

# Conservations Laws and Solitary Wave Solutions for Generalized Schamel Equations

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## Abstract

The solitary wave solution is given for nonlinear equations, generalizing the standard and modified Korteweg–de Vries and Schamel equations, as recently investigated by Xiao. A search for conservation laws of a slightly more general class of nonlinear evolution equations reveals that the generalized Schamel equations can have no more than three polynomial invariants. The method is based on obtaining suitable building blocks for conserved densities under scalings which leave the evolution equations invariant.

## 1. Introduction

In view of the fundamental role played by integrable systems in nonlinear science, many new paradigms have been proposed to generate and study exactly solvable systems. Many of the new models are inspired by the complete integrability of the standard and modified Korteweg–de Vries (KdV and mKdV) equations [1]. Recently Xiao [2] has reported on the integrability of the nonlinear evolution equation

$$16u_t + 30u^{1/2}u_x + u_{xxx} = 0. \quad (1)$$

In fact, this is a rescaled version of the Schamel equation [3],

$$u_t + u^{1/2}u_x + \alpha u_{xxx} = 0, \quad (2)$$

governing the propagation of ion-acoustic waves in a cold-ion plasma where some of the electrons do not behave isothermally during the passage of the wave but are trapped in it. The square root in the nonlinear term then translates to lowest order some of the kinetic effects, associated with electron trapping, which dominate over the fluid effects giving rise themselves to the well-studied KdV or mKdV equations.

Equation (1) is a special case of what we shall call the generalized Schamel (GS) equation, given by Xiao [2] as

$$n^2u_t + (n + 1)(n + 2)u^{2/n}u_x + u_{xxx} = 0, \quad (3)$$

where  $n$  is a positive integer. Obviously, for  $n = 4$  (3) reduces to the scaled Schamel equation (1). Upon a trivial rescaling of the coefficients, the KdV equation is obtained for  $n = 2$ , and the MKdV equation follows for  $n = 1$ .

Moreover, the nonlinear equations mentioned so far, with KdV-like dispersion but different nonlinearities, are special

cases of generalized KdV equations

$$u_t + (a + bu^c)u^c u_x + du_{xxx} = 0, \quad (4)$$

for which Hereman and Takaoka [4] constructed a solitary wave solution in closed form, via a systematic approach detailed earlier in Hereman *et al.* [5].

From Painlevé analysis, Xiao [2] concluded that (1) can be classed as integrable, but that was promptly disputed by Ramani and Grammaticos [6]. For values of  $n$  other than 1, 2, or 4 the analysis of (3) was not carried through anyway, and no statements about integrability were made. However, in view of the remarks by Ramani and Grammaticos concerning Xiao's treatment of the original Schamel equation (1), complete integrability of (3) is doubtful.

Nevertheless, for the GS equation it is straightforward to derive by direct integration a single solitary wave solution of the typical sech-form, which depends on the coordinate  $\xi = x - vt$ , for a frame of reference travelling at speed  $v$ . For  $u(x, t) = u(\xi)$  we find from (3) that

$$-vn^2u' + u''' + (n + 1)(n + 2)u^{2/n}u' = 0, \quad (5)$$

denoting derivatives of  $u$  with respect to  $\xi$  by primes. Equation (5) can be integrated twice. Under the usual assumptions that  $u$  and its derivatives vanish at infinity we get

$$u'^2 = n^2u^2(v - u^{2/n}), \quad (6)$$

with the solution

$$u = (\sqrt{v})^n \operatorname{sech}^n [\sqrt{v}(x - vt) + \delta]. \quad (7)$$

For  $n = 1$  and  $n = 2$  we recover the usual solitary wave solution of the mKdV and KdV equation. For  $n = 4$  we find

$$u = v^2 \operatorname{sech}^4 [\sqrt{v}(x - vt) + \delta], \quad (8)$$

a solitary wave solution of the Schamel equation as studied by Xiao [2]. Again, (7) is a special case of the solution of (4) given by Hereman and Takaoka [4], Weinstein [7], and most recently by Conte and Musette [8].

