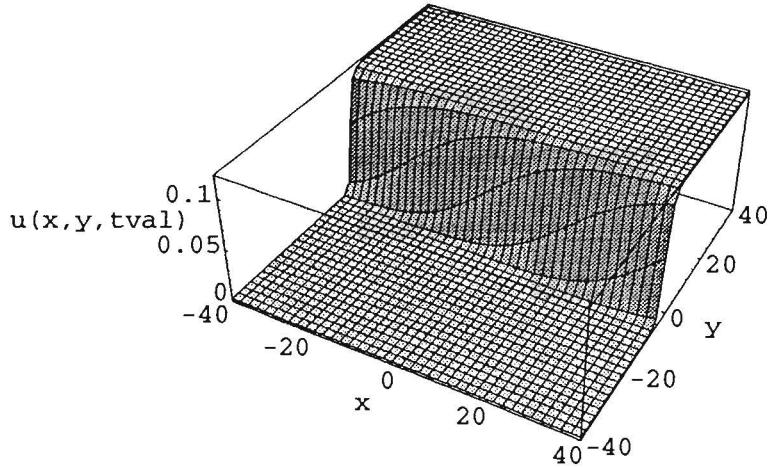


Figure 8. The solution (2.69) to the two-dimensional KdV-Burgers equation for $tval=0$.

Kawahara [56] in the analysis of ion-acoustic waves in a plasma consisting of cold ions and warm electrons. According to their result, this equation has no exact solution for $m = 1$. In trying to solve the same equation for $m \neq 1$, the following two cases show up.

Case 1 : $m \neq 0$ and $m \neq 1$

For this case the transformation is

$$u = -\frac{2b}{am} \frac{g_x}{g}. \quad (2.72)$$

After substituting (2.72) into (2.71) one gets

$$\begin{aligned} & 2g^2 g_x^2 (abng_{2x} - a^2 mg_t + bg_x) - g(2abmg_x g_{3x} + abmg_{2x}^2 \\ & - a^2 mg_t g_{2x} - 2a^2 mg_x g_{tx} + ang_x g_t + 2bg_x g_{2x}) \\ & + am(bg_{4x} - ag_{txx} + g_{tx}) = 0. \end{aligned} \quad (2.73)$$

Now, let

$$g = 1 + c \exp(kx - \omega t + \delta), \quad (2.74)$$

and solve for k and ω to get

$$\omega = -\frac{bk}{a^2 m - a^2 m^2} \quad (2.75)$$

and

$$k = -\frac{1}{am - a}. \quad (2.76)$$

Therefore, the solution is

$$u = \frac{2cb \left[\tanh\left(\frac{kx - \omega t + \delta_1}{2}\right) + 1 \right]}{a^2(m-1) \left[c \tanh\left(\frac{kx - \omega t + \delta_1}{2}\right) - \tanh\left(\frac{kx - \omega t + \delta_1}{2}\right) + c + 1 \right]} \quad (2.77)$$

For the special value $c = 1$, one gets

$$u = \frac{2b \left[\tanh\left(\frac{kx - \omega t + \delta_1}{2}\right) + 1 \right]}{2a^2(m-1)}. \quad (2.78)$$

Figure 9, illustrates the solution in (2.78) for $a = 1, b = -1, m = 2$, and $\delta = 0$.

Case 2: $m = 0$.

For $m = 0$, the equation reduces to

$$bu_{3x} + uu_x - au_{tx} + u_t = 0, \quad (2.79)$$

and the transformation that follows from the truncated Painlevé expansion is

$$u = \frac{12g(5bg_{xx} - ag_t) - 60bg_x^2}{5g^2}. \quad (2.80)$$

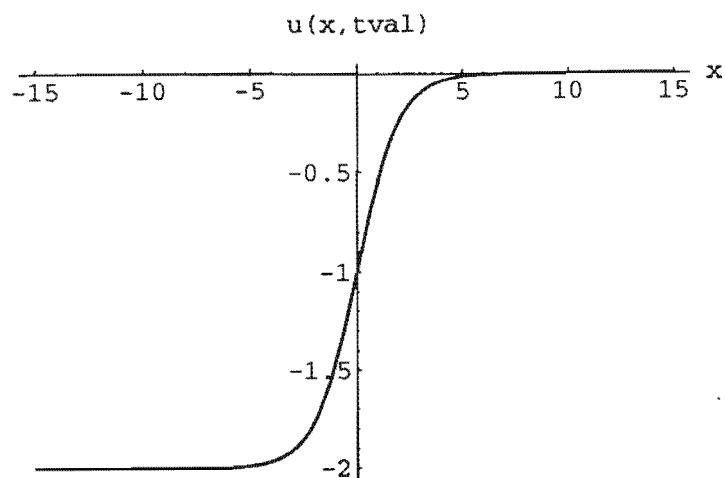


Figure 9a. The solution (2.78) to the equation (2.71) for $tval=0$.

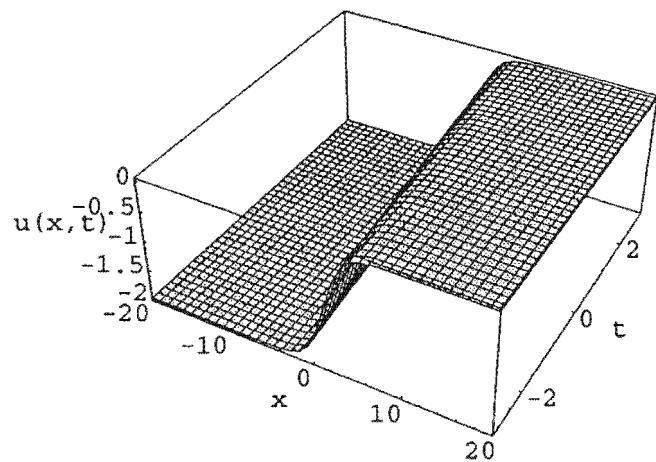


Figure 9b. The solution (2.78) to the equation (2.71) in 3-dimensions.

Substituting u in the equation and solving for

$$g = 1 + c \exp(kx - \omega t + \delta), \quad (2.81)$$

again, leads to two cases:

Case 2.1: Corresponding to

$$\omega = \frac{125b}{36a} \quad (2.82)$$

and

$$k = \frac{5}{6a}, \quad (2.83)$$

the final solution is

$$u = \frac{25cb \exp(kx - \omega t + \delta) [c \exp(kx - \omega t + \delta) + 2]}{3a^2 [c \exp(kx - \omega t + \delta) + 1]^2}. \quad (2.84)$$

If $c = 1$, the solution can be rewritten in terms of tanh as

$$u = -\frac{25b [\tanh(\frac{kx-\omega t+\delta}{2}) - 3] [\tanh(\frac{kx-\omega t+\delta}{2}) + 1]}{12a^2}. \quad (2.85)$$

Figure 10, demonstrates the solution in (2.85) for $a = 1, b = -1, m = 2$, and $\delta = 0$.

Case 2.2: For

$$\omega = -\frac{125b}{36a} \quad (2.86)$$

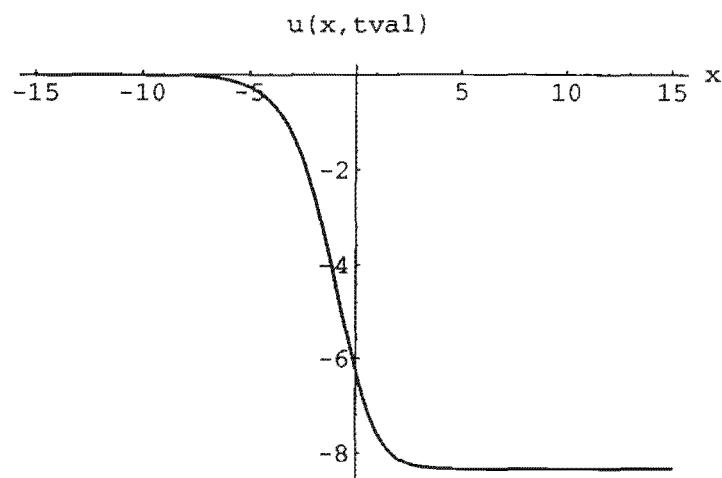


Figure 10a. The solution (2.85) to the equation (2.71) for $tval=0$.

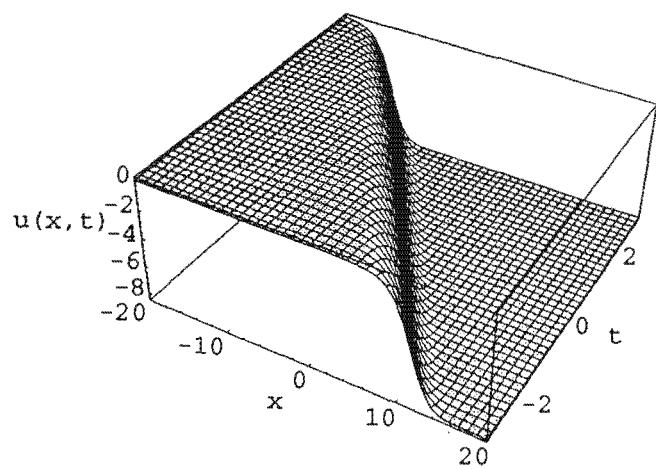


Figure 10b. The solution (2.85) to the equation (2.71) in 3-dimensions.

and

$$k = \frac{5}{6a}, \quad (2.87)$$

I obtained the solution

$$u = -\frac{25c^2b \exp^2(kx - \omega t + \delta)}{3a^2 [c \exp(kx - \omega t + \delta) + 1]^2}, \quad (2.88)$$

which for $c = 1$, can be written as

$$u = -\frac{25b [\tanh(\frac{kx - \omega t + \delta}{2}) + 1]^2}{12a^2}. \quad (2.89)$$

Figure 11 represents the solution in (2.89) for $a = 1, b = -1, m = 2$, and $\delta = 0$. Malfliet and Hereman [45] had solved this equation for the case where $m = 0$, using the tanh-method. They only achieved the solution (2.85).

Example 6: The Evolution Equation due to Calogero

I will now try to find an exact solution to an evolution equation due to Calogero [46]:

$$u_t - u_{3x} - 3(u^2 u_{xx} + 3u u_x^2 + u^4 u_x) = 0. \quad (2.90)$$

The truncated Painlevé expansion method implies that $\alpha = -\frac{1}{2}$. To use the Painlevé truncated method, let

$$u = v^{\frac{1}{2}}. \quad (2.91)$$

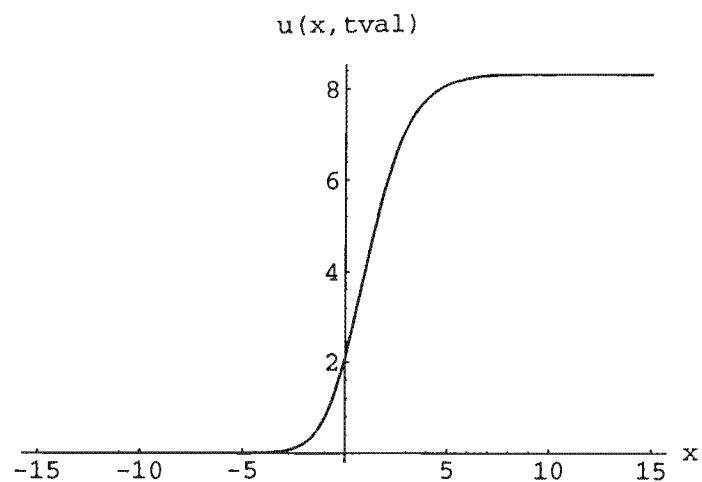


Figure 11a. The solution (2.89) to the equation (2.71) for $tval=0$.

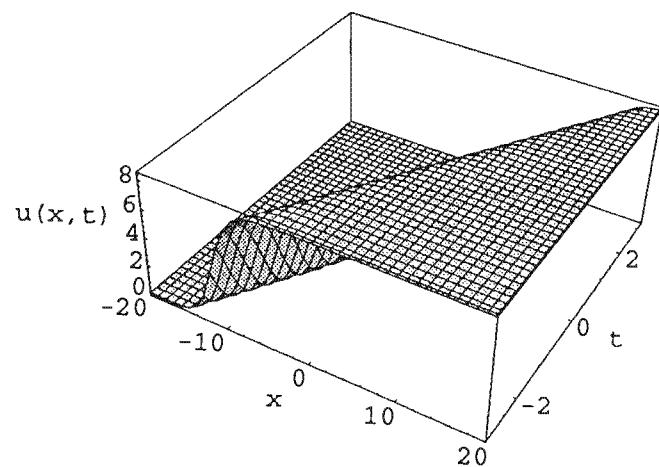


Figure 11b. The solution (2.89) to the equation (2.71) in 3-dimensions.

So, the new equation will be

$$\frac{1}{2}v^2v_t - \frac{3}{2}v^4v_x - \frac{3}{2}v^3v_{2x} - \frac{3}{2}v^2v_x^2 - \frac{1}{2}v^2v_{3x} + \frac{3}{4}vv_xv_{2x} - \frac{3}{8}v_x^3 = 0, \quad (2.92)$$

and, obviously, for this equation $\alpha = -1$. Proceeding as before, the transformation is

$$v = \frac{1}{2} \frac{g_x}{g} = \frac{1}{2} \frac{\partial}{\partial x} \ln g. \quad (2.93)$$

Therefore, taking

$$g = 1 + \exp(kx - \omega t + \delta) \quad (2.94)$$

one finds that $\omega = -\frac{k^3}{4}$.

The solution is then readily obtained:

$$v = \frac{k}{4} \left[1 + \tanh\left(\frac{4kx + k^3t + \delta}{8}\right) \right]. \quad (2.95)$$

Hence, a particular solution for (2.90) is

$$u = \sqrt{\frac{k}{4}} \sqrt{1 + \tanh\left(\frac{4kx + k^3t + \delta}{8}\right)}. \quad (2.96)$$

Figure 12, illustrates the solution in (2.96) for $k = 2$, and $\delta = 0$. Calogero [46] solved the same equation by linearizing it using different transformations.

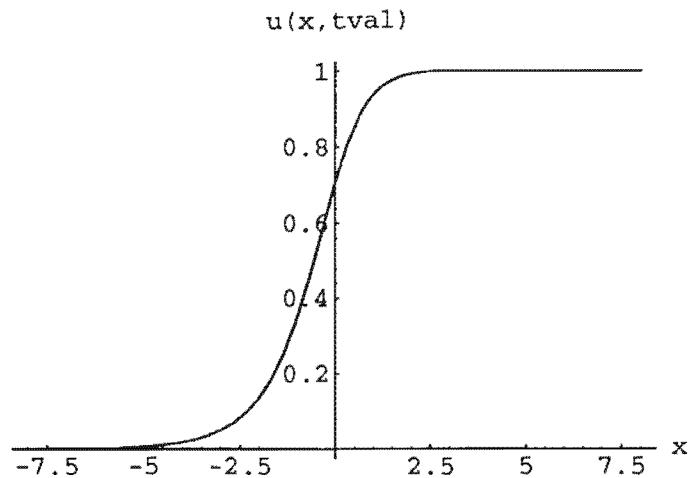


Figure 12a. The solution (2.96) to the equation (2.90) for $t\text{val}=0$.

