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Complete Integrability of a Modified Vector Derivative Nonlinear Schrödinger Equation

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Abstract

Oblique propagation of magnetohydrodynamic waves in warm plasmas is described by a modified vector derivative nonlinear Schrödinger equation, if charge separation in Poisson's equation and the displacement current in Ampère's law are properly taken into account. This modified equation cannot be reduced to the standard derivative nonlinear Schrödinger equation and hence its possible integrability and related properties need to be established afresh. Indeed, the new equation is shown to be integrable by the existence of a bi-Hamiltonian structure, which yields the recursion operator needed to generate an infinite sequence of conserved densities. Some of these have been found explicitly by symbolic computations based on the symmetry properties of the new equation.

1. Introduction

The derivative nonlinear Schrödinger equation (DNLS) was first given by Rogister [1] for the nonlinear evolution of parallel Alfvén waves in plasmas, and later encountered in many different contexts by other authors, emerging as one of the canonical nonlinear equations in physics. The DNLS could also account for slightly oblique propagation of Alfvén waves (see e.g. [2]), albeit at the price of neglecting two effects which might be important in strongly magnetized astrophysical plasmas. One is the deviation from charge neutrality between the different plasma species, the other is the influence of the displacement current in Ampère's law. Retaining these effects results in a nonlinear vector evolution equation which differs from the standard vector form of the DNLS by an extra linear term, and therefore was called the modified vector derivative nonlinear Schrödinger equation (MVDNLS) [3, 4].

Recently, there has been a renewed interest in the use of the DNLS for certain astrophysical plasmas [5, 6], assuming that the case of slightly oblique propagation could easily be reduced to that of parallel propagation. To do so, one modifies the dependent variable, in this case the perpendicular magnetic field, by including the static part as well. Since the MVDNLS cannot be transformed into the DNLS itself, such an easy transition from parallel to oblique propagation is not possible. The reverse is true, of course, the MVDNLS includes the DNLS as a special case, when we go from oblique to parallel propagation, which amounts to dropping the bothersome extra term.

That term, which distinguishes the MVDNLS from the DNLS, has implications for the discussion of integrability and the possibility of deriving solitary wave solutions for the MVDNLS. The DNLS is well known to be completely integrable [7], whereas for the MVDNLS we could only get certain indications about its integrability [4]. The applicability of the prolongation method [8], adapted to a vector nonlinear equation, and the existence of some invariants [4] were indicative of complete integrability, without giving a watertight proof.

In the present paper we show that the MVDNLS possesses a bi-Hamiltonian structure (see §§ 4 and 5 for more details), and hence through the resulting recursion operator an infinite sequence of conserved densities. That the DNLS itself has a bi-Hamiltonian structure was proved by Kulish [9] and Strampp and Oevel [10], although an infinite sequence of conserved densities had been derived earlier [7] without showing the bi-Hamiltonian character. Our constructions, formulas and conclusions for the MVDNLS immediately include these known results for the DNLS.

In § 2 we recall the form of the MVDNLS and list in § 3 some of the conserved densities, which were found in an ad hoc fashion with the help of a symbolic program and by looking at the symmetry properties of the equation. The knowledge of these conserved densities turns out to be beneficial for an easy construction of the appropriate Hamiltonians in § 4. In § 5 the bi-Hamiltonian structure and the recursion operator are derived and with it we establish the existence of an infinite sequence of conserved densities, needed to guarantee complete integrability. In § 6 we draw some conclusions.

2. MVDNLS

The MVDNLS is, after the necessary scaling and Galilean transforms to cast it in its simplest dimensionless form, given by

$$\frac{\partial \boldsymbol{B}_\perp}{\partial t} + \frac{\partial}{\partial x}(B_\perp^2 \boldsymbol{B}_\perp) + \alpha \boldsymbol{B}_{\perp 0} \boldsymbol{B}_{\perp 0} \cdot \frac{\partial \boldsymbol{B}_\perp}{\partial x} + \boldsymbol{e}_x \times \frac{\partial^2 \boldsymbol{B}_\perp}{\partial x^2} = \boldsymbol{0}, \tag{1}$$

where the parameter α characterizes the extra term which distinguishes the MVDNLS from the DNLS. In eq. (1) $\boldsymbol{B}_\perp$ stands for the perpendicular magnetic field, which includes both the wave contributions and the static perpendicular field due to the oblique propagation with respect to the total external magnetic field. The direction of wave propagation is along the x-axis. If the third term is absent, we can project eq. (1) onto axes perpendicular to the direction of wave propagation, introduce a new complex variable from the components of $\boldsymbol{B}_\perp$,

$$\phi_\pm = B_y \pm \mathrm{i}B_z, \tag{2}$$

and combine the projections to obtain the DNLS in standard scalar form,

$$\frac{\partial \phi_\pm}{\partial t} + \frac{\partial}{\partial x}(|\phi_\pm|^2 \phi_\pm) \pm \mathrm{i}\,\frac{\partial^2 \phi_\pm}{\partial x^2} = 0. \tag{3}$$