In Section 2 we outline a simple strategy to construct conserved densities, and apply it to KdV-like equations. Returning to the GS equation (3) in Section 3, we prove that for  $n \neq 1, 2$  there are no more than three conserved densities of polynomial type. Finally, some conclusions are drawn in Section 4.

**2. A direct search for conservation laws**

In this Section we extend some of the work by Coffey [9] on Schamel's equation, which was shown to possess at least three conservation laws. Recall that the completely integrable KdV ( $n = 2$ ) and mKdV ( $n = 1$ ) equations possess infinite sequences of conserved densities, and this property is used as the criterion to characterize complete integrability. The generalized KdV and Schamel equations are of the form

$$u_t + f(u)u_x + \alpha u_{xxx} = 0, \tag{9}$$

with  $f(u)$  sufficiently smooth to assure the existence of its first few derivatives or integrals.

Now we discuss in detail how the scaling or symmetry properties of the equations can be used to obtain information about both the number of polynomial conserved densities, and the building blocks they are made off. The scaling of (9) is such that

$$f(u) \sim \frac{\partial^2}{\partial x^2}, \quad \frac{\partial}{\partial t} \sim \frac{\partial^3}{\partial x^3}. \tag{10}$$

Let  $F(u)$  be a primitive of  $f(u)$ , i.e.

$$f(u) = \frac{dF(u)}{du}, \tag{11}$$

then (9) can be written as a conservation law

$$u_t + (F(u) + \alpha u_{xx})_x = 0. \tag{12}$$

In general, for any density  $\rho$  such that there exists an associated flux  $J$  obeying

$$\rho_t + J_x = 0, \tag{13}$$

one finds that

$$P = \int_{-\infty}^{+\infty} \rho \, dx = \text{constant}, \tag{14}$$

provided  $J$  vanishes at infinity. Regarding (12), it is clear that  $\rho_1 = u$  is a first conserved density. The trivial conservation law (12) expresses conservation of momentum, and is well known for most of the nonlinear evolution equations studied so far.

Next comes

$$(u^2)_t + (2uF(u) - 2\mathcal{F}(u) + 2\alpha uu_{xx} - \alpha u_x^2)_x = 0, \tag{15}$$

where we now also need  $\mathcal{F}(u)$ , the second primitive of  $f(u)$ ,

$$f(u) = \frac{dF(u)}{du} = \frac{d^2\mathcal{F}(u)}{du^2}. \tag{16}$$

Hence,  $\rho_2 = u^2$  is a second conserved density associated with the conservation of energy. For the special case  $f(u) = u^{-1}$  (15) has to be replaced by

$$(u^2)_t + (2u + 2\alpha uu_{xx} - \alpha u_x^2)_x = 0. \tag{17}$$

The existence of other conserved densities is less obvious and their construction is less trivial. If one looks at the ordinary KdV equation, for which  $f(u) = u$ , the third conserved density is [10]  $\rho_3 = u^3 - 3\alpha u_x^2$ , up to an irrelevant numerical factor. For the KdV equation one can continue like this, for every order there is a conserved density of the structure  $\rho_n = u^n + \dots + \beta u_{(n-2)x}^2$ . Note that all the terms in each of

the conserved densities agree with the scaling

$$f(u) = u \sim \frac{\partial^2}{\partial x^2}. \tag{18}$$

We may restrict ourselves to building blocks which belong to the same class under the mentioned scaling, since for any mixed conserved quantity which one could derive, the freedom implied in the scaling would split that quantity in several conserved quantities, each with building blocks of the same scaling. Thus there is a straightforward and logical way to construct invariant quantities [10].

Starting with the density containing the building block  $u^3$ , one has three factors  $u$ . Keeping the scaling in mind, this is equivalent to two factors  $u$  and two derivations (to be distributed over these factors), or one factor  $u$  and four derivations (obviously  $u_{xxxx}$ ). Two factors  $u$  and two derivations can be written in two ways: either as  $uu_{xx}$  or as  $u_x^2$ . But  $uu_{xx}$  can be integrated by parts to yield  $(uu_x)_x - u_x^2$ . Without loss of generality, we may remove any density (or part thereof) that is a total  $x$ -derivative, for these are trivially conserved. Doing so, only  $u_x^2$  is to be considered as the next building block. As a result of the scaling (18) for the KdV equation, the conserved density  $\rho_3$  must be a linear combination of the building blocks  $u^3$  and  $u_x^2$ . It is then straightforward to find the appropriate numerical coefficients, resulting in  $\rho_3 = u^3 - 3\alpha u_x^2$ .