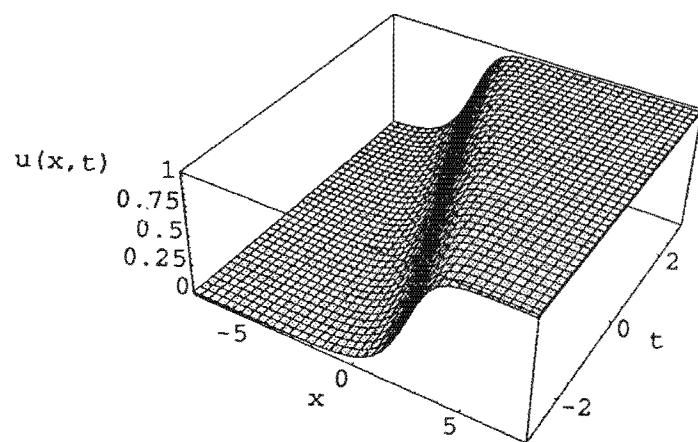


Figure 12b. The solution (2.96) to the equation (2.90) in 3-dimensions.

Example 7 : A General Class of Evolution Equations

I will now illustrate how the method can be used to find particular solutions to a general class of evolution equations, that can be written in the form

$$u_t = (u^n)_{xx} + f(u), \quad (2.97)$$

where $f(u)$ is a polynomial of degree p , and n is a positive integer [57]. Explicitly, this equation can be written as

$$u_t = n(n-1)u^{n-2}(u_x)^2 + nu^{n-1}u_xx + a_p u^p + a_{p-1} u^{p-1} + \dots + a_0. \quad (2.98)$$

For an equation like this, with unknown p and n , there is no algorithmic way to find the value of α . So, α will be calculated manually.

Let,

$$u = g^\alpha(x, t)u_0(x, t) \quad (2.99)$$

and substitute it into (2.98). This gives

$$\begin{aligned} \alpha g^{\alpha-1}g_t u_0 + g^\alpha u_{0t} &= n(n-1)u_0^{n-2} [\alpha^2 u_0^2 g_x^2 g^{\alpha n-2} + u_{0x}^2 g^{\alpha n} + 2\alpha u_0 u_{0x} g_x g^{\alpha n-1}] \\ &+ nu_0^{n-1} [\alpha(\alpha-1)u_0 g_x^2 g^{\alpha n-2} + 2\alpha g_x u_{0x} g^{\alpha n-1} + \alpha g_{xx} u_0 g^{\alpha n-1} \\ &+ u_{0xx} g^{\alpha n}] + a_p u_0^p g^{\alpha p} + a_{p-1} u_0^{p-1} g^{\alpha p-\alpha} + \dots + a_0. \end{aligned} \quad (2.100)$$

By equating the lowest power terms in g , one gets

$$\alpha n - 2 = \alpha p, \quad (2.101)$$

i.e.

$$\alpha = \frac{2}{n - p}. \quad (2.102)$$

From (2.102), α will be a negative number if $p > n$.

To find u_0 , substitute α into (2.100) and set the coefficient of the lowest power term in g equal to zero, which yields

$$n(n-1)\alpha^2 u_0^2 g_x^2 + n(\alpha^2 - \alpha)u_0^n g_x^2 + a_p u_0^p = 0. \quad (2.103)$$

This implies that

$$u_0^{p-n} = -\frac{2n(n+p)}{(n-p)^2 a_p} g_x^2. \quad (2.104)$$

For (2.97) with α negative, I have worked out some cases for special values of p and n . These cases are:

Case 1: $p = n + 1$.

In this case $\alpha = -2$, and

$$u_0 = -\frac{2n(2n+1)}{a_{n+1}} g_x^2. \quad (2.105)$$

The following example represents the case when $n = 3$. The equation becomes

$$u_t = (u^3)_{xx} + a_4 u^4 + a_3 u^3 + a_2 u^2 + a_1 u + a_0, \quad (2.106)$$

and the transformation is

$$u = -\frac{42}{a_4} \frac{\partial^2}{\partial x^2} \ln g. \quad (2.107)$$

Replacing u from the above into (2.106), solving the resulting equations for g , and assuming the form

$$g = 1 + \exp(kx - \omega t + \delta), \quad (2.108)$$

one gets the following constraints:

$$a_0 = 0, \quad \omega = a_1 = 0 \quad \text{and} \quad k = \frac{\sqrt{-a_3}}{3}. \quad (2.109)$$

Using the transformation linking u and g , the solution comes out to be

$$u = -\frac{14a_3 \exp(kx + \delta)}{3a_4[1 + \exp(kx + \delta)]^2}. \quad (2.110)$$

It can also be written as

$$u = \frac{7a_3}{6a_4} \left[\tanh^2\left(\frac{kx + \delta}{2}\right) - 1 \right]. \quad (2.111)$$

Figure 13 shows the solution (2.111) for $a_3 = -1$, $a_4 = 2$, and $\delta = 0$.

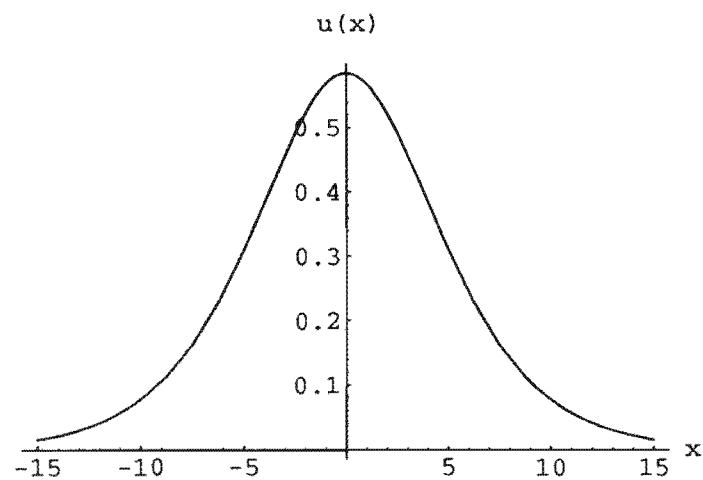


Figure 13. The solution (2.111) to the equation (2.97).

Case 2: $p = n + 2$.

For this case $\alpha = -1$, and

$$u_0 = \sqrt{-\frac{n^2 + n}{a_{n+2}}} g_x. \quad (2.112)$$

The transformation will be

$$u = \frac{u_0}{g} = \sqrt{-\frac{n^2 + n}{a_{n+2}}} \frac{g_x}{g}. \quad (2.113)$$

For u to be real, a_{n+2} must be negative. Therefore, u can be written as

$$u = b \frac{g_x}{g}, \quad (2.114)$$

where

$$b = \sqrt{-\frac{n^2 + n}{a_{n+2}}}. \quad (2.115)$$

Now, by substituting for u into (2.98), and picking the coefficients of different power terms in g , one gets the following

$$g^0 : -a_0$$

$$g^1 : -a_1 b g_x + b g_{xt}$$

$$\begin{aligned}
g^2 & : -a_2 b^2 g_x^2 - b g_x g_t \\
& \vdots \\
g^n & : -a_n b^n g_x^n - n b^n g_x^{n-1} g_{3x} - n(n-1) b^n g_x^{n-2} g_{xx}^2 \\
g^{n+1} & : -a_{n+1} b^{n+1} g_x^{n+1} + 3n b^n g_x^n g_{xx} + 2n(n-1) b^n g_x^n g_{xx} \\
g^{n+2} & : -a_{n+2} b^{n+2} g_x^{n+2} - 2n b^n g_x^{n+2} - n(n-1) b^n g_x^{n+2}. \tag{2.116}
\end{aligned}$$

Setting these equal to zero implies that $a_0 = 0$ for any p and n . The following is an example for specific values of p and n :

Example: $n = 2$.

For this case, the equation becomes

$$u_t = (u^2)_{xx} - a_4 u^4 - a_3 u^3 - a_2 u^2 - a_1 u, \tag{2.117}$$

and the transformation is

$$u = \sqrt{\frac{6}{a_4}} \frac{g_x}{g}. \tag{2.118}$$

By substituting (2.118) into (2.117), and solving for

$$g = 1 + \exp(kx - \omega t + \delta) \tag{2.119}$$

one has

$$\begin{aligned}\omega &= a_1, \\ k &= -\frac{\sqrt{6}}{10\sqrt{a_4}}a_3, \\ \text{and } a_1 &= \frac{3a_3}{125a_4^2}(25a_2a_4 - 6a_3^2).\end{aligned}\tag{2.120}$$

Therefore, the solution is

$$u = -\frac{3a_3}{10a_4} \left[1 + \tanh\left(\frac{kx - \omega t + \delta}{2}\right) \right].\tag{2.121}$$

Figure 14 shows the solution (2.121) for $a_2 = 1$, $a_3 = \frac{2}{3}$, $a_4 = 2$, and $\delta = 0$.

2.3 Classical and Nonclassical Symmetry

In this section, an alternative way based on group techniques that could be used to obtain the same exact solutions of the PDEs solved in this chapter is discussed. These techniques lead to solutions in special forms, and are obtained by exploiting the symmetries of the original equation. Symmetry techniques provide a method for getting exact solutions of a certain PDE in terms of solutions of lower dimensional equations. An advantage of these techniques is that they are applicable to all PDEs, irrespective whether or not the equations are integrable.

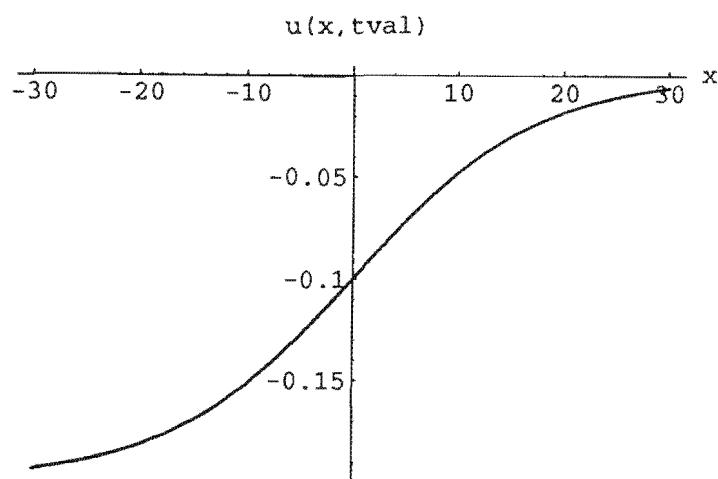


Figure 14a. The solution (2.121) to the equation (2.117) for $tval=0$.

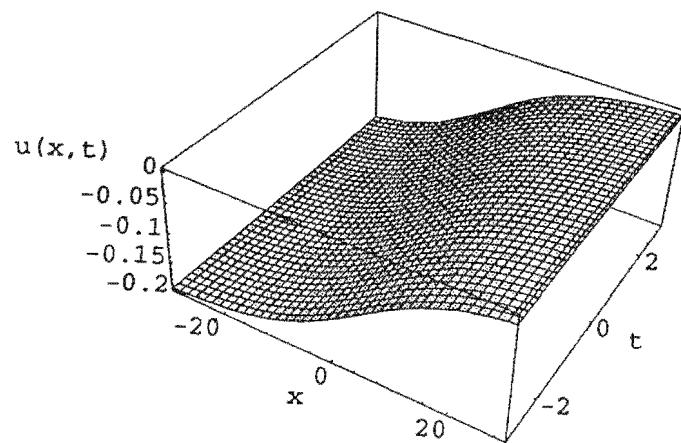


Figure 14b. The solution (2.121) to the equation (2.117) in 3-dimensions.

The Lie group method of infinitesimal transformations is the classical method of finding symmetry reductions of PDEs. To apply this method to a second-order equation of the form

$$\mathcal{G}(x, t, u, u_x, u_t, u_{xx}) = 0, \quad (2.122)$$

where x and t are the independent variables, one considers the one-parameter (ϵ) Lie group of infinitesimal transformations in (x, t, u) , given by

$$\begin{aligned} \bar{x} &= x + \epsilon \xi(x, t, u) + O(\epsilon^2), \\ \bar{t} &= t + \epsilon \tau(x, t, u) + O(\epsilon^2), \\ \bar{u} &= u + \epsilon \eta(x, t, u) + O(\epsilon^2). \end{aligned} \quad (2.123)$$

The requirement that (2.122) is invariant under this transformation yields an over-determined, linear system of equations for the infinitesimals $\tau(x, t, u)$, $\xi(x, t, u)$, and $\eta(x, t, u)$. The corresponding Lie algebra is realized by vector fields of the form

$$X = \tau(x, t, u) \frac{\partial}{\partial t} + \xi(x, t, u) \frac{\partial}{\partial x} + \eta(x, t, u) \frac{\partial}{\partial u}. \quad (2.124)$$

Similarity reductions are then obtained by solving the characteristic equations

$$\frac{dx}{\xi(x, t, u)} = \frac{dt}{\tau(x, t, u)} = \frac{du}{\eta(x, t, u)} \quad (2.125)$$

or, equivalently, the invariant surface condition

$$\xi(x, t, u)u_x + \tau(x, t, u)u_t - \eta(x, t, u) = 0. \quad (2.126)$$

Solving the system of the overdetermined equations for ξ , τ , and η involves large amounts of algebra and calculus, and requires the use of symbolic manipulation programs. Many programs have been developed to facilitate these calculations. Hereman [58] gives a survey of different packages, and discusses their strengths, weaknesses, and range of application.

In their study of reductions of the linear heat equation, Bluman and Cole [59] proposed a generalization of Lie's method, which they called "nonclassical method of group-invariant solutions". In the literature, the method occurs with two names: "the nonclassical method" [60, 61] and the "method of conditional symmetries" [62].

In the nonclassical method, the original PDE (2.122) is augmented with the invariant surface condition (2.126) associated with the vector field (2.124). An overdetermined, non-linear system of equations for the infinitesimals $\tau(x, t, u)$, $\xi(x, t, u)$, and $\eta(x, t, u)$ that appear both in the transformation (2.123) and in the invariant surface condition (2.126), is obtained upon requiring that both the PDE (2.122) and invariant surface condition (2.126) are invariant under the transformation (2.123). The set of solutions for the infinitesimals is larger than that produced by the classical method. However, since the invariant surface condition (2.126) depends upon the particular

reduction, it should be emphasized that the vector fields no longer form a vector space, much less a Lie algebra.

Compared to the classical method, the nonclassical method involves more algebra due to the fact that the associated determining equations are nonlinear. For some PDEs, solving the nonlinear system can be too difficult to be done explicitly, especially if attempted manually. It is well known that, for some equations like the KdV equation, the nonclassical method does not lead to any additional symmetry reductions to those obtained using the classical method. In spite of that, sometimes the nonclassical method yields more symmetry reductions for PDEs than the classical one. For example, Clarkson [63] found new symmetry reductions for the generalized nonlinear Schrödinger equation using the nonclassical method. Levi and Winternitz [64] found more reductions for the Boussinesq equation using the nonclassical method than what they could obtain using the classical method.

Many people worked on the relation between the classical symmetry technique and the truncated Painlevé expansion method to study which method leads to more general solutions than the other. Clarkson and Mansfield [36] showed that the nonclassical method and the truncated Painlevé expansion method do not, in general, yield the same solution set. They solved the shallow water wave equation, using both the truncated Painlevé expansion and the nonclassical method and obtained two different solutions. Neither of the two solutions obtained, leads to the other.