The $\pm$ signs in eqs (2) and (3) are correlated. The DNLS can account for oblique propagation, provided α is zero, which means that we impose charge neutrality to all orders and neglect the displacement current in Ampère's law [3]. The bothersome third term in eq. (1) also disappears if $\boldsymbol{B}_{\perp 0}$ is zero, in the case of strictly parallel propagation.

At this stage, it is worth recalling that the DNLS has constant-amplitude solutions of the form

$$B_y = a \cos(kx - \omega t),$$
$$B_z = \pm a \sin(kx - \omega t), \tag{4}$$

with a an arbitrary constant. As can easily be checked, there are no constant amplitude solutions to eq. (1) besides the trivial case with $\boldsymbol{B}_\perp = \boldsymbol{B}_{\perp 0}$, if $\boldsymbol{B}_{\perp 0} \neq 0$. This means that any sort of separation into left and right circularly polarized waves as for the DNLS is doomed to fail. Of course, for oblique propagation with $\alpha = 0$, the circular polarization given by the DNLS is only apparent, since $\boldsymbol{B}_\perp$ includes the static part $\boldsymbol{B}_{\perp 0}$, and this shift leads in reality to elliptical polarization for the perpendicular wave field [6]. In addition, the MVDNLS has a class of stationary solitary wave solutions which the DNLS does not have, the subalfvénic modes, which have totally different properties compared to the known stationary solutions of the DNLS [4].

We know that the DNLS is completely integrable in the sense that it possesses an infinite series of conserved densities, which can be constructed explicitly. As the MVDNLS reduces for $\alpha \to 0$ to the vector form of the DNLS, there is hope to prove integrability for the MVDNLS. There are, however, two major complications with the MVDNLS: it has a vector character which contrary to the DNLS cannot be transformed away, and the boundary values at infinity are not zero (at least not in the physical model for which the nonlinear evolution equation was derived!).

One encounters in the literature quite a variety of methods to investigate symmetries, to construct conservation laws, or to establish integrability of nonlinear equations via direct or inverse methods. While in principle these methods could be used, one rarely sees worked examples involving vector equations. Furthermore, the nonzero boundary conditions for the MVDNLS are an additional hurdle.

3. Integrability and conserved densities

The first step in any treatment is to try to prove integrability, or at least collect sufficiently compelling evidence. For instance, one could check if the equation passes the Painlevé test. However, here again the nonscalar character of the MVDNLS prevents straightforward application of this otherwise so useful test [11]. A different way to ascertain integrability, at least as convincing, is indicated by Kaup [8] and Fordy [11]. It involves the application of the prolongation method due to Estabrook and Wahlquist. This method was adapted successfully to the nonlinear equation at hand [4].

Yet another important tool to determine integrability of a nonlinear PDE is finding a sufficiently large number of conservation laws of the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial J}{\partial x} = 0, \tag{5}$$

where

$$\int_{-\infty}^{+\infty} \rho \, \mathrm{d}x$$

is the conserved quantity with density ρ and associated flux J. We have assumed functions u and v which are fast decreasing at infinity to symmetric but in the case of u nonzero boundary conditions. As is well known from many other examples, the first conservation law comes from rewriting the equations themselves in the form of a conservation law. For the MVDNLS this yields after projection

$$u_t + [u(u^2 + v^2) + \beta u - v_x]_x = 0,$$
$$v_t + [v(u^2 + v^2) + u_x]_x = 0, \tag{6}$$

where u and v denote the components of $\boldsymbol{B}_\perp$ parallel and perpendicular to $\boldsymbol{B}_{\perp 0}$, and $\beta = \alpha B_{\perp 0}^2$. As usual, subscripts refer to partial derivatives with respect to t and x. Whereas eq. (6) amounts to a vector conservation law, the other ones we have derived through constructive procedures are all scalar ones, involving powers of $(u^2 + v^2)$, as presented in eqs (9)–(13).