For the mKdV equation, with  $f(u) = u^2$ , again  $\rho_1 = u$  and  $\rho_2 = u^2$ . The next conserved density is [10]  $\rho_3 = u^4 - 6\alpha u_x^2$ . Analogous to the KdV-case, for every order there is a conserved density of the structure  $\rho_n = u^{2n} + \dots + \beta u_{(n-1)x}^2$ , where the terms in each conserved density agree with the scaling

$$f(u) = u^2 \sim \frac{\partial^2}{\partial x^2}. \tag{19}$$

There is no conserved density starting with  $u^3$ . More generally, only even powers of  $u$  give results, with the exception of  $u$  itself, which is thus somewhat outside the normal range. Starting with the density containing the building block  $u^4$ , one has either four factors  $u$ , or due to the scaling, three factors  $u$  and one derivation, two factors  $u$  and two derivations, or one factor  $u$  and three derivations. Rejecting partially integrable building blocks, only the building blocks  $u^4$  and  $u_x^2$  remain, leading to  $\rho_3 = u^4 - 6\alpha u_x^2$ .

Returning now to the scaling (10) and using similar arguments, one sees that  $u_x^2$  and  $u^2 f(u)$  must occur together in a third conserved density. Indeed, using (9) we obtain

$$(u_x^2)_t + (f(u)u_x^2 + 2\alpha u_x u_{xxx} - \alpha u_{xx}^2)_x = -u_x^3 \frac{df(u)}{du}, \tag{20}$$

and similarly

$$\begin{aligned} (u^2 f(u))_t + \left( \frac{1}{2} u^2 f^2(u) + \alpha u_{xx} \frac{d}{du} [u^2 f(u)] \right. \\ \left. - \frac{\alpha}{2} u_x^2 \frac{d^2}{du^2} [u^2 f(u)] \right)_x \\ = -uu_x f^2(u) - \frac{\alpha}{2} u_x^3 \frac{d^3}{du^3} [u^2 f(u)]. \end{aligned} \tag{21}$$

A linear combination of  $u_x^2$  and  $u^2 f(u)$  will give rise to a conserved density provided (i) a linear combination with

suitable constant  $\gamma$ ,

$$\frac{\alpha}{2} \frac{d^3}{du^3} [u^2 f(u)] + \gamma \frac{df(u)}{du} = 0, \tag{22}$$

allows to remove the bothersome terms in  $u_x^3$ , and (ii)  $uu_x f^2(u)$  is a perfect  $x$ -derivative.

The second requirement is easy to satisfy. For instance, for  $f(u) = u^p$ , with  $p$  positive rational, one has  $uu_x f^2(u) = u^{2p+1}u_x = 1/2(p+1) \partial/(\partial x) u^{2(p+1)}$ . After one integration with respect to  $u$ , (22) is the Euler equation:

$$u^2 \frac{d^2 f(u)}{du^2} + 4u \frac{df(u)}{du} + \frac{2(\gamma + \alpha)}{\alpha} f(u) = 0. \tag{23}$$

We set the integration constant equal to zero since we reject solutions  $f(u)$  containing an additive constant, for it would lead to a single standing  $u_x$  term in (9) which could be removed via a simple change of coordinates. The ordinary differential equation (23) admits solutions  $f(u) = u^p$ , with  $p$  positive rational, provided

$$\gamma = -\frac{\alpha}{2} (p+1)(p+2). \tag{24}$$

Hence

$$\rho_3 = u^{2+p} - \frac{\alpha}{2} (p+1)(p+2)u_x^2 \tag{25}$$

is the third conserved density, with corresponding flux

$$J_3 = \frac{p+2}{2(p+1)} u^{2p+2} + \alpha(p+2)u^{p+1}u_{xx} - \alpha(p+1)(p+2)u^p u_x^2 - \alpha^2(p+1)(p+2)u_x u_{xxx} + \frac{1}{2}\alpha^2(p+1)(p+2)u_{xx}^2. \tag{26}$$

In conclusion, eq. (9) with  $f(u) = u^p$ ,  $p$  positive rational, has always three conserved densities.