Cariello and Tabor [65] showed that the solution for the real Newell-Whitehead equation (also known as the Kolmogorov-Petrovsky-Piskunov equation) obtained from the truncated Painlevé expansion method cannot be obtained from the classical symmetry method but corresponds to the singular solution of the nonclassical method. Estévez and Gordoa [66] studied the relation between the nonclassical method and the truncated Painlevé expansion method for six equations. These equations are the KdV equation, Burgers' equation, the Boussinesq equation, the mKdV equation, the Burgers-Huxley equation, the Fitzhugh-Nagumo equation, and a diffusion equation in two dimensions. They concluded that, for a PDE with the conditional Painlevé property, the truncated Painlevé expansion method allows the identification of all nonclassical symmetries whose associated symmetry reductions lead to an ordinary differential equation having the Painlevé property.

In conclusion, I would like to mention that after studying the symmetry reduction methods in connection with the truncated Painlevé expansion method, and consulting the available literature, I found that neither method can be said to be more general than the other. For some equations, the two methods coincide, and produce the same solution set, while for others they produce totally different solutions.

In this chapter, I have illustrated how to construct a solitary wave solution for various nonlinear PDEs using the truncated Painlevé expansion method. Unfortunately, this method only leads to particular solutions, and at best leads to one-soliton

solutions when applied to integrable nonlinear PDEs that are known to admit multi-soliton solutions. Therefore, in the next chapter, I will introduce and use another more powerful method to construct N-soliton solutions.

Chapter 3

Simplified Version of Hirota's Method

In this chapter, the second method used for finding soliton solutions for nonlinear PDEs will be introduced. This method is a simplified version of Hirota's method [7]. It will be applied for finding one- and multi-soliton solutions for some nonlinear PDEs, which are known to be completely integrable and therefore admit soliton solutions. Contrary to Hirota's original method, which has been effectively used to derive soliton solutions of large classes of nonlinear PDEs, this simplified version is more efficient in the sense that it does not require the knowledge of the bilinear forms in order to find the exact solutions.

3.1 Hirota's Method

Hirota's method has been one of the most successful direct techniques for constructing exact solutions of various nonlinear PDEs from soliton theory. It also allows the testing of whether or not a given PDE satisfies the necessary requirements to admit solitary wave and soliton solutions.

The drawback of Hirota's method is that the bilinear form of the PDE must be known in advance. In other words, the technique can only be applied to equations that can be written in a bilinear form, either as a single bilinear equation, or as a system of coupled bilinear equations.

In essence, Hirota's method requires (i) a clever change of the dependent variable, (ii) the introduction of a novel differential operator, and (iii) a perturbation expansion to solve the resulting bilinear equation.

Hirota introduced the differential operator D_x , which is defined on ordered pairs of functions $f(x)$ and $g(x)$ as follows:

$$D_x(f \cdot g) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right) f(x) g(x') \Big|_{x'=x}. \quad (3.1)$$

More generally, Hirota defined

$$D_x^m D_t^n (f \cdot g) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^n f(x, t) g(x', t') \Big|_{x'=x, t'=t}, \quad (3.2)$$

for non-negative integers m and n . This type of differential operator is called a *bilinear operator*, due to the obvious linearity in both of its arguments. Bilinear operators D_y^n, D_z^n , etc. and their products can be defined in a similar way. These operators

have the following properties:

$$D_x^m(f \cdot 1) = \frac{\partial^m f}{\partial x^m}, \quad (3.3)$$

$$D_x^m(f \cdot g) = (-1)^m D_x^m(g \cdot f), \quad (3.4)$$

$$D_x^m(f \cdot f) = 0, \quad \text{for } m \text{ odd}, \quad (3.5)$$

$$D_x^m D_t^n (e^{k_1 x - \omega_1 t} \cdot e^{k_2 x - \omega_2 t}) = (k_1 - k_2)^m (-\omega_1 + \omega_2)^n e^{(k_1 + k_2)x - (\omega_1 + \omega_2)t}. \quad (3.6)$$

In solving a PDE

$$\mathcal{F}(u, u_x, u_{xx}, \dots, u_t, u_{tt}, \dots, u_{xt}, \dots) = 0, \quad (3.7)$$

a change of the dependent variable is made,

$$u(x, t) = \mathcal{G}(f(x, t)) \quad (3.8)$$

where \mathcal{G} defines the new transformation, usually involving derivatives with respect to either x , t or both. A typical example is $u = \partial_x^2 \ln f$.

Using the differential operators, the equation for f can then be written in bilinear form. Once in bilinear form, the equation is then solved by assuming that f is of the form

$$f(x, t) = 1 + \sum_{n=1}^{\infty} \epsilon^n f^{(n)}(x, t). \quad (3.9)$$

The term ϵ in (3.9) is a bookkeeping parameter (not a small parameter), and $f^{(i)}(x, t)$, for $i = 1, 2$, etc., are unknown functions to be found by substituting (3.9) into the bilinear form of the PDE and solving the equations obtained by equating the different powers of ϵ to zero.

3.2 The Simplified Version of Hirota's Method

As mentioned earlier, Hirota's method can be used to find exact solutions of nonlinear PDEs by transforming these equations into bilinear form. Finding bilinear forms for most nonlinear PDEs, if they exist, is a hard task. Due to this difficulty, I will introduce a simplified version of Hirota's method without ever using the bilinear forms. Furthermore, the simplified method can easily be implemented in any symbolic manipulation package.

To demonstrate the simplified method, consider the ubiquitous KdV equation

$$u_t + 6uu_x + u_{3x} = 0. \quad (3.10)$$

First, a change of dependent variable

$$u(x, t) = 2 \frac{\partial^2 \ln f(x, t)}{\partial x^2} = 2 \frac{(ff_{xx} - f_x^2)}{f^2} \quad (3.11)$$

allows the transformation of (3.10) into a quadratic equation in f and its derivatives,

$$f(f_{xt} + f_{4x}) - f_x f_t - 4f_x f_{3x} + 3f_{2x}^2 = 0. \quad (3.12)$$

Note that the transformation (3.11) follows from the truncated Painlevé expansion introduced in Chapter 1.

Compared to some other equations, the KdV equation is considered to be simple in terms of its bilinear form [67] which can be written as

$$D_x \left[D_t + D_x^3 \right] (f(x, t) \cdot f(x, t)) = 0. \quad (3.13)$$

To make Hirota's technique applicable to equations that cannot be written in bilinear form, Hirota's bilinear operators will intentionally not be introduced. Instead, I write (3.12) as

$$f\mathcal{L}(f) + \mathcal{N}(f, f) = 0, \quad (3.14)$$

where \mathcal{L} denotes the linear differential operator

$$\mathcal{L} = \frac{\partial^2}{\partial x \partial t} + \frac{\partial^4}{\partial x^4}, \quad (3.15)$$

and where the nonlinear differential operator \mathcal{N} is defined as

$$\mathcal{N}(f, g) = -\frac{\partial f}{\partial x} \frac{\partial g}{\partial t} - 4 \frac{\partial f}{\partial x} \frac{\partial^3 g}{\partial x^3} + 3 \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 g}{\partial x^2}. \quad (3.16)$$

Now, I seek a formal solution of the type

$$f(x, t) = 1 + \sum_{n=1}^{\infty} \epsilon^n f^{(n)}(x, t), \quad (3.17)$$

where ϵ serves as a bookkeeping parameter, not a small quantity. Proceeding as in Hirota's method, I substitute (3.17) into (3.14), and equate to zero the different powers of ϵ , so that

$$O(\epsilon^1) : \mathcal{L}f^{(1)} = 0 \quad (3.18)$$

$$O(\epsilon^2) : \mathcal{L}f^{(2)} = -\mathcal{N}(f^{(1)}, f^{(1)}) \quad (3.19)$$

$$O(\epsilon^3) : \mathcal{L}f^{(3)} = -f^{(1)}\mathcal{L}f^{(2)} - \mathcal{N}(f^{(1)}, f^{(2)}) - \mathcal{N}(f^{(2)}, f^{(1)}) \quad (3.20)$$

⋮

$$O(\epsilon^n) : \mathcal{L}f^{(n)} = -\sum_{j=1}^{n-1} [\mathcal{N}(f^{(j)}, f^{(n-j)}) + f^{(j)}\mathcal{L}f^{(n-j)}]. \quad (3.21)$$

The N-soliton solution of the KdV is then generated from

$$f^{(1)} = \sum_{i=1}^N f_i = \sum_{i=1}^N \exp(\theta_i) = \sum_{i=1}^N \exp(k_i x - \omega_i t + \delta_i), \quad (3.22)$$

where k_i , ω_i and δ_i are real constants. Substitution of (3.22) into (3.18) yields

$$P(k_i, \omega_i) = 0, \quad (3.23)$$

where

$$P(k_i, \omega_i) = -\omega_i k_i + k_i^4, \quad i = 1, 2, \dots, N. \quad (3.24)$$

The dispersion law is thus given by $\omega_i = k_i^3$. Using (3.22), one then computes the right-hand side (RHS) of (3.19):

$$-\sum_{i,j=1}^N 3k_i k_j^2 (k_i - k_j) f_i f_j = \sum_{1 \leq i < j \leq N} 3k_i k_j (k_i - k_j)^2 f_i f_j. \quad (3.25)$$

Observe that the terms in f_i^2 drop out, which is typical for equations admitting solitons. Furthermore, (3.25) determines the form of $f^{(2)}$:

$$\begin{aligned} f^{(2)} &= \sum_{1 \leq i < j \leq N} a_{ij} f_i f_j \\ &= \sum_{1 \leq i < j \leq N} a_{ij} \exp[(k_i + k_j)x - (\omega_i + \omega_j)t + (\delta_i + \delta_j)]. \end{aligned} \quad (3.26)$$

With (3.15), (3.24), and (3.26) the left hand side (LHS) of (3.19) is readily computed:

$$\begin{aligned}\mathcal{L}f^{(2)} &= \sum_{1 \leq i < j \leq N} P(k_i + k_j, \omega_i + \omega_j) a_{ij} f_i f_j \\ &= \sum_{1 \leq i < j \leq N} 3k_i k_j (k_i + k_j)^2 a_{ij} f_i f_j.\end{aligned}\quad (3.27)$$

Equating (3.25) and (3.27) determines

$$a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}, \quad 1 \leq i < j \leq N. \quad (3.28)$$

Proceeding in a similar fashion with (3.20) leads to the explicit form of $f^{(3)}$. For example, for $N = 3$ I will have

$$\begin{aligned}f^{(3)} &= b_{123} \exp(\theta_1 + \theta_2 + \theta_3) \\ &= b_{123} \exp[(k_1 + k_2 + k_3)x - (\omega_1 + \omega_2 + \omega_3)t + (\delta_1 + \delta_2 + \delta_3)],\end{aligned}\quad (3.29)$$

with

$$b_{123} = a_{12} a_{13} a_{23} = \frac{(k_1 - k_2)^2 (k_1 - k_3)^2 (k_2 - k_3)^2}{(k_1 + k_2)^2 (k_1 + k_3)^2 (k_2 + k_3)^2}. \quad (3.30)$$

Note that $f^{(3)}$ has no terms in $f_i^2 f_j$, with $i, j = 1, \dots, N$, $i \neq j$. For $N = 3$, calculate $f^{(n)}$ for $n > 3$ by using the RHS of (3.21) to find that $f^{(n)} = 0$ for $n > 3$. Therefore,

the expansion (3.17) truncates, and

$$\begin{aligned}
 f = & 1 + \exp \theta_1 + \exp \theta_2 + \exp \theta_3 \\
 & + a_{12} \exp(\theta_1 + \theta_2) + a_{13} \exp(\theta_1 + \theta_3) + a_{23} \exp(\theta_2 + \theta_3) \\
 & + b_{123} \exp(\theta_1 + \theta_2 + \theta_3),
 \end{aligned} \tag{3.31}$$

with $\epsilon = 1$. Upon substitution of (3.31) into (3.11), the well-known three-soliton solution of the KdV equation follows. Figure 15 shows the interaction between the three-soliton solution for the KdV equation for $k_1 = 1, k_2 = \frac{5}{4}, k_3 = \frac{3}{2}$, and for (a) $t = -5$, (b) $t = 0$, and (c) $t = 5$ in two-dimensions. Figure 16 shows the three-soliton solution in three-dimensions. The N -soliton solution for any $N > 3$ can also be constructed in a similar way. However, the calculations become very lengthy, and it is more elegant to find the form of the N -soliton solution via mathematical induction [14].

A few other equations will now be solved using this technique, focusing on equations that do not necessarily have soliton solutions.

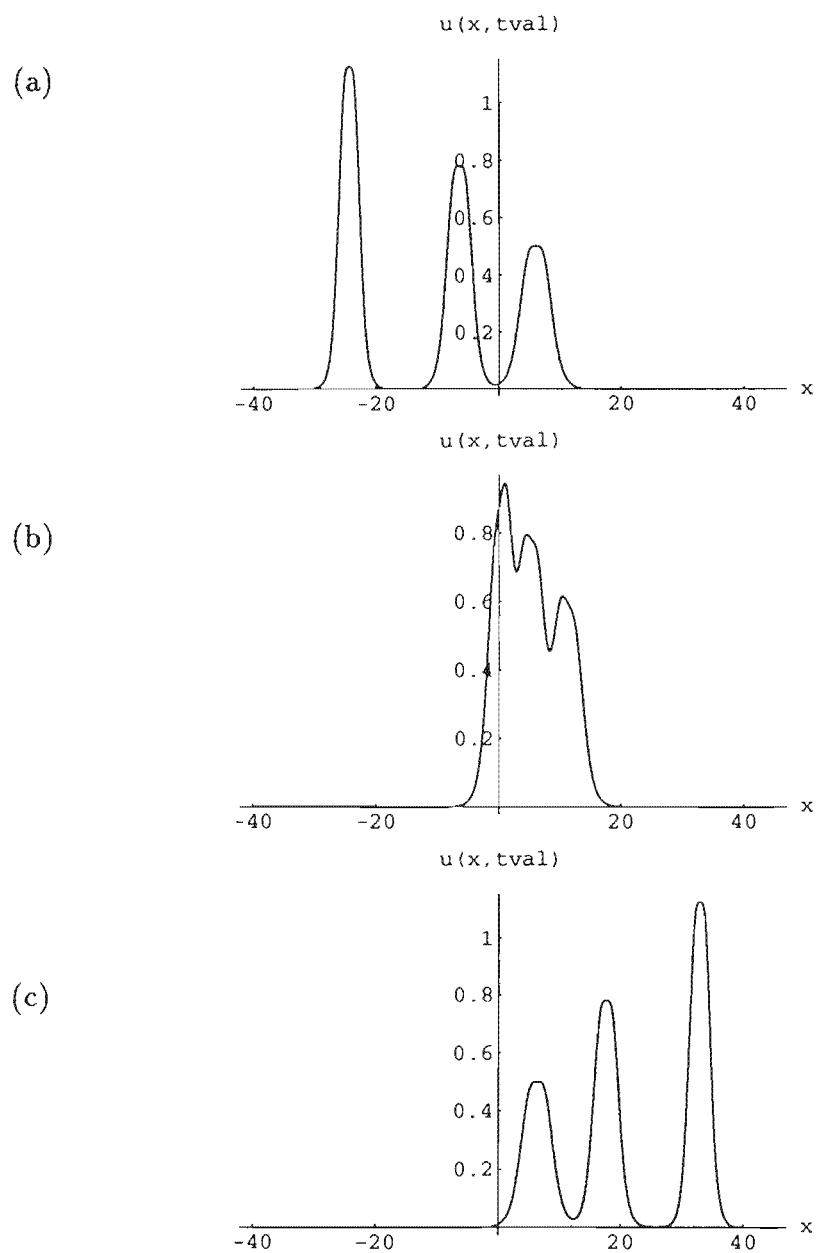


Figure 15. The three-soliton solution for the KdV equation for $tval = -5, 0, 5$ respectively.