To see how to proceed, we follow ideas exposed in more detail in Verheest and Hereman [12] and briefly discuss the scaling or symmetry properties of the equations. These can be used to obtain information about polynomial conserved densities and the building blocks they are made of. The scaling of eq. (6) is such that

$$u \sim v, \quad \frac{\partial}{\partial t} \sim \frac{\partial^2}{\partial x^2}, \quad u^2 \sim v^2 \sim \beta \sim \frac{\partial}{\partial x}. \tag{7}$$

We may restrict ourselves to building blocks which belong to the same class under the mentioned scaling, since for any mixed conserved quantity which one could derive, the freedom implied in the scaling would split that quantity in several conserved quantities, each with building blocks of the same scaling. Thus there is a straightforward and logical way to construct invariant quantities [13]. Moreover and without loss of generality, we may remove any density (or part thereof) that is a total x-derivative, for these are trivially conserved. In addition, for $\beta = 0$ eq. (6) is invariant under substitution

$$u \to v, \quad v \to -u. \tag{8}$$

In every conserved density the part without factors β will have to obey this additional symmetry.

At the quadratic level in u and v, the only possibility is $u^2 + v^2$ or eq. (9) given below, due to the rule in eq. (8). Cubic terms in u and v are not possible at all.

Starting then with a candidate density containing the building block $(u^2 + v^2)^2$, one has four factors u or v, and one could add a combination of the form $u^3v - uv^3$. Keeping the scaling eq. (7) in mind, quartic terms are also equivalent to two factors u and/or v and one derivation, or to two factors u and/or v and one factor β. Two factors u and/or v with one derivation can only lead to a non-trivial building block of the structure $uv_x - vu_x$, if we keep all the preceding remarks in mind. With one factor β we could in principle have a linear combination of u^2, uv and v^2. This exhausts the building blocks at this order and it is then for the MATHEMATICA program we wrote to determine the necessary coefficients. This leads to eq. (10). Obviously, we can go on like this for higher orders.

Labelling the conserved densities ρ_n, with n the corresponding power of $(u^2 + v^2)$, we obtained

$$\rho_1 = u^2 + v^2, \tag{9}$$

$$\rho_2 = \tfrac{1}{2}(u^2 + v^2)^2 - uv_x + u_x v + \beta u^2, \tag{10}$$

$$\rho_3 = \tfrac{1}{4}(u^2 + v^2)^3 + \tfrac{1}{2}(u_x^2 + v_x^2) - u^3 v_x + v^3 u_x + \frac{\beta}{4}(u^4 - v^4), \tag{11}$$

$$\begin{aligned}\rho_4 = {} & \tfrac{1}{4}(u^2 + v^2)^4 - \tfrac{2}{5}(u_x v_{xx} - u_{xx} v_x) + \tfrac{4}{5}(uu_x + vv_x)^2 \\ & + \tfrac{6}{5}(u^2 + v^2)(u_x^2 + v_x^2) - (u^2 + v^2)^2(uv_x - u_x v) \\ & + \frac{\beta}{5}(2u_x^2 - 4u^3 v_x + 2u^6 + 3u^4 v^2 - v^6) + \frac{\beta^2}{5}u^4,\end{aligned} \tag{12}$$

$$\begin{aligned}\rho_5 = {} & \tfrac{7}{16}(u^2 + v^2)^5 + \tfrac{1}{2}(u_{xx}^2 + v_{xx}^2) - \tfrac{5}{2}(u^2 + v^2) \\ & \times (u_x v_{xx} - u_{xx} v_x) + 5(u^2 + v^2)(uu_x + vv_x)^2 \\ & + \tfrac{15}{4}(u^2 + v^2)^2(u_x^2 + v_x^2)^2 - \tfrac{35}{16}(u^2 + v^2)^3(uv_x - u_x v) \\ & + \frac{\beta}{8}(5u^8 + 10u^6 v^2 - 10u^2 v^6 - 5v^8 + 20u^2 u_x^2 \\ & - 12u^5 v_x + 60uv^4 v_x - 20v^2 v_x^2) + \frac{\beta^2}{4}(u^6 + v^6).\end{aligned} \tag{13}$$

Note that we have not yet included the renormalization constants needed to ensure the boundedness of the conserved quantities obtained from the above listed densities. We will come back to this point in the following paragraph.

4. Hamiltonian structure

Let us start by pointing out some of the principal ingredients of the Hamiltonian structure of evolution equations, adapted and generalized where necessary to vector quantities.