### 3. Application: the generalized Schamel equation

Our main result in the previous Section is applicable to the GS equation. Therefore, (3) has at least three conserved densities. Indeed, identification of (3) with (9) gives, after proper scaling of the coefficients,  $\rho_1 = u$ ,  $\rho_2 = u^2$ , and  $\rho_3 = \frac{1}{2}u_x^2 - (n^2/2)u^{2+2/n}$ .

Another way to derive the third conserved density for (3) is based on Coffey's ideas [9], originally applied to the Schamel equation (1). As a Lagrangian density for (3) one can take

$$\mathcal{L} = \frac{n^2}{2} \vartheta_x \vartheta_t - \frac{1}{2} \vartheta_{xx}^2 + \frac{n^2}{2} \vartheta_x^{2+2/n}, \tag{27}$$

where  $\vartheta$  is a potential function such that  $\vartheta_x = u$ . The corresponding Hamiltonian density is

$$\mathcal{H} = \vartheta_t \Pi - \mathcal{L} = \frac{1}{2} u_x^2 - \frac{n^2}{2} u^{2+2/n}. \tag{28}$$

Using (3) and integrating by parts reveals that the Hamiltonian

$$H = \int_{-\infty}^{+\infty} \mathcal{H} dx \tag{29}$$

is a third invariant, with the corresponding flux

$$J_3 = \frac{1}{n^2} [u_x u_{xxx} - \frac{1}{2}u_{xx}^2 - n(n+1)u^{1+2/n}u_{xx} + (n+1)(n+2)u^{2/n}u_x^2 - \frac{1}{2}n^2(n+1)^2u^{2+4/n}]. \tag{30}$$

In total we have three conserved quantities: momentum, energy and the Hamiltonian. For  $n = 1, 2, 4$  these reduce to the known invariants for the KdV, modified KdV and Schamel equations. For the KdV equation a scaling is needed to bring the invariants in their customary form, as was the case with the equation itself from (3).

With the help of the symbolic manipulation program MATHEMATICA, we searched for further conserved densities of (3), but these do not seem straightforward to find, if they exist at all. Everything we tried indicated strongly that the above mentioned three invariants are the only polynomial ones in  $u$  and its derivatives. This can be inferred as follows.

It is easy to see that (3) implies a scaling of  $u$  and  $x$  such that

$$u^{2/n} \sim \frac{\partial^2}{\partial x^2}. \tag{31}$$

That scaling puts  $u^{2+2/n}$  and  $u_x^2$  in the same category, and with proper coefficients their combination leads to the conserved density in (28).

Adhering to the scaling (31), the next classes then consist of

$$u^{3+2/n} \text{ and } uu_x^2, \tag{32}$$

$$u^{4+2/n} \text{ and } u^2u_x^2, \tag{33}$$

and

$$u^{2+4/n}, u^{2/n}u_x^2 \text{ and } u_{xx}^2, \tag{34}$$

$$u^{2+6/n}, u^{4/n}u_x^2, u^{2/n}u_{xx}^2 \text{ and } u_{xxx}^2. \tag{35}$$

For polynomial invariants we restrict ourselves to building blocks which belong to the same class under the mentioned scaling, as explained already before. There is thus a logical way to anticipate and also build conserved densities.

Some care should be taken in collecting members of the same family linked under the scaling. Clearly, (32) and (34) should be joined for  $n = 2$ , that is for the KdV equation, whereas (33) and (34) belong together for  $n = 1$ , corresponding to the mKdV equation. For other values of  $n$ , like  $n = 4$  (Schamel's equation), (32) then combines with (35), etc. Upon testing whether a linear combination of those terms can give a conserved quantity, one finds this is only possible for  $n = 1$  or  $n = 2$ , yielding the fourth invariants for the mKdV and KdV equations, respectively.