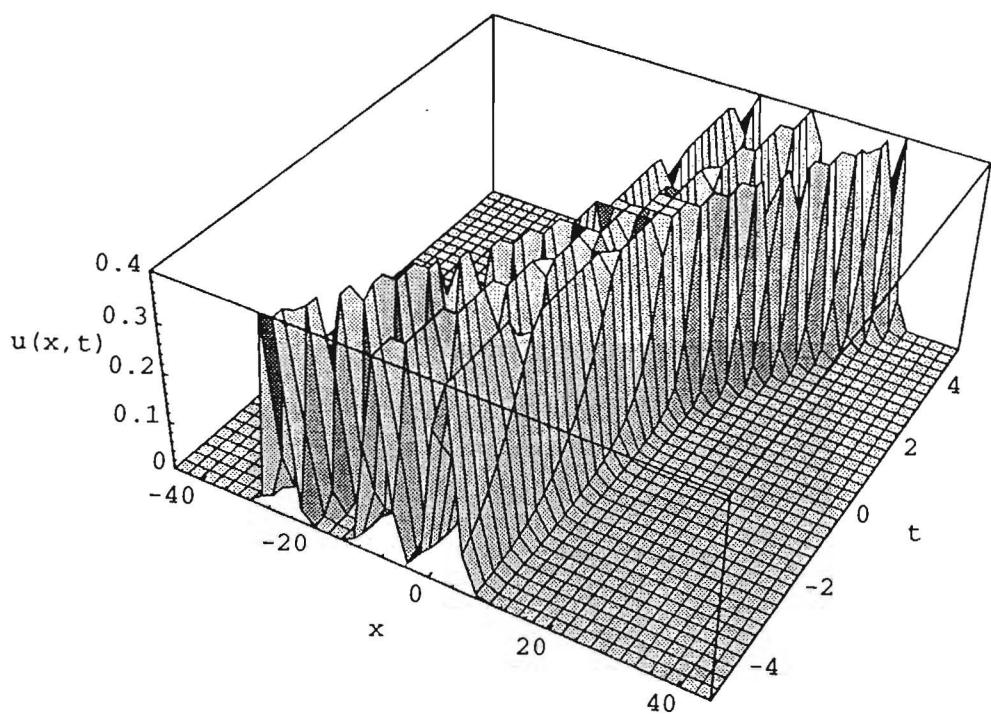


Figure 16. The three-soliton solution for the KdV equation in 3-dimensions.

3.3 Example: The Fisher Equation With Convection

The simplified version of Hirota's method will now be applied to the Fisher equation with convection [54]

$$u_t + kuu_x - u_{xx} - u(1 - u) = 0. \quad (3.32)$$

This equation describes the propagation of nerve pulses. Painlevé analysis reveals that

$$u(x, t) = -\frac{2}{k} \frac{\partial \ln f(x, t)}{\partial x} = -\frac{2}{k} \left(\frac{f_x}{f} \right). \quad (3.33)$$

Using this transformation, (3.32) can be replaced by the quadratic equation

$$f(f_{3x} + f_x - f_{xt}) + (f_x)(f_t - f_{2x} + \frac{2}{k} f_x) = 0, \quad (3.34)$$

or written compactly,

$$f\mathcal{L}(f) + \mathcal{N}(f, f) = 0. \quad (3.35)$$

Here, the nonlinear operator is defined as

$$\mathcal{N}(f, g) = \left(\frac{\partial f}{\partial x} \right) \left(\frac{\partial g}{\partial t} - \frac{\partial^2 g}{\partial x^2} + \frac{2}{k} \frac{\partial g}{\partial x} \right). \quad (3.36)$$

Seeking a solution of type (3.17), the scheme (3.18)–(3.21) remains the same, but \mathcal{L} must be replaced by

$$\mathcal{L} \bullet = \frac{\partial^3 \bullet}{\partial x^3} + \frac{\partial \bullet}{\partial x} - \frac{\partial^2 \bullet}{\partial x \partial t}. \quad (3.37)$$

From (3.18), one obtains

$$\omega_i = -(1 + k_i^2), \quad i = 1, 2, \dots, N, \quad (3.38)$$

and consequently, (3.19) reduces to

$$\mathcal{L} f^{(2)} = - \sum_{i=1}^N k_i (1 + \frac{2}{k} k_i) f_i^2 - \sum_{1 \leq i < j \leq N} \frac{4}{k} k_i k_j (k_i + k_j) f_i f_j. \quad (3.39)$$

An essential difference with the KdV case is the presence of terms in f_i^2 . It is no longer possible to derive a two- or three-soliton solution, much less an N -soliton solution. Furthermore, inclusion of terms in f_i^2 in $f^{(2)}$ would not help, because then $f^{(n)} \neq 0$, for all n , and f would be an infinite series.

The alternative is to get rid of the first sum in the RHS of (3.39) by setting $k_i = -\frac{k}{2}$. Rejecting equal values for k_i leads to the case $N = 1$. Taking into account (3.17), (3.22) and the dispersion law, one finds that

$$f(x, t) = 1 + \exp \theta = 1 + \exp[-\frac{k}{2}x + \frac{1}{4}(4 + k^2)t + \delta]. \quad (3.40)$$

Note that cf , with c constant, is also a solution of (3.35) and therefore, with (3.33),

$$u(x, t) = \frac{\exp \theta}{c + \exp \theta} = \frac{\exp[-\frac{k}{2}x + \frac{1}{4}(4 + k^2)t + \delta]}{c + \exp[-\frac{k}{2}x + \frac{1}{4}(4 + k^2)t + \delta]}, \quad (3.41)$$

where $|c|$ has been absorbed in the phase δ . The final solution can then be written in a more pleasing form as

$$u(x, t) = \frac{1 + \tanh \frac{1}{2}\theta}{(1 + c) + (1 - c)\tanh \frac{1}{2}\theta} \quad (3.42)$$

with

$$\theta = -\frac{k}{2}x + \frac{1}{4}(4 + k^2)t + \delta. \quad (3.43)$$

The special case corresponding to $c = 1$ was also derived by Malfliet via a tanh-method [54].

The solution of the Fisher equation without convection term ($k = 0$) can be obtained in a similar way. From the truncated Painlevé expansion one determines the transformation,

$$\begin{aligned} u(x, t) &= -6 \frac{\partial^2 \ln f(x, t)}{\partial x^2} + \frac{6}{5} \frac{\partial \ln f(x, t)}{\partial t} \\ &= -6 \frac{(ff_{xx} - f_x^2)}{f^2} + \frac{6}{5} \left(\frac{f_t}{f} \right), \end{aligned} \quad (3.44)$$

which allows the production of a quadratic equation in f and its derivatives (not

explicitly shown here), which can be rewritten as

$$f\mathcal{L}(f) + \mathcal{N}(f, f) = 0, \quad (3.45)$$

where the nonlinear operator \mathcal{N} is defined as

$$\begin{aligned} \mathcal{N}(f, g) = & -100 \frac{\partial f}{\partial x} \frac{\partial^3 g}{\partial x^3} + 75 \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 g}{\partial x^2} - 30 \frac{\partial f}{\partial t} \frac{\partial^2 g}{\partial x^2} - 25 \frac{\partial f}{\partial x} \frac{\partial g}{\partial x} \\ & + 60 \frac{\partial^2 f}{\partial x \partial t} \frac{\partial g}{\partial x} + \frac{\partial f}{\partial t} \frac{\partial g}{\partial t} \end{aligned} \quad (3.46)$$

and the linear operator \mathcal{L} is given by

$$\mathcal{L}\bullet = 25 \frac{\partial^4 \bullet}{\partial x^4} + 25 \frac{\partial^2 \bullet}{\partial x^2} + 5 \frac{\partial^2 \bullet}{\partial t^2} - 30 \frac{\partial^3 \bullet}{\partial t \partial x^2} - 5 \frac{\partial \bullet}{\partial t}. \quad (3.47)$$

Proceeding with the method as in the previous examples, one gets

$$u(x, t) = c^2 \frac{[1 + \tanh \frac{1}{2}\theta]^2}{[(1+c) - (1-c)\tanh \frac{1}{2}\theta]^2}, \quad (3.48)$$

with c any constant, and with

$$\theta = \frac{1}{\sqrt{6}}x + \frac{5}{6}t + \delta. \quad (3.49)$$

Note that (3.48) does not follow from (3.42) in the limit for $k \rightarrow 0$.

Chapter 4

The General Class of 5th-Order Evolution Equations

In this chapter, the truncated Painlevé expansion and the simplified version of Hirota's method will be applied to construct one- and multi-soliton solutions of a class of 5th-order evolution equations of the form

$$u_t + \alpha u u_{3x} + \beta u_x u_{2x} + \gamma u^2 u_x + u_{5x} = 0, \quad (4.1)$$

where α , β and γ are real constants. This class includes the well-known Lax [68], Sawada-Kotera (SK) [19], Kaup-Kupershmidt (KK) [69, 74, 47, 71] and Ito equations [72].

As the constants α , β and γ change, the properties of the equation (4.1) drastically change. For instance, the Lax equation with $\alpha = 10$, $\beta = 20$, and $\gamma = 30$, and the SK equation where $\alpha = \beta = \gamma = 5$, are completely integrable. These two equations have N-soliton solutions and an infinite set of conserved densities. Another example is the KK equation with $\alpha = 10$, $\beta = 25$, and $\gamma = 20$, which is known to be

integrable [47], and has bilinear representations [73, 74], but for which the explicit form of the N-soliton solutions is not known. A fourth equation in this class is the Ito equation, with $\alpha = 3$, $\beta = 6$, and $\gamma = 2$, which is not completely integrable, but has a limited number of special conserved densities [79].

Obviously, for arbitrary values of the constants α , β , and γ , the equation (4.1) is not completely integrable, and therefore does not admit soliton solutions, which does not exclude the existence of solitary wave solutions. Note that with scales on u , x , and t , the equations cannot be transformed into one another; they are fundamentally different. Using the simplified version of Hirota's method, I will construct exact two-soliton and three-soliton solutions for the KK equation.

4.1 Application of the Truncated Painlevé Expansion Method

In the early stages of this study, I tried to solve (4.1) using the truncated Painlevé expansion method. It was no surprise that some constraints on the coefficients must be satisfied in order for the equation to have solitary-wave solutions. Starting with the truncated Painlevé expansion, one gets the following form of the transformation:

$$u = K \frac{\partial^2}{\partial x^2} \ln g, \quad (4.2)$$

where $g(x, t)$, and K is a constant. Upon substitution of (4.2) into (4.1), and solving the resulting equations for the choice

$$g = 1 + \exp(\theta), \quad \text{where } \theta = kx - \omega t + \delta, \quad (4.3)$$

one is forced to take

$$\gamma = \frac{\alpha^2 + \alpha\beta}{10}, \quad (4.4)$$

and

$$K = \frac{60}{\alpha} \quad \text{or} \quad K = \frac{60}{\alpha + \beta} \quad (4.5)$$

to be able to proceed with the computation of a solitary-wave solution.

The first choice for K results in an incompatible equation, hence no solution is possible. For the second choice of K , the solitary-wave solution is

$$u = \frac{15k^2}{\alpha + \beta} \operatorname{sech}^2 \left(\frac{kx - k^5t + \delta}{2} \right). \quad (4.6)$$

It is worth mentioning that while (4.6) is a solution for the Lax and SK equations, the values for the constants in the KK and Ito equations do not satisfy the constraint (4.4); hence (4.6) does not satisfy the KK and Ito equations. Therefore, these equations will be solved separately using a slightly more general form of the truncated Painlevé expansion method. I will first show why a generalization is needed.

Consider the Kaup-Kupershmidt equation

$$u_t + 10uu_{3x} + 25u_xu_{2x} + 20u^2u_x + u_{5x} = 0. \quad (4.7)$$

Notwithstanding the fact that this equation differs from the Lax and SK equations by the choice of the constants, it is still completely integrable. It also has the Painlevé property; in other words, its solution (as a Laurent series) has no worse singularities than movable poles. From the truncated Painlevé expansion, one gets the transformations

$$u = \frac{3}{2} \frac{\partial^2}{\partial x^2} \ln g, \quad (4.8)$$

and

$$u = 12 \frac{\partial^2}{\partial x^2} \ln g. \quad (4.9)$$

If these transformations are applied to the equation, a solution cannot be constructed due to inconsistencies in the resulting equations for $g(x, t)$. If, however, constants are added like A and B to the transformations, i.e.

$$u = \frac{3}{2} \frac{\partial^2}{\partial x^2} \ln g + A, \quad (4.10)$$

$$u = 12 \frac{\partial^2}{\partial x^2} \ln g + B, \quad (4.11)$$

then these transformations can be used to solve the equation. With

$$g = 1 + \exp(\theta), \quad \theta = kx - \omega t + \delta, \quad (4.12)$$

the solutions are

$$u = -\frac{1}{8}k^2 \left[3 \tanh^2 \left(\frac{kx - \frac{k^5}{16}t + \delta}{2} \right) - 2 \right], \quad (4.13)$$

and

$$u = -k^2 \left[3 \tanh^2 \left(\frac{kx - 11k^5t + \gamma}{2} \right) - 2 \right], \quad (4.14)$$

respectively.

Furthermore, I have studied the Ito equation,

$$u_t + 3uu_{3x} + 6u_xu_{2x} + 2u^2u_x + u_{5x} = 0, \quad (4.15)$$

which was derived by Ito in 1980 [72].

Ito's equation is not completely integrable. From a truncated Painlevé expansion, one gets the transformation

$$u = 30 \frac{\partial^2}{\partial x^2} \ln g, \quad (4.16)$$

which does not yet lead to a closed form solution. Again, adding a constant such as

A to the transformation, i.e.

$$u = 30 \frac{\partial^2}{\partial x^2} \ln g + A, \quad (4.17)$$

allows the construction of an exact solution, namely,

$$u = -\frac{5k^2}{2} \left[3 \tanh^2 \left(\frac{kx - 6k^5 + \delta}{2} \right) - 2 \right], \quad (4.18)$$

where k and δ are constant.