The system (6) is said to possess a Hamiltonian structure [14], if there exists a so-called Hamiltonian operator Θ [15] (sometimes called implectic operator [16]) and a (2-component) gradient vector function $\gamma_H(u, v)$ such that eq. (6) can be written in the form:

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \Theta \cdot \gamma_H. \tag{14}$$

The operator Θ is a Hamiltonian operator if it is skew-symmetric

$$\langle \Theta \cdot \boldsymbol{a}, \boldsymbol{b} \rangle = -\langle \boldsymbol{a}, \Theta \cdot \boldsymbol{b} \rangle \tag{15}$$

with respect to the scalar product

$$\langle f, g \rangle = \int_{-\infty}^{+\infty} f(x) \cdot g(x)\, \mathrm{d}x, \tag{16}$$

and if it satisfies a "Jacobi-like" identity. The precise form [eq. (A.1)] of that identity is given in the Appendix, since it is not of immediate importance here. A vector function $\gamma_H(u, v)$ is a gradient function if its Fréchet derivative is symmetric with respect to the scalar product (16):

$$\langle \gamma'_H[\boldsymbol{a}], \boldsymbol{b} \rangle = \langle \boldsymbol{a}, \gamma'_H[\boldsymbol{b}] \rangle. \tag{17}$$

Recall that the Fréchet derivative of a vector function $\gamma_H(u, v) = [\gamma_1(u, v)\ \gamma_2(u, v)]^T$ in a direction $[\xi\ \eta]^T$ is given by

$$\gamma'_H\left[\begin{pmatrix} \xi \\ \eta \end{pmatrix}\right] = \frac{\partial}{\partial \varepsilon}\begin{pmatrix} \gamma_1(u + \varepsilon\xi, v) + \gamma_1(u, v + \varepsilon\eta) \\ \gamma_2(u + \varepsilon\xi, v) + \gamma_2(u, v + \varepsilon\eta) \end{pmatrix}\Bigg|_{\varepsilon = 0} \tag{18}$$

The Hamiltonian, or Hamiltonian functional,

$$H = \int_{-\infty}^{+\infty} \rho(u, v)\, \mathrm{d}x$$

giving rise to the gradient function $\gamma_H(u, v)$ through the identity [17]

$$H'[\zeta] = \int_{-\infty}^{+\infty} \rho'(\zeta)\, \mathrm{d}x \equiv \langle \gamma_H, \zeta \rangle, \tag{19}$$

can be recovered from this gradient in the following way:

$$H = \int_0^1 \left[\left\langle \gamma_H(\lambda u, \lambda v), \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle - C\right] \mathrm{d}\lambda, \tag{20}$$

where the constant C is chosen such that the integral giving H is bounded. Such a Hamiltonian formulation admits a Poisson bracket,

$$\{A, B\} \equiv \langle \gamma_A, \Theta \cdot \gamma_B \rangle, \tag{21}$$

defining the time evolution of a functional A of u and v:

$$\begin{aligned}\frac{\mathrm{d}A}{\mathrm{d}t} & = \frac{\partial A}{\partial t} + A'\left[\begin{pmatrix} u \\ v \end{pmatrix}_t\right] = \frac{\partial A}{\partial t} + \left\langle \gamma_A, \begin{pmatrix} u \\ v \end{pmatrix}_t \right\rangle \\ & = \frac{\partial A}{\partial t} + \langle \gamma_A, \Theta \cdot \gamma_H \rangle = \frac{\partial A}{\partial t} + \{A, H\},\end{aligned} \tag{22}$$

using formulas (19) and (14).

Since Θ is skew-symmetric, and due to identity (A.1), the bracket (21) possesses all the characteristics of a standard Poisson bracket, except for the "Leibniz-like" expulsion property which cannot be properly defined in the case of functionals. Clearly, if autonomous, the Hamiltonian H itself is a conserved quantity for the evolution equation (14). Since eq. (6) can be written as a conservation law, an apparent Hamiltonian formulation is the following:

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \Theta_2 \cdot \gamma_2,$$

$$\Theta_2 = -\begin{pmatrix} \partial_x & 0 \\ 0 & \partial_x \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} (u^2 + v^2)u - v_x + \beta u \\ (u^2 + v^2)v + u_x \end{pmatrix}, \tag{23}$$

where ∂_x denotes the partial derivative with respect to x.

Θ_2 is easily seen to be a Hamiltonian operator since it is a constant (i.e. not depending on u or v), skew-symmetric operator (see Appendix).