If one tries to go higher up in the families of building blocks, one quickly generates, upon taking the time derivatives of these terms, many more remainders. Some of these unwanted terms occur only once among the derivatives of building blocks of the same family under the scaling. Furthermore, those remaining terms cannot be rewritten as total space derivatives. Thus, there is no hope that they could be balanced by terms coming from other members of the same family. As one can easily check, the problem becomes worse with increasing degree of  $u$  in the family, even for Schamel's equation where  $n = 4$ . We can conclude

that all hope is lost to find other polynomial invariants. To illustrate the above argument, we show what happens for the Schamel equation. For simplicity we set  $\alpha = 1$  in (2).

The ordinary Schamel equation has conserved densities  $u$ ,  $u^2$  and  $u^{5/2} - 15u_x^2/8$ . It is readily verified that  $u^{3/2}$  has no corresponding building blocks and is not conserved. The building blocks (34) come next, specifically  $u^3$ ,  $u^{1/2}u_x^2$  and  $u_{xx}^2$ . If we compute their total time derivatives separately, and for economy of notation only indicate those parts which cannot be written as total space derivatives, we get

$$(u^3)_t + (\dots)_x = -3u_x^3, \quad (36)$$

$$(u^{1/2}u_x^2)_t + (\dots)_x = \frac{3u_x^5}{32u^{5/2}} - \frac{3u_x u_{xx}^2}{2u^{1/2}} - \frac{1}{2} u_x^3, \quad (37)$$

$$(u_{xx}^2)_t + (\dots)_x = \frac{3u_x^5}{16u^{5/2}} - \frac{5u_x u_{xx}^2}{2u^{1/2}}. \quad (38)$$

The irreducible terms in  $u_x^5$  and  $u_x u_{xx}^2$  only occur in (37) and (38), with coefficients which prevent their simultaneous elimination via linear combination. Although the terms in  $u_x^3$  in (36) and (37) can easily be got rid of, there still is no conserved density at this stage.

The following step is to consider the building blocks from (32) and (35), namely  $u^{7/2}$ ,  $uu_x^2$ ,  $u^{1/2}u_{xx}^2$  and  $u_{xxx}^2$ . An argument similar to the one just above yields equally disappointing results:

$$(u^{7/2})_t + (\dots)_x = -\frac{105}{16} u^{1/2} u_x^3, \quad (39)$$

$$(uu_x^2)_t + (\dots)_x = -\frac{1}{2} u^{1/2} u_x^3 - 3u_x u_{xx}^2, \quad (40)$$

$$(u^{1/2}u_{xx}^2)_t + (\dots)_x = \frac{u_x^5}{8u^2} - \frac{5}{2} u_x u_{xx}^2 + \frac{3u^{1/2}u_x^3 u_{xx}^2}{8u^3} - \frac{3u_x u_{xx}^3}{4u^{3/2}} - \frac{3u_x u_{xxx}^2}{2u^{1/2}}, \quad (41)$$

$$(u_{xxx}^2)_t + (\dots)_x = -\frac{35u_x^7}{32u^{9/2}} + \frac{21u_x^3 u_{xx}^2}{4u^{5/2}} - \frac{7u_x u_{xxx}^3}{2u^{3/2}} - \frac{7u_x u_{xxx}^2}{2u^{1/2}}. \quad (42)$$

By going higher up in the families of building blocks like this, the number of irreducible remainders increases rapidly. Some of these occur only once among the derivatives of building blocks of the same family, thus, it is impossible to

obtain conserved densities. As we have checked for a couple of more orders by symbolic manipulation, the problem becomes worse and worse. No other polynomial invariants except the known three can be found.

#### 4. Conclusions

In this paper we have given the extension of the invariants derived by Coffey [9] for the Schamel equation to generalized Schamel equations, which include the class of KdV-like equations with broken nonlinearities studied by Xiao [2]. We also derived a solitary wave solution to these equations. We argued that the Schamel equations and its generalizations can only have three polynomial invariants, by referring to the building blocks needed to construct conserved densities that are invariant under the scaling which leaves the original equations invariant. As a final remark, it is not known nor readily seen whether the GS equations possesses a two-solitary wave solution. Soliton solutions are unlikely, for the equations have only three polynomial conservation laws, in contrast to soliton equations with an unlimited number of conserved densities.

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