4.2 Direct Construction of Two-Soliton Solution

In this section I will investigate the constraints on α , β and γ under which a two-soliton solution of the class (4.1) can be constructed. I still assume that the two-soliton solution will have the form

$$u = \frac{60}{\alpha + \beta} \frac{\partial^2}{\partial x^2} \ln f, \quad (4.19)$$

for unknown function $f(x, t)$. For a two-soliton, I start with f of the form

$$f = 1 + f_1 + f_2 + a_{12}f_1f_2 \quad (4.20)$$

where a_{12} is still unknown, and

$$f_i = \exp(\theta_i), \text{ where } \theta_i = k_i x - \omega_i t + \delta_i \text{ for } i = 1, 2. \quad (4.21)$$

Note that (4.20) implies that the two-soliton solution is a KdV-type solution. Substitute (4.19) into (4.1), integrate once with respect to x , and this time do not split the resulting equation into different powers of f . Adherence to the choice of f in (4.20) leads to the dispersion relation, relating k_i and ω_i

$$\omega_i = k_i^5 \quad \text{for } i = 1, 2. \quad (4.22)$$

Cancellation of power terms in f_1 and f_2 requires that

$$\gamma = \frac{\alpha^2 + \alpha\beta}{10} \text{ and } \beta = \alpha \text{ or } \beta = 2\alpha. \quad (4.23)$$

These two cases will now be investigated separately.

Case 1: $\beta = \alpha$

For this case

$$a_{12} = \frac{(k_2 - k_1)^2(k_2^2 - k_1 k_2 + k_1^2)}{(k_2 + k_1)^2(k_2^2 + k_1 k_2 + k_1^2)}. \quad (4.24)$$

Notice that this case reduces to the SK equation.

Case 2: $\beta = 2\alpha$

For this case

$$a_{12} = \frac{(k_2 - k_1)^2}{(k_2 + k_1)^2}, \quad (4.25)$$

which corresponds to the Lax equation. In conclusion, these are the only possible cases in this class that are integrable and for which the two-soliton solution is a KdV-type solution.

4.3 Application of The Simplified Version of Hirota's Method

In this section, the simplified version of Hirota's method will be used to find one- and multi-soliton solutions of the well-known integrable equations mentioned in the introduction to this chapter. This technique no longer requires the assumption of the form of f as in (4.20).

4.3.1 The Sawada-Kotera Equation

The first example is the SK equation [19]

$$u_t + 5uu_{3x} + 5u_x u_{2x} + 5u^2 u_x + u_{5x} = 0. \quad (4.26)$$

When Painlevé analysis is applied to the equation, one gets $\alpha = -2$, and the transformation

$$u(x, t) = 6 \frac{\partial^2 \ln f(x, t)}{\partial x^2}. \quad (4.27)$$

Substitution of (4.27) into (4.26) gives a quadratic equation in f and its derivatives:

$$f(f_{6x} + f_{tx}) - 6f_x f_{5x} + 15f_{xx} f_{4x} - 10f_{3x}^2 - f_t f_x = 0. \quad (4.28)$$

Equation (4.28) can compactly be written as

$$f\mathcal{L}(f) + \mathcal{N}(f, f) = 0, \quad (4.29)$$

where the linear operator \mathcal{L} is defined as

$$\mathcal{L}\bullet = \frac{\partial^2 \bullet}{\partial x \partial t} + \frac{\partial^6 \bullet}{\partial x^6}, \quad (4.30)$$

and the nonlinear operator \mathcal{N} is given by

$$\mathcal{N}(f, g) = -6f_x g_{5x} + 15f_{xx} g_{4x} - 10f_{3x} g_{3x} - f_t g_x. \quad (4.31)$$

Since this equation reduces to a quadratic equation, I can use the perturbation scheme mentioned in Chapter Three.

$$O(\epsilon^1) : \mathcal{L}f^{(1)} = 0 \quad (4.32)$$

$$O(\epsilon^2) : \mathcal{L}f^{(2)} = -\mathcal{N}(f^{(1)}, f^{(1)}) \quad (4.33)$$

$$O(\epsilon^3) : \mathcal{L}f^{(3)} = -f^{(1)}\mathcal{L}f^{(2)} - \mathcal{N}(f^{(1)}, f^{(2)}) - \mathcal{N}(f^{(2)}, f^{(1)}) \quad (4.34)$$

⋮

$$O(\epsilon^n) : \mathcal{L}f^{(n)} = -\sum_{j=1}^{n-1} [\mathcal{N}(f^{(j)}, f^{(n-j)}) + f^{(j)}\mathcal{L}f^{(n-j)}]. \quad (4.35)$$

To find a three-soliton solution for the SK equation, the function $f^{(1)}$ must be of the form

$$f^{(1)} = \exp(\theta_1) + \exp(\theta_2) + \exp(\theta_3), \quad (4.36)$$

where $\theta_i = k_i x - \omega_i t + \delta_i$. Solving (4.32) for $f^{(1)}$ one gets

$$\omega_i = k_i^5 \quad \text{for } i = 1, 2, 3. \quad (4.37)$$

To find the form of $f^{(2)}$, one computes the RHS of (4.33) with $f^{(1)}$ in (4.36). The result is

$$\begin{aligned} -\mathcal{N}(f^{(1)}, f^{(1)}) &= 5 [k_1 k_2 (k_2 - k_1)^2 (k_2^2 - k_1 k_2 + k_1^2) \exp(\theta_1 + \theta_2) \\ &\quad + k_1 k_3 (k_3 - k_1)^2 (k_3^2 - k_1 k_3 + k_1^2) \exp(\theta_1 + \theta_3) \\ &\quad + k_2 k_3 (k_3 - k_2)^2 (k_3^2 - k_2 k_3 + k_2^2) \exp(\theta_2 + \theta_3)] \end{aligned} \quad (4.38)$$

which determines the form of $f^{(2)}$:

$$f^{(2)} = a_{12} \exp(\theta_1 + \theta_2) + a_{13} \exp(\theta_1 + \theta_3) + a_{23} \exp(\theta_2 + \theta_3). \quad (4.39)$$

Now, solving the LHS of (4.33) for $f^{(2)}$ yields

$$\begin{aligned} \mathcal{L}f^{(2)} = & 5 \left[a_{12}k_1k_2(k_2+k_1)^2(k_2^2+k_1k_2+k_1^2)\exp(\theta_1+\theta_2) \right. \\ & + a_{13}k_1k_3(k_3+k_1)^2(k_3^2+k_1k_3+k_1^2)\exp(\theta_1+\theta_3) \\ & \left. + a_{23}k_2k_3(k_3+k_2)^2(k_3^2+k_2k_3+k_2^2)\exp(\theta_2+\theta_3) \right]. \end{aligned} \quad (4.40)$$

The coefficients a_{12} , a_{13} and a_{23} can be found by equating (4.40) to (4.38).

$$a_{ij} = \frac{(k_i - k_j)^2(k_i^2 - k_i k_j + k_j^2)}{(k_i + k_j)^2(k_i^2 + k_i k_j + k_j^2)} \quad 1 \leq i < j \leq 2, \quad (4.41)$$

Note that the coefficients a_{13} and a_{23} can be obtained from a_{12} via cyclical permutation of the wave numbers, i.e. $k_1 \rightarrow k_2$, $k_2 \rightarrow k_3$ and $k_3 \rightarrow k_1$.

To find $f^{(3)}$, the RHS of (4.34) is computed with the known functions $f^{(1)}$ and $f^{(2)}$, revealing the necessary form for $f^{(3)}$. After computation of the LHS of (4.34) and equating it with the RHS, the solution turns out to be

$$f^{(3)} = a_{12}a_{13}a_{23} \exp(\theta_1 + \theta_2 + \theta_3), \quad (4.42)$$

with a_{ij} as in (4.41). However, proceeding in the same way to find $f^{(4)}$, $f^{(5)}$, etc. results in $f^{(n)} = 0$ for $n > 3$. So, with (4.36), (4.39) and (4.42), the form of f is explicitly known. The three-soliton solution of the SK equation then follows from (4.27).

4.3.2 The Lax Equation

The equation

$$u_t + 10uu_{3x} + 20u_xu_{2x} + 30u^2u_x + u_{5x} = 0 \quad (4.43)$$

belongs to the infinite family of integrable higher-order KdV equations, discovered by Lax [68].

Painlevé analysis determines the transformation

$$u(x, t) = 2 \frac{\partial^2 \ln f(x, t)}{\partial x^2}. \quad (4.44)$$

A cubic equation in f and its derivatives is obtained after substitution of (4.44) into (4.43), and one integration with respect to x ,

$$f^2[f_{xt} + f_{6x}] - f[f_tf_x + 6f_xf_{5x} - 5f_{2x}f_{4x}] + 10[f_x^2f_{4x} - 2f_xf_{2x}f_{3x} + f_{2x}^3] = 0. \quad (4.45)$$

Obviously, (4.45) can be written as

$$f^2 \mathcal{L}(f) + f \mathcal{N}_1(f, f) + \mathcal{N}_2(f, f, f) = 0, \quad (4.46)$$

where the linear operator \mathcal{L} is defined as

$$\mathcal{L}\bullet = \frac{\partial^2 \bullet}{\partial x \partial t} + \frac{\partial^6 \bullet}{\partial x^6}, \quad (4.47)$$

and the nonlinear operators are

$$\mathcal{N}_1(f, g) = -(f_t g_x + 6f_x g_{5x} - 5f_{2x} g_{4x}), \quad (4.48)$$

$$\mathcal{N}_2(f, g, h) = 10(f_x g_x h_{4x} - 2f_x g_{2x} h_{3x} + f_{2x} g_{2x} h_{2x}). \quad (4.49)$$

Again, I will try to find a soliton solution via the series expansion

$$f(x, t) = 1 + \sum_{n=1}^{\infty} \epsilon^n f^{(n)}(x, t), \quad (4.50)$$

in the hope that the series will truncate at some power in ϵ . Substitute (4.50) into (4.46) and equate the different powers of ϵ to zero:

$$O(\epsilon^1) : \mathcal{L}f^{(1)} = 0 \quad (4.51)$$

$$O(\epsilon^2) : \mathcal{L}f^{(2)} = -2f^{(1)}\mathcal{L}f^{(1)} - \mathcal{N}_1(f^{(1)}, f^{(1)}) \quad (4.52)$$

$$\begin{aligned} O(\epsilon^3) : \mathcal{L}f^{(3)} &= -2f^{(1)}\mathcal{L}f^{(2)} - 2f^{(2)}\mathcal{L}f^{(1)} - f^{(1)}{}^2\mathcal{L}f^{(1)} \\ &\quad - \mathcal{N}_1(f^{(1)}, f^{(2)}) - \mathcal{N}_1(f^{(2)}, f^{(1)}) \\ &\quad - f^{(1)}\mathcal{N}_1(f^{(1)}, f^{(1)}) - \mathcal{N}_2(f^{(1)}, f^{(1)}, f^{(1)}) \end{aligned} \quad (4.53)$$

⋮

The N-soliton solution of the Lax equation is then generated by

$$f^{(1)} = \sum_{i=1}^N f_i = \sum_{i=1}^N \exp(\theta_i) = \sum_{i=1}^N \exp(k_i x - \omega_i t + \delta_i). \quad (4.54)$$

Substituting (4.54) into (4.51) yields

$$P(k_i, \omega_i) = -\omega_i k_i + k_i^6 = 0, \quad (4.55)$$

or

$$\omega_i = k_i^5, \quad i = 1, 2, \dots, N. \quad (4.56)$$

With (4.54), the RHS of (4.52) will be

$$-\mathcal{N}_1(f^{(1)}, f^{(1)}) = \sum_{i,j=1}^N k_i k_j (k_j - k_i) (2k_j + k_i) (3k_j^2 - k_i k_j + k_i^2) f_i f_j$$

$$= \sum_{1 \leq i < j \leq N} 5k_i k_j (k_i - k_j)^2 (k_i^2 + k_i k_j + k_j^2) f_i f_j. \quad (4.57)$$

Observe that the terms in f_i^2 in (4.57) dropped out thus, $f^{(2)}$ must be of the form

$$f^{(2)} = \sum_{1 \leq i < j \leq N} a_{ij} f_i f_j = \sum_{1 \leq i < j \leq N} a_{ij} \exp[(k_i + k_j)x - (\omega_i + \omega_j)t + (\delta_i + \delta_j)]. \quad (4.58)$$

Now, computing the LHS of (4.52) gives

$$\begin{aligned} \mathcal{L}f^{(2)} &= \sum_{1 \leq i < j \leq N} P(k_i + k_j, \omega_i + \omega_j) a_{ij} f_i f_j \\ &= \sum_{1 \leq i < j \leq N} 5k_i k_j (k_i + k_j)^2 (k_i^2 + k_i k_j + k_j^2) a_{ij} f_i f_j. \end{aligned} \quad (4.59)$$

Equating (4.57) and (4.59) leads to

$$a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}, \quad 1 \leq i < j \leq N. \quad (4.60)$$

For the two-soliton solution, one can show that $f^{(n)} = 0$ for $n > 2$, and thus

$$f = 1 + \exp(\theta_1) + \exp(\theta_2) + a_{12} \exp(\theta_1 + \theta_2), \quad (4.61)$$

with

$$a_{12} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}, \quad \text{and} \quad \theta_i = k_i x - k_i^5 t + \delta_i. \quad (4.62)$$

The three-soliton solution of the Lax equation is constructed in a similar way.

The form of $f^{(3)}$ can be found by solving the RHS of (4.53) and assuming that

$$f^{(1)} = \sum_{i=1}^3 \exp(k_i x - \omega_i t + \delta_i) = \sum_{i=1}^3 \exp(\theta_i). \quad (4.63)$$

Proceeding as before, the RHS of (4.53) determines the form of $f^{(3)}$,

$$f^{(3)} = b_{123} \exp(\theta_1 + \theta_2 + \theta_3), \quad (4.64)$$

and the balance of terms in $\exp(\theta_1 + \theta_2 + \theta_3)$ in LHS and RHS of (4.53) leads to

$$\begin{aligned} b_{123} &= \frac{(k_1 - k_2)^2(k_2 - k_3)^2(k_1 - k_3)^2}{(k_1 + k_2)^2(k_2 + k_3)^2(k_1 + k_3)^2} \\ &= a_{12} a_{23} a_{13}. \end{aligned} \quad (4.65)$$

Therefore, the three-soliton solution corresponds to

$$f = 1 + \sum_{i=1}^3 \exp(\theta_i) + \sum_{1 \leq i < j \leq 3} a_{ij} \exp(\theta_i + \theta_j) + a_{12} a_{23} a_{13} \exp(\theta_1 + \theta_2 + \theta_3), \quad (4.66)$$

since $f^{(n)} = 0$ for $n > 3$. Using the same procedure, the N-soliton solution can be found for any selected value of N . In practice, the computations become more complex as higher equations in the perturbation scheme are solved.

4.4 The Kaup-Kupershmidt Equation

Although the KK equation is known to be completely integrable [47] and has bilinear representations [73, 74], to my knowledge (and after consulting with the leading experts) its soliton solutions have not been constructed before. There are two reasons for that. First, the computation of the soliton solutions is complicated and lengthy, to the point that without a symbolic manipulation program the computations would be impossible. Second, the structure of the soliton solution is quite different from the one for the Lax [68] and SK [75, 69] equations, which all belong to the same class of fifth-order evolution equations.

I will now focus on the construction of the soliton solutions of the KK equation,

$$u_t + 10uu_{3x} + 25u_xu_{2x} + 20u^2u_x + u_{5x} = 0, \quad (4.67)$$

which is the hardest to solve among the integrable cases.