It is straightforward to verify that γ_2 satisfies eq. (17) and thus leads to a Hamiltonian functional H_2 given by formula (20):

$$H_2 = \int_0^1 \mathrm{d}\lambda \int_{-\infty}^{+\infty} [\lambda^3(u^2+v^2)^2 + \lambda(u_x v - uv_x + \beta u^2) - C_2]\,\mathrm{d}x$$
$$= \int_{-\infty}^{+\infty} \left[\frac{(u^2+v^2)^2}{4} + \frac{u_x v - uv_x}{2} + \frac{\beta}{2}u^2 - C_2\right]\mathrm{d}x. \quad (24)$$

Comparison with formula (10) shows that it is the conserved density ρ_2 which gives rise to this Hamiltonian. One may wonder if the conserved density ρ_1 in eq. (9) can be linked to a Hamiltonian functional as well. Let us define H_1 by

$$H_1 = \int_{-\infty}^{+\infty} \left[\frac{\rho_1}{2} - C_1\right]\mathrm{d}x = \int_{-\infty}^{+\infty} \left[\frac{u^2+v^2}{2} - C_1\right]\mathrm{d}x. \quad (25)$$

Using eq. (19), the gradient γ_1 of this functional is found to be:

$$\gamma_1 = \begin{pmatrix} u \\ v \end{pmatrix}. \quad (26)$$

The challenge is to find a Hamiltonian operator Θ_1 such that eq. (6) can be recast into the form (14) using γ_1. The MVDNLS equation would then have a second Hamiltonian formulation and thus possess a bi-Hamiltonian structure.

5. Bi-Hamiltonian structure and recursion operator

If an evolution equation admits a bi-Hamiltonian formulation [18]

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \Theta_1 \cdot \gamma_1 = \Theta_2 \cdot \gamma_2, \quad (27)$$

and if it is possible to show that Θ_1 and Θ_2 are compatible Hamiltonian operators (i.e. $\Theta_1 + \Theta_2$ is again a Hamiltonian operator), and if one of the operators, say Θ_2 is invertible, then the operator

$$R = \Theta_1 \cdot \Theta_2^{-1} \quad (28)$$

is a hereditary recursion operator [16, 19] for that evolution equation.

The formal adjoint of R with respect to the scalar product (16)

$$R^\dagger = \Theta_2^{-1} \cdot \Theta_1 \quad (29)$$

then maps gradients of conserved quantities into gradients, provided the operator R is injective (see [16] and also Appendix). This adjoint thus defines an infinite sequence of conserved quantities, all in involution with respect to the Poisson bracket (21).

In our case we have already found one Hamiltonian structure with an invertible Hamiltonian operator Θ_2:

$$\Theta_2^{-1} = -\begin{pmatrix} \partial_x^{-1} & 0 \\ 0 & \partial_x^{-1} \end{pmatrix}, \quad (30)$$

where ∂_x^{-1} denotes the inverse of the ∂_x operator, such that $\partial_x \partial_x^{-1} = \partial_x^{-1}\partial_x = 1$.

A first step towards finding a second Hamiltonian structure is the construction of a skew-symmetric operator Θ_1 which casts eq. (6) into Hamiltonian form with the gradient γ_1 in eq. (26). The most general parametrization (involving a single ∂_x^{-1} operator) which satisfies the above constraints is

$$\Theta_1 = -\begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix}, \quad (31)$$

with

$$\theta_{11} = \beta\partial_x + c_1 v\partial_x v + c_3 u\partial_x u + (6-4c_3)u_x\partial_x^{-1}u_x,$$
$$\theta_{12} = -\partial_x^2 + \left(\frac{3}{2} - \frac{c_1+c_2}{2} - \frac{c_5}{4}\right)uv\partial_x + \left(\frac{1}{2} - \frac{c_1-c_2}{2} + \frac{c_5}{4}\right)uv_x + \left(1 - c_1 - \frac{c_5}{2}\right)u_x v + c_5 u_x\partial_x^{-1}v_x,$$
$$\theta_{21} = \partial_x^2 + \left(\frac{3}{2} - \frac{c_1+c_2}{2} - \frac{c_5}{4}\right)uv\partial_x + \left(\frac{1}{2} + \frac{c_1-c_2}{2} + \frac{c_5}{4}\right)u_x v + \left(1 - c_2 - \frac{c_5}{2}\right)uv_x + c_5 v_x\partial_x^{-1}u_x,$$
$$\theta_{22} = c_2 u\partial_x u + c_4 v\partial_x v + (6-4c_4)v_x\partial_x^{-1}v_x. \quad (32)$$

At this point one could of course try to find values of the parameters for which this operator is actually a Hamiltonian operator. However, this proves to be a formidable task (see Appendix). A better approach is to concentrate on the action of $R^\dagger$ on the gradient functions obtain so far. Because of the bi-Hamiltonian structure (27) of the equations, $R^\dagger$ maps γ_1 into γ_2:

$$\gamma_2 = \Theta_2^{-1} \cdot \Theta_