Using the transformation

$$u = \frac{3}{2} \frac{\partial^2}{\partial x^2} \ln f, \quad (4.68)$$

which follows from a truncated Painlevé expansion, one gets a fourth degree equation

in f and its derivatives,

$$\begin{aligned} & 4f^3[f_{xt} + f_{6x}] - f^2[4f_t f_x - 5f_{3x}^2 + 24f_x f_{5x}] \\ & - 30f[f_x f_{2x} f_{3x} - 2f_x^2 f_{4x}] + 15[3f_x^2 f_{2x}^2 - 4f_x^3 f_{3x}] = 0. \end{aligned} \quad (4.69)$$

Equation (4.69) can be written in operator form as

$$f^3\mathcal{L}(f) + f^2\mathcal{N}_1(f, f) + f\mathcal{N}_2(f, f, f) + \mathcal{N}_3(f, f, f, f) = 0, \quad (4.70)$$

where the linear and nonlinear operators are defined as

$$\mathcal{L}\bullet = 4\left(\frac{\partial^2 \bullet}{\partial x \partial t} + \frac{\partial^6 \bullet}{\partial x^6}\right), \quad (4.71)$$

and

$$\mathcal{N}_1(f, g) = -4f_t g_x + 5f_{3x} g_{3x} - 24f_x g_{5x}, \quad (4.72)$$

$$\mathcal{N}_2(f, g, h) = -30f_x g_{2x} h_{3x} + 60f_x g_x h_{4x}, \quad (4.73)$$

$$\mathcal{N}_3(f, g, h, j) = 45f_x g_x h_{2x} j_{2x} - 60f_x g_x h_x j_{3x}, \quad (4.74)$$

for auxiliary functions $f(x, t)$ through $j(x, t)$.

In seeking a solution of the form

$$f(x, t) = 1 + \sum_{n=1}^{\infty} \epsilon^n f^{(n)}(x, t), \quad (4.75)$$

I substitute (4.75) into (4.70) and equate the coefficients of different powers of ϵ to zero. The following perturbation scheme follows:

$$O(\epsilon^1) : \mathcal{L}f^{(1)} = 0 \quad (4.76)$$

$$O(\epsilon^2) : \mathcal{L}f^{(2)} = -\mathcal{N}_1(f^{(1)}, f^{(1)}) \quad (4.77)$$

$$\begin{aligned} O(\epsilon^3) : \mathcal{L}f^{(3)} &= -3f^{(1)}\mathcal{L}f^{(2)} - 2f^{(1)}\mathcal{N}_1(f^{(1)}, f^{(1)}) - \mathcal{N}_1(f^{(2)}, f^{(1)}) \\ &\quad - \mathcal{N}_1(f^{(1)}, f^{(2)}) - \mathcal{N}_2(f^{(1)}, f^{(1)}, f^{(1)}) \end{aligned} \quad (4.78)$$

$$\begin{aligned} O(\epsilon^4) : \mathcal{L}f^{(4)} &= -3f^{(1)}\mathcal{L}f^{(3)} - 3f^{(2)}\mathcal{L}f^{(2)} - 3f^{(1)^2}\mathcal{L}f^{(2)} - \mathcal{N}_1(f^{(1)}, f^{(3)}) - \mathcal{N}_1(f^{(3)}, f^{(1)}) \\ &\quad - \mathcal{N}_1(f^{(2)}, f^{(2)}) - 2f^{(1)}\mathcal{N}_1(f^{(1)}, f^{(2)}) - 2f^{(1)}\mathcal{N}_1(f^{(2)}, f^{(1)}) \\ &\quad - 2f^{(2)}\mathcal{N}_1(f^{(1)}, f^{(1)}) - f^{(1)^2}\mathcal{N}_1(f^{(1)}, f^{(1)}) - \mathcal{N}_2(f^{(1)}, f^{(1)}, f^{(2)}) \\ &\quad - \mathcal{N}_2(f^{(1)}, f^{(2)}, f^{(1)}) - \mathcal{N}_2(f^{(2)}, f^{(1)}, f^{(1)}) - f^{(1)}\mathcal{N}_2(f^{(1)}, f^{(1)}, f^{(1)}) \\ &\quad - \mathcal{N}_3(f^{(1)}, f^{(1)}, f^{(1)}, f^{(1)}) \end{aligned} \quad (4.79)$$

⋮

Noticeably, the number of terms in the right hand side (RHS) of the equations grows rapidly as the order in ϵ increases.

4.4.1 The One-Soliton Solution

To find the one-soliton solution, take

$$f^{(1)} = \exp(\theta), \quad \text{with } \theta = kx - \omega t + \delta. \quad , \quad (4.80)$$

Equation (4.76) gives the dispersion law $\omega = k^5$. To solve (4.77), I first compute its RHS,

$$-\mathcal{N}(f^{(1)}, f^{(1)}) = 15k^6 \exp(2\theta). \quad (4.81)$$

Thus, $f^{(2)}$ will be of the form

$$f^{(2)} = a \exp(2\theta). \quad (4.82)$$

The value $a = \frac{1}{16}$ follows from calculating the LHS of (4.77),

$$\mathcal{L}f^{(2)} = 240ak^6 \exp(2\theta), \quad (4.83)$$

and equating it with (4.81).

It is straightforward to check that $f^{(n)} = 0$ for $n \geq 3$. Therefore, using (4.68)

with $\epsilon = 1$, the one-soliton solution generated by

$$f = 1 + \exp(\theta) + \frac{1}{16} \exp(2\theta) \quad (4.84)$$

is

$$u = 24k^2 \frac{\exp(\theta)[4\exp(\theta) + \exp(2\theta) + 16]}{[16\exp(\theta) + \exp(2\theta) + 16]^2}, \quad (4.85)$$

which can be written as

$$u = -24k^2 \frac{\left[\tanh\left(\frac{\theta}{2}\right) - 1\right] \left[\tanh\left(\frac{\theta}{2}\right) + 1\right] \left[13\tanh^2\left(\frac{\theta}{2}\right) - 30\tanh\left(\frac{\theta}{2}\right) + 21\right]}{\left[\tanh^2\left(\frac{\theta}{2}\right) - 30\tanh\left(\frac{\theta}{2}\right) + 33\right]^2}, \quad (4.86)$$

where $\theta = kx - k^5t + \delta$, and k and δ are arbitrary real constants.

4.4.2 The Two-Soliton Solution

For the two-soliton solution, one starts with

$$f^{(1)} = \exp(\theta_1) + \exp(\theta_2), \quad (4.87)$$

where $\theta_i = k_i x - \omega_i t + \delta_i$ for $i = 1, 2$. From (4.76) one gets

$$\omega_i = k_i^5, \quad \text{for } i = 1, 2. \quad (4.88)$$

To find $f^{(2)}$, the RHS of (4.77) has to be calculated. One obtains

$$\begin{aligned} -\mathcal{N}_1(f^{(1)}, f^{(1)}) = & \quad 15k_1^6 \exp(2\theta_1) + 15k_2^6 \exp(2\theta_2) \\ & + 10k_1 k_2 (2k_1^4 - k_1^2 k_2^2 + 2k_2^4) \exp(\theta_1 + \theta_2). \end{aligned} \quad (4.89)$$

Obviously, $f^{(2)}$ must be of the form

$$f^{(2)} = a \exp(2\theta_1) + b \exp(2\theta_2) + a_{12} \exp(\theta_1 + \theta_2). \quad (4.90)$$

In contrast to what happened for the KdV, the Lax, and SK equations the terms in $\exp(2\theta_i)$ no longer drop out. Computation of the LHS of (4.77) yields

$$\begin{aligned} \mathcal{L}f^{(2)} = & \quad 240ak_1^6 \exp(2\theta_1) + 240bk_2^6 \exp(2\theta_2) \\ & + 20a_{12}k_1 k_2 (k_1 + k_2)^2 (k_1^2 + k_1 k_2 + k_2^2) \exp(\theta_1 + \theta_2). \end{aligned} \quad (4.91)$$

Equating (4.89) and (4.91) gives $a = b = \frac{1}{16}$, and

$$a_{12} = \frac{2k_1^4 - k_1^2 k_2^2 + 2k_2^4}{2(k_1 + k_2)^2 (k_1^2 + k_1 k_2 + k_2^2)}. \quad (4.92)$$

Therefore,

$$f^{(2)} = \frac{1}{16} \exp(2\theta_1) + \frac{1}{16} \exp(2\theta_2) + \frac{2k_1^4 - k_1^2 k_2^2 + 2k_2^4}{2(k_1 + k_2)^2(k_1^2 + k_1 k_2 + k_2^2)} \exp(\theta_1 + \theta_2). \quad (4.93)$$

Proceeding in a similar way with (4.78) one finds

$$f^{(3)} = b_{12} [\exp(2\theta_1 + \theta_2) + \exp(\theta_1 + 2\theta_2)], \quad (4.94)$$

with

$$b_{12} = \frac{(k_1 - k_2)^2(k_1^2 - k_1 k_2 + k_2^2)}{16(k_1 + k_2)^2(k_1^2 + k_1 k_2 + k_2^2)}. \quad (4.95)$$

To find $f^{(4)}$, equation (4.80) has to be solved, which leads to

$$f^{(4)} = b_{12}^2 \exp(2\theta_1 + 2\theta_2) = \frac{(k_1 - k_2)^4(k_1^2 - k_1 k_2 + k_2^2)^2}{256(k_1 + k_2)^4(k_1^2 + k_1 k_2 + k_2^2)^2} \exp(2\theta_1 + 2\theta_2). \quad (4.96)$$

After verifying that all $f^{(n)}$ will be zero, for $n \geq 5$, the form of f will be

$$\begin{aligned} f &= 1 + \exp(\theta_1) + \exp(\theta_2) + \frac{1}{16} \exp(2\theta_1) + \frac{1}{16} \exp(2\theta_2) + a_{12} \exp(\theta_1 + \theta_2) \\ &\quad + b_{12} [\exp(2\theta_1 + \theta_2) + \exp(\theta_1 + 2\theta_2)] + b_{12}^2 \exp(2\theta_1 + 2\theta_2), \end{aligned} \quad (4.97)$$

with a_{12} and b_{12} as defined in (4.92) and (4.95). The two-soliton solution could be obtained by substituting (4.97) into (4.68).

4.4.3 The Three-Soliton Solution

The computation of the three-soliton solution is more difficult because the first six equations in the perturbation scheme must be solved. However, the procedure is the same as before, the only difference is that the computations become very lengthy.

Starting with

$$f^{(1)} = \sum_{i=1}^3 \exp(\theta_i) = \exp(\theta_1) + \exp(\theta_2) + \exp(\theta_3), \quad (4.98)$$

the RHS of (4.77) is evaluated to determine the form of $f^{(2)}$. Next, the coefficients of the terms in $f^{(2)}$ are computed by balancing the LHS and RHS of (4.77). After some algebra one obtains

$$f^{(2)} = \frac{1}{16} \sum_{i=1}^3 \exp(2\theta_i) + \sum_{1 \leq i < j \leq 3} a_{ij} \exp(\theta_i + \theta_j), \quad (4.99)$$

with

$$a_{ij} = \frac{2k_i^4 - k_i^2 k_j^2 + 2k_j^4}{2(k_i + k_j)^2(k_i^2 + k_i k_j + k_j^2)}, \quad 1 \leq i < j \leq 3. \quad (4.100)$$

Subsequently, $f^{(3)}$ through $f^{(6)}$ are determined; beyond that level the expansion for f truncates, i.e. $f^{(n)} = 0$ for $n \geq 7$. In summary, the results are

$$f^{(3)} = \sum_{1 \leq i < j \leq 3} b_{ij} [\exp(2\theta_i + \theta_j) + \exp(\theta_i + 2\theta_j)] + c_{123} \exp(\theta_1 + \theta_2 + \theta_3) \quad (4.101)$$

with

$$b_{ij} = \frac{(k_i - k_j)^2(k_i^2 - k_i k_j + k_j^2)}{16(k_i + k_j)^2(k_i^2 + k_i k_j + k_j^2)}, \quad (4.102)$$

and

$$\begin{aligned} c_{123} = & \frac{1}{D}[(2k_1^4 - k_1^2 k_2^2 + 2k_2^4)(k_3^8 + k_1^4 k_2^4) + (2k_1^4 - k_1^2 k_3^2 + 2k_3^4)(k_2^8 + k_1^4 k_3^4) \\ & + (2k_2^4 - k_2^2 k_3^2 + 2k_3^4)(k_1^8 + k_2^4 k_3^4)] - \frac{1}{2D}[(k_1^2 + k_2^2)(k_1^4 + k_2^4)(k_3^6 + k_1^2 k_2^2 k_3^2) \\ & + (k_1^2 + k_3^2)(k_1^4 + k_2^4)(k_2^6 + k_1^2 k_2^2 k_3^2) + (k_2^2 + k_3^2)(k_2^4 + k_3^4)(k_1^6 + k_1^2 k_2^2 k_3^2) \\ & + 12k_1^4 k_2^4 k_3^4], \end{aligned} \quad (4.103)$$

where

$$D = 4 \prod_{1 \leq i < j \leq 3} (k_i + k_j)^2(k_i^2 + k_i k_j + k_j^2). \quad (4.104)$$

Further,

$$\begin{aligned} f^{(4)} = & \sum_{1 \leq i < j \leq 3} b_{ij}^2 \exp(2\theta_i + 2\theta_j) + 16[a_{23}b_{12}b_{13} \exp(2\theta_1 + \theta_2 + \theta_3) \\ & + a_{13}b_{12}b_{23} \exp(\theta_1 + 2\theta_2 + \theta_3) + a_{12}b_{13}b_{23} \exp(\theta_1 + \theta_2 + 2\theta_3)], \end{aligned} \quad (4.105)$$

$$\begin{aligned} f^{(5)} = & 16^2 b_{12}b_{13}b_{23}[b_{12} \exp(2\theta_1 + 2\theta_2 + \theta_3) + b_{13} \exp(2\theta_1 + \theta_2 + 2\theta_3) \\ & + b_{23} \exp(\theta_1 + 2\theta_2 + 2\theta_3)], \end{aligned} \quad (4.106)$$

and, finally,

$$f^{(6)} = 16^3 b_{12}^2 b_{13}^2 b_{23}^2 \exp(2\theta_1 + 2\theta_2 + 2\theta_3). \quad (4.107)$$

The actual form of the soliton solution $u(x, t)$ of the KK equation then follows from substitution of

$$f = 1 + f^{(1)} + f^{(2)} + f^{(3)} + f^{(4)} + f^{(5)} + f^{(6)} \quad (4.108)$$

into (4.68). The resulting long expression is not shown here. Instead, I computed it for the values $k_1 = 1$, $k_2 = \frac{5}{4}$ and $k_3 = \frac{3}{2}$ and graphed the result for $t = -5, t = 0$, and $t = 5$ in Figure 17. The pictures show clearly how the solitons, which are interacting near $t = 0$, emerge from the interaction without change of shape. In Figure 18, the actual three-soliton solution is plotted in three-dimensional space.

As stated previously, the two- and three-soliton solutions for the KK equation have never been derived. The simplified Hirota method turns out to be an effective tool to solve soliton equations. The main difference with other equations in this class is that the terms in $\exp(2\theta_i)$ in $f^{(2)}$ no longer drop out. It is commonly believed that such terms have to vanish in order for the series for f to truncate. The above construction shows that this statement, while true for the KdV-type equations and also for the Lax and SK equations, is false for the KK equation.

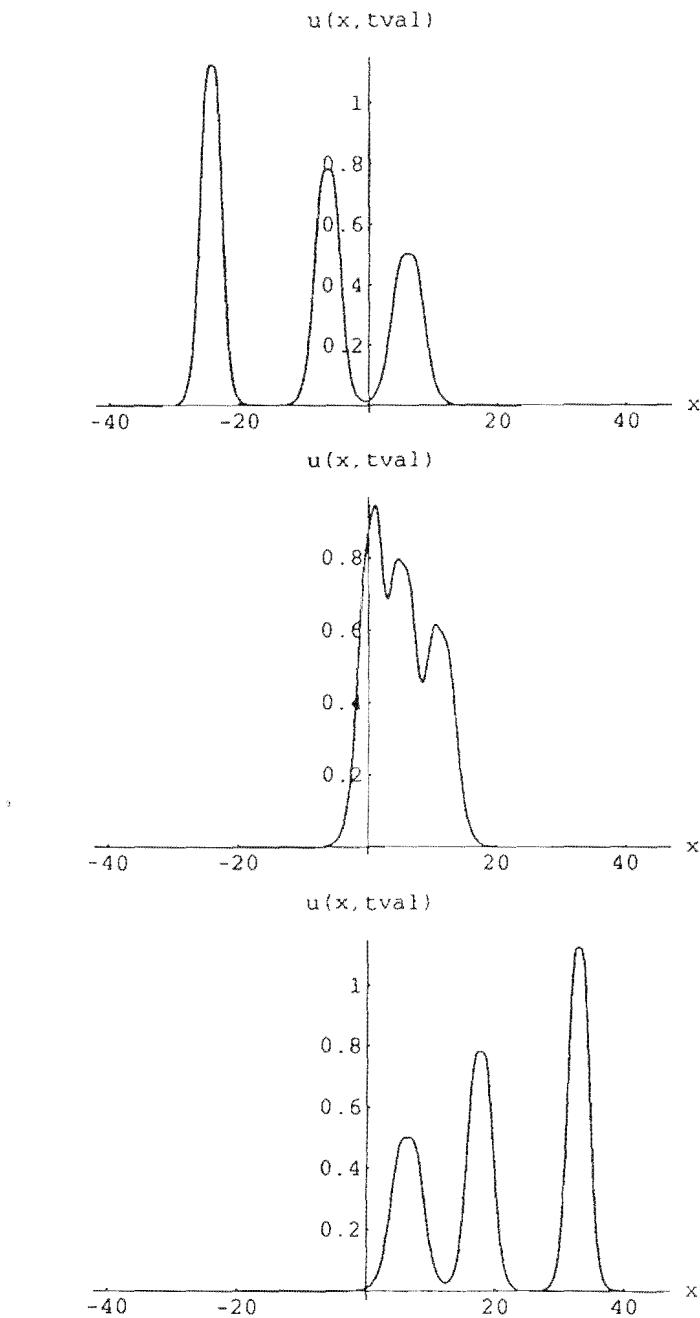


Figure 17. The three-soliton solution for KK equation in 2-dimensions for $tval=-5, 0, 5$, respectively.

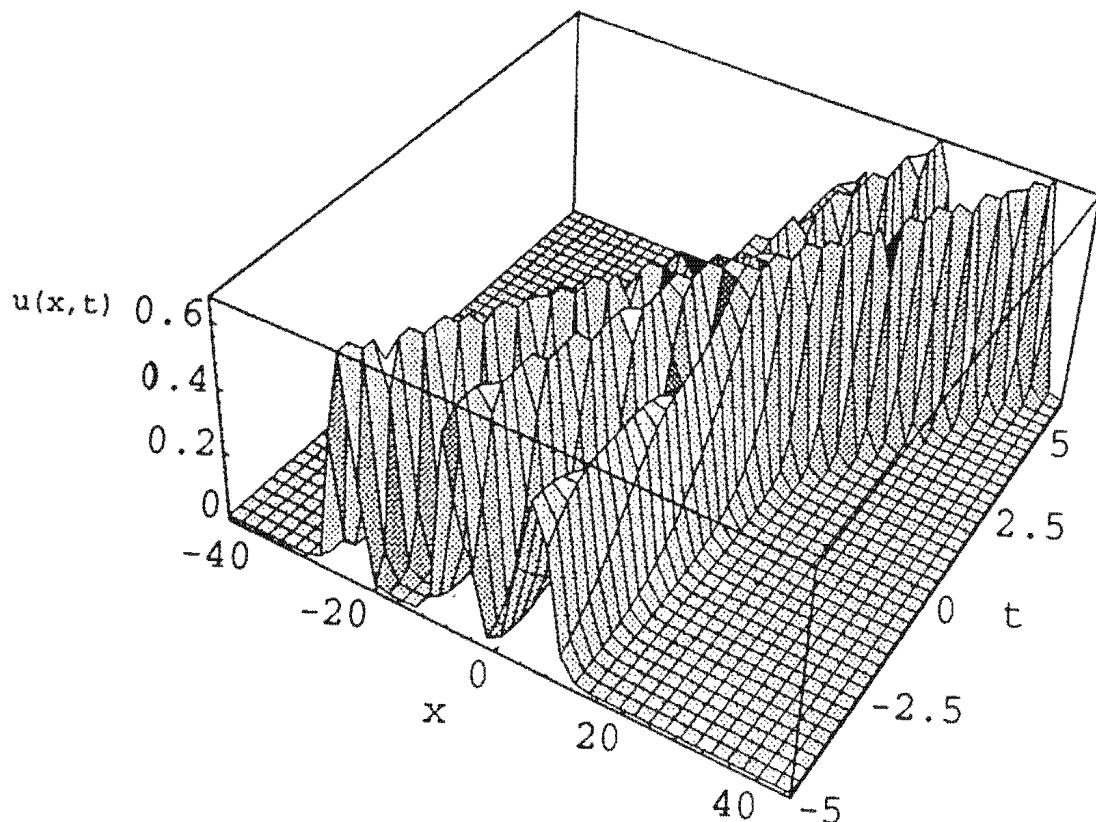


Figure 18. The three-soliton solution for the KK equation in 3-dimensions.

Chapter 5

Conservation Laws

In Chapters Two and Three, two methods for finding solitary-wave solutions and soliton solutions for nonlinear PDEs were presented. In this chapter the conservation laws for nonlinear evolution equations of KdV type will be discussed. The existence of a sufficiently large (in principle infinite) number of conservation laws of type (5.2) assures complete integrability of an evolution equation. In turn, completely integrable PDEs have N-soliton solutions. The study of the relation between integrability, the existence of conservation laws, and the existence of N-soliton solutions is beyond the scope of this thesis (see for example [76]).

I will use the existence of conservation laws as a tool to confirm the results for the class of fifth-order evolution equations obtained in the previous chapter. Conditions for obtaining polynomial conservation laws will be established, some of the conserved densities will be computed explicitly, and the connections with the requirements that lead to multi-soliton solutions will be explored.

The availability of symbolic manipulation programs such as MACSYMA, MAPLE,

and MATHEMATICA, which allow the performance of tedious algebra and routine computations to be done by a computer, makes the search for special properties such as integrability, exact solutions, and conserved densities of nonlinear PDEs quite feasible.

5.1 Conservation Laws

A conservation law for a nonlinear PDE can be expressed in the form

$$\rho_t + J_x = 0, \quad (5.1)$$

where ρ is called the conserved density, and J is the flux. Both are functionals of u . Furthermore, ρ is a *local* conserved density if ρ is a local functional of u , i.e., if the value of ρ at any x depends only on the values of u in an arbitrary small neighborhood of x . If J is also local, then (5.1) is a local conservation law. In particular, if ρ is a polynomial in u and its x derivatives, and does not dependent explicitly on x or t , then ρ is called a polynomial conserved density. If J is also a polynomial in u, u_x, u_{xx} , etc., then (5.1) is called a polynomial conservation law.

5.2 An Algorithmic Method For Finding Conservation Laws

One of the most important discoveries in the early development of the Inverse Scattering Technique was made by Miura, Gardner and Kruskal [77]. They showed

that the KdV equation has an infinite set of local conservation laws.

Recall that for a density ρ , there exists an associated flux J , obeying the conservation law,

$$\rho_t + J_x = 0. \quad (5.2)$$

So, one finds that

$$P = \int_{-\infty}^{+\infty} \rho dx = \text{constant}, \quad (5.3)$$

provided J vanishes at infinity.

The method of finding conservation laws is based on ideas of Kruskal and co-workers [78, 77], Ito [79], Ito and Kako [81], and Verheest and Hereman [82].

As the leading example, I will use the KdV equation once more

$$u_t + uu_x + u_{3x} = 0. \quad (5.4)$$

Information about the building blocks of $\rho(u, u_x, u_{2x}, \dots, u_{nx})$ can be obtained from the scaling or symmetry properties of the equation. The scaling of (5.4) is such that

$$u \sim \frac{\partial^2}{\partial x^2}, \quad \frac{\partial}{\partial t} \sim \frac{\partial^3}{\partial x^3}. \quad (5.5)$$

The first conserved density is trivial, since it corresponds to $\rho_1 = u$. From

(5.4), it is clear that

$$u_t + uu_x + u_{3x} = \frac{\partial \rho_1}{\partial t} + \frac{\partial}{\partial x} \left(\frac{1}{2}u^2 + u_{2x} \right) = 0 \quad (5.6)$$

therefore, $J_1 = \frac{1}{2}u^2 + u_{2x}$. The second conserved density corresponds to $\rho_2 = u^2$.

Compute $\frac{\partial \rho_2}{\partial t} = 2uu_t$ and substitute $-(uu_x + u_{3x})$ for u_t to obtain

$$\begin{aligned} \frac{\partial \rho_2}{\partial t} &= -(2u^2u_x + 2uu_{3x}) \\ &= -\left(\frac{2}{3}u^3 + 2uu_{2x}\right)_x + 2u_xu_{2x} \\ &= -\left(\frac{2}{3}u^3 + 2uu_{2x} - u_x^2\right)_x. \end{aligned} \quad (5.7)$$

Therefore,

$$J_2 = \frac{2}{3}u^3 + 2uu_{2x} - u_x^2. \quad (5.8)$$

To find ρ_3 , start with building block u^3 , which has three factors u . Keeping (5.5) in mind, this is equivalent to two factors u and two derivatives (to be distributed over these two factors as uu_{2x} or as u_x^2), or one factor u and four derivatives (obviously u_{4x}). But uu_{2x} can be integrated by parts to yield $(uu_x)_x - u_x^2$ and $u_{4x} = (u_{3x})_x$. By removing any density (or part thereof) that can be written as a total x -derivative, only u_x^2 remains. Therefore, ρ_3 is a linear combination of the building blocks u^3 and u_x^2 . So, finding the appropriate numerical coefficient becomes straightforward and

results in $\rho_3 = u^3 - 3u_x^2$.

Using a MATHEMATICA program, currently being developed by Hereman and Göktas, the building blocks of $\rho(u, u_x, u_{2x}, \dots, u_{nx})$ can easily be computed. For example, to compute the building blocks of ρ_6 , one proceeds as follows:

(i) Start with u^6 , divide it by u , and taking into account the scaling (5.5), then compute $(u^5)_{2x}$ to produce the following list of terms $[u^3u_x^2, u^4u_{2x}] \rightarrow [u^3u_x^2]$. Note that, the terms in the second list that can be written either as a total derivative with respect to x or a total derivative up to terms appearing earlier in the list, were removed.

Next, divide u^6 by u^2 , and compute $(u^4)_{4x}$. The corresponding lists are

$$[u_x^4, uu_x^2u_{2x}, u^2u_{2x}^2, u^2u_xu_{3x}, u^3u_{4x}] \rightarrow [u_x^4, u^2u_{2x}^2].$$

In a similar way, proceed with $(\frac{u^6}{u^3})_{6x} = (u^3)_{6x}$, $(\frac{u^6}{u^4})_{8x} = (u^2)_{8x}$, and $(\frac{u^6}{u^5})_{10x} = (u)_{10x}$, to obtain the lists:

$$[u_{2x}^3, u_xu_{2x}u_{3x}, uu_{3x}^2, u_x^2u_{4x}, uu_{2x}u_{4x}, uu_xu_{5x}, u^2u_{6x}] \rightarrow [u_{2x}^3, uu_{3x}^2],$$

$$[u_{4x}^2, u_{3x}u_{5x}, u_{2x}u_{6x}, u_xu_{7x}, uu_{8x}] \rightarrow [u_{4x}^2], \text{ and } [u_{10x}] \rightarrow [],$$

where $[]$ refers to the empty list.

Gathering the terms in the simplified lists yields

$$\rho_6 = u^6 + c_1u^3u_x^2 + c_2u_x^4 + c_3u^2u_{2x}^2 + c_4u_{2x}^3 + c_5uu_{3x}^2 + c_6u_{4x}^2, \quad (5.9)$$

where the constants c_i have to be determined.

(ii) Now, compute $\frac{\partial \rho_6}{\partial t}$, and replace $u_t, u_{xt}, \dots, u_{nx,t}$ by $-(uu_x + u_{xxx}), -(uu_x + u_{xxx})_x, \dots, -(uu_x + u_{xxx})_{nx}$, respectively.

(iii) Integrate the result with respect to x , carry out all integrations by parts, and require that the part which cannot be further integrated vanishes. By setting all the coefficients of the independent combinations involving powers of u and its derivatives with respect to x (in the part that is required to vanish) equal to zero leads to a linear system for the unknowns c_1, c_2, \dots, c_6 . After solving that system, one finds

$$\rho_6 = u^6 - 60u^3u_x^2 - 30u_x^4 + 108u^2u_{2x}^2 + \frac{720}{7}u_{2x}^3 - \frac{648}{7}uu_{3x}^2 + \frac{216}{7}u_{4x}^2. \quad (5.10)$$

(iv) Once the constants c_i are determined, the form of the flux J_6 can be computed by substituting these constants into the integrable part of $\frac{\partial \rho_6}{\partial t}$, and reversing the sign.

For simplicity, let me illustrate steps (ii)-(iv) for ρ_3 . Applying step (i) leads to $\rho_3 = u^3 + c_1 u_x^2$. After the replacement of u_t and u_{xt} in $\frac{\partial \rho_3}{\partial t}$ by $-(uu_x + u_{xxx})$ and $-(uu_x + u_{xxx})_x$, one obtains

$$\frac{\partial \rho_3}{\partial t} = -3u^3u_x - 3u^2u_{3x} - 2c_1u_x^3 - 2c_1uu_xu_{2x} - 2c_1u_xu_{4x}. \quad (5.11)$$

Now, by writing each term as a total derivative with respect to x , one observes that

$$-3u^3u_x = \left(-\frac{3}{4}u^4\right)_x \quad (5.12)$$

and

$$-3u^2u_{3x} = (-3u^2u_{2x})_x + 6uu_xu_{2x} = (-3u^2u_{2x} + 3uu_x^2)_x - 3u_x^3. \quad (5.13)$$

The fourth term in (5.11) can be written as

$$-2c_1uu_xu_{2x} = -c_1[(uu_x^2)_x - u_x^3]. \quad (5.14)$$

Finally, in the same manner, the last term will be

$$\begin{aligned} -2c_1u_xu_{4x} &= -2c_1[(u_xu_{3x})_x - u_{2x}u_{3x}] \\ &= -2c_1[u_xu_{3x} - \frac{1}{2}u_{2x}^2]_x. \end{aligned} \quad (5.15)$$

Therefore, (5.11) can be written as

$$\frac{\partial\rho_3}{\partial t} = -[\frac{3}{4}u^4 + (c_1 - 3)uu_x^2 + 3u^2u_{xx} - c_1u_{xx}^2 + 2c_1u_xu_{3x}]_x - (c_1 + 3)u_x^3. \quad (5.16)$$

Since the last term in (5.16) cannot be written as a total derivative of x , it must

vanish. Hence, $c_1 = -3$, and the expression [...] in (5.16) simplifies into

$$J_3 = \frac{3}{4}u^4 - 6uu_x^2 + 3u^2u_{xx} + 3u_{xx}^2 - 6u_xu_{3x}, \quad (5.17)$$

corresponding to $\rho_3 = u^3 - 3u_x^2$.

5.3 Conservation Laws for the General Class of Evolution Equations

In this section, the conservation laws will be calculated for the general class of evolution equations of fifth-order that was solved in Chapter Four. The goal is to retrieve the conditions for complete integrability and the existence of multi-soliton solutions from the construction of conservation laws.

In investigating the conservation laws for the equation

$$u_t + \alpha uu_{3x} + \beta u_xu_{2x} + \gamma u^2u_x + u_{5x} = 0, \quad (5.18)$$

which can be written as

$$u_t + \left(\frac{\gamma}{3}u^3 + \frac{\beta - \alpha}{2}u_x^2 + \alpha uu_{2x} + u_{4x} \right)_x = 0, \quad (5.19)$$

one can see that the forms for the density and flux are:

$$\rho_1 = u \quad \text{and}, \quad J_1 = \frac{\gamma}{3}u^3 + \frac{\beta - \alpha}{2}u_x^2 + \alpha uu_{2x} + u_{4x}. \quad (5.20)$$

To find the second conservation law, assume $\rho_2 = u^2$. Following the same procedure as for the KdV case, one computes

$$\begin{aligned} \frac{\partial \rho_2}{\partial t} + \frac{\partial}{\partial x} & \left[\frac{\alpha}{2}u^4 + 2\alpha u^2 u_{2x} + 2uu_{4x} - u_x u_{3x} + \frac{1}{2}u_{2x}^2 + (\beta - 2\alpha)uu_x^2 \right] \\ & - (\beta - 2\alpha)u_x^3 = 0, \end{aligned} \quad (5.21)$$

which implies that for $\beta = 2\alpha$, a second conservation law is obtained with flux

$$J_2 = \frac{\gamma}{2}u^4 + 2\alpha u^2 u_{2x} + 2uu_{4x} - u_x u_{3x} + \frac{1}{2}u_{2x}^2. \quad (5.22)$$

Note that the condition $\beta = 2\alpha$ needed for the second conservation law is satisfied for the Lax and Ito equations, but not for the SK and KK equations.

To find the third conserved density, start with

$$\rho_3 = \frac{1}{3}u^3 + Au_x^2. \quad (5.23)$$

After a tedious computation, one finds

$$\begin{aligned} \frac{\partial \rho_3}{\partial t} + \frac{\partial}{\partial x} \left[\frac{\gamma}{5} u^5 - \alpha u^3 u_{2x} - \left(-\frac{\beta}{2} + \frac{3\alpha}{2} - A\gamma \right) u^2 u_x^2 + (2 + 2A\beta) u_x^2 u_{2x} - \right. \\ \left. (1 + A\alpha) u u_{2x}^2 + (2 + 2A\alpha) u u_x u_{3x} - u^2 u_{4x} + A u_{3x}^2 - 2A u_{2x} u_{4x} + 2A u_x u_{5x} \right] \\ + \left[(-\beta + 3\alpha + 2A\gamma) u u_x^3 - (5 + 2A\beta - A\alpha) u_x u_{2x}^2 \right] = 0. \end{aligned} \quad (5.24)$$

The last term will vanish, provided

$$A = \frac{\beta - 3\alpha}{2\gamma} = \frac{5}{\alpha - 2\beta}, \quad (5.25)$$

which leads to

$$\gamma = \frac{1}{10}(\alpha - 2\beta)(\beta - 3\alpha). \quad (5.26)$$

This condition, (5.26), is satisfied for the Lax, SK and KK equations. But the constants in the Ito equation do not satisfy the condition.

To find the next conservation law, take

$$\rho_4 = \frac{1}{4} u^4 + A u u_x^2 + B u u_{2x}^2, \quad (5.27)$$

and compute

$$\begin{aligned}
& \frac{\partial \rho_4}{\partial t} + \frac{\partial}{\partial x} \left[\frac{\gamma}{6} u^6 + \frac{1}{2} (2A\gamma + \beta - 4\alpha) u^3 u_x^2 - \frac{1}{4} (6 - 4\gamma B + A\beta - A\alpha) u_x^4 + \alpha u^4 u_{2x} \right. \\
& + (\alpha - 2A\beta - 2\alpha) u u_x^2 u_{2x} - \frac{1}{2} (2A\alpha - 2B\gamma - 3) u^2 u_{2x}^2 - \frac{4}{3} (A - B\beta) u_{2x}^3 \\
& - (3 - 2A\alpha) u^2 u_x u_{3x} + (4A + 2B\beta - 2B\alpha) u_x u_{2x} u_{3x} + (A - B\alpha) u u_{3x}^2 + u^3 u_{4x} \\
& \left. - A u_x u_{4x} - (2A - 2B\alpha) u u_{2x} u_{4x} + B u_{4x}^2 + 2A u u_x u_{5x} - 2B u_{3x} u_{5x} + 2B u_{2x} u_{6x} \right] \\
& - \left[\frac{1}{2} (-4A\gamma + 3\beta - 12\alpha) u^2 u_x^3 + (15 - 10\gamma B + 2A\beta - 4A\alpha) u u_x u_{2x}^2 \right. \\
& \left. + (5A + 2B\beta + B\alpha) u_x u_{3x}^2 \right] = 0. \tag{5.28}
\end{aligned}$$

Rejecting non-integrable terms leads to the following conditions:

$$-4A\gamma + 3\beta - 12\alpha = 0, \tag{5.29}$$

$$15 - 10\gamma B + 2A\beta - 4A\alpha = 0, \tag{5.30}$$

$$5A + 2B\beta + B\alpha = 0. \tag{5.31}$$

Solving the first condition in (5.29) for A results in

$$A = \frac{3(\beta - 4\alpha)}{4\gamma}. \tag{5.32}$$

Furthermore, solving (5.30) and (5.31) for B yields

$$B = \frac{2A\beta - 4A\alpha + 15}{10\gamma} \quad (5.33)$$

$$= -\frac{5A}{2\beta + \alpha}. \quad (5.34)$$

Equating (5.33) and (5.34), followed by the substitution for A from (5.32), leads to two cases, which must be investigated separately.

Case 1: When

$$\beta = 2\alpha, \quad (5.35)$$

the values of A and B are

$$A = -\frac{3}{2}\frac{\alpha}{\gamma}, \quad (5.36)$$

and

$$B = \frac{3}{2}\frac{1}{\gamma}. \quad (5.37)$$

With the above values, the form of ρ_4 is

$$\rho_4 = \frac{1}{4}u^4 - \frac{3}{2}\frac{\alpha}{\gamma}uu_x^2 + \frac{3}{2}\frac{1}{\gamma}uu_{2x}^2, \quad (5.38)$$

and this case corresponds to the Lax equation.

Case 2 : For this case

$$\gamma = -\frac{1}{45}(\beta - 4\alpha)(2\beta + \alpha), \quad (5.39)$$

which determines A and B as

$$A = -\frac{135}{4(2\beta + \alpha)}, \quad (5.40)$$

and

$$B = \frac{675}{4(2\beta + \alpha)^2}. \quad (5.41)$$

Hence,

$$\rho_4 = \frac{1}{4}u^4 - \frac{135}{4(2\beta + \alpha)}uu_x^2 + \frac{675}{4(2\beta + \alpha)^2}uu_{2x}^2, \quad (5.42)$$

which is the fourth density for the SK, KK, and Ito equations. As a result, one can say that all the well-known equations of the class presented in (5.18) have a conservation law of the type (5.27).

Obviously, the computation of the higher degree conservation laws becomes long, complicated, and hard to be done by hand. Therefore, to find additional conservation laws for cases mentioned in this Chapter, the use of a computer program is necessary. A MATHEMATICA program written by Hereman [6], and being improved by Hereman and Göktas, is used to accomplish this task. In Tables 1, and 2, I summarize the densities up to degree eight for the Lax, SK, KK and Ito equations. Note that

the integrable Lax, SK, and KK equations have many conserved densities, whereas the non-integrable Ito equation has only three conserved densities. This result supports my findings of Chapter Four, where it was shown that integrable equations such as the Lax, SK, and KK equations have N-soliton solutions, and the non-integrable equation (such as the Ito equation) has, at most, a solitary-wave solution.

In conclusion, the calculations in this chapter confirm that completely integrable equations share two properties: the presence of a large number of conserved densities and having N-soliton solutions.

Table 1. Conserved Densities for Sawada-Kotera and Lax equations.

Density	Sawada-Kotera equation	Lax equation
ρ_1	u	u
ρ_2	----	$\frac{1}{2}u^2$
ρ_3	$\frac{1}{3}u^3 - u_x^2$	$\frac{1}{3}u^3 - \frac{1}{6}u_x^2$
ρ_4	$\frac{1}{4}u^4 - \frac{9}{4}uu_x^2 + \frac{3}{4}u_{2x}^2$	$\frac{1}{4}u^4 - \frac{1}{2}uu_x^2 + \frac{1}{20}u_{2x}^2$
ρ_6	----	$\frac{1}{5}u^5 - u^2u_x^2 + \frac{1}{5}uu_{2x}^2 - \frac{1}{70}u_{3x}^2$
ρ_6	$\frac{1}{6}u^6 - \frac{25}{4}u^3u_x^2 - \frac{17}{8}u_x^4 + 6u^2u_{2x}^2$ $+ 2u_{2x}^3 - \frac{21}{8}uu_{3x}^2 + \frac{3}{8}u_{4x}^2$	$\frac{1}{6}u^6 - \frac{5}{3}u^3u_x^2 - \frac{5}{36}u_x^4 + \frac{1}{2}u^2u_{2x}^2$ $+ \frac{5}{63}u_{2x}^3 - \frac{1}{14}uu_{3x}^2 + \frac{1}{252}u_{4x}^2$
ρ_7	$\frac{1}{7}u^7 - 9u^4u_x^2 - \frac{54}{5}uu_x^4 + \frac{57}{5}u^3u_{2x}^2$ $+ \frac{648}{35}u_x^2u_{2x}^2 + \frac{489}{35}uu_{2x}^3 - \frac{261}{35}u^2u_{3x}^2$ $- \frac{288}{35}u_{2x}u_{3x}^2 + \frac{81}{35}uu_{4x}^2 - \frac{9}{35}u_{5x}^2$	$\frac{1}{7}u^7 - \frac{5}{2}u^4u_x^2 - \frac{5}{6}uu_x^4 + u^3u_{2x}^2$ $+ \frac{1}{2}u_x^2u_{2x}^2 + \frac{10}{21}uu_{2x}^3 - \frac{3}{14}u^2u_{3x}^2$ $- \frac{5}{42}u_{2x}u_{3x}^2 + \frac{1}{42}uu_{4x}^2 - \frac{1}{924}u_{5x}^2$
ρ_8	----	$\frac{1}{8}u^8 - \frac{7}{2}u^5u_x^2 - \frac{35}{12}u^2u_x^4 + \frac{7}{4}u^4u_{2x}^2$ $+ \frac{7}{2}uu_x^2u_{2x}^2 + \frac{5}{3}u^2u_{2x}^3 + \frac{7}{24}u_{2x}^4 + \frac{1}{2}u^3u_{3x}^2$ $- \frac{1}{4}u_x^2u_{3x}^2 - \frac{5}{6}uu_{2x}u_{3x}^2 + \frac{1}{12}u^2u_{4x}^2$ $+ \frac{7}{132}u_{2x}u_{4x}^2 - \frac{1}{132}uu_{5x}^2 + \frac{1}{3432}u_{6x}^2$

Table 2. Conserved Densities for Kaup-Kupershmidt and Ito equations.

Density	Kaup-Kupershmidt equation	Ito equation
ρ_1	u	u
ρ_2	----	$\frac{u^2}{2}$
ρ_3	$\frac{u^3}{3} - \frac{1}{8}u_x^2$	----
ρ_4	$\frac{u^4}{4} - \frac{9}{16}uu_x^2 + \frac{3}{64}u_{2x}^2$	$\frac{u^4}{4} - \frac{9}{4}uu_x^2 + \frac{3}{4}u_{2x}^2$
ρ_5	----	----
ρ_6	$\frac{u^6}{6} - \frac{35}{16}u^3u_x^2 - \frac{31}{256}u_x^4 + \frac{51}{64}u^2u_{2x}^2$ $+ \frac{37}{256}u_{2x}^3 - \frac{15}{128}uu_{3x}^2 + \frac{3}{512}u_{4x}^2$	----
ρ_7	$\frac{u^7}{7} - \frac{27}{8}u^4u_x^2 - \frac{369}{320}uu_x^4 + \frac{69}{40}u^3u_{2x}^2$ $+ \frac{2619}{4480}u_x^2u_{2x}^2 + \frac{2211}{2240}uu_{2x}^3 - \frac{477}{1120}u^2u_{3x}^2$ $- \frac{171}{640}u_{2x}u_{3x}^2 + \frac{27}{560}uu_{4x}^2 - \frac{9}{4480}u_{5x}^2$	----
ρ_8	----	----

Chapter 6

Conclusion

The truncated Painlevé expansion method was used to construct solitary-wave solutions of nonlinear evolution equations. The solutions obtained via this method were more general than those already known in the literature. The algorithmic nature of the method makes it possible to use a computer algebra package to do the tedious but routine calculations.

The simplified version of Hirota's method, as developed in the thesis, proved to be an effective and straightforward technique for finding solitary-wave and soliton solutions for various nonlinear PDEs. The method allows one to construct solutions for PDEs that have no bilinear representation or for which the bilinear form is not known. Using this method the two-soliton and three-soliton solutions for the Kaup-Kupershmidt equation were found. Finally, the relation between the existence of an infinite number of conservation laws, the complete integrability and the existence of soliton solutions of a class of fifth-order PDEs were studied.

The techniques developed in this thesis could be applied to study nonlinear

integrable and non-integrable models for wave phenomena in plasma and particle physics, fluid dynamics, nonlinear optics, meteorology, and mathematical biology.

As a continuation to this study, first I would like to suggest the applications of the two methods, presented in this thesis, to systems of nonlinear PDEs. Second, I propose the study of theoretical aspects of the two methods, in particular, their range of applicability to classes of PDEs.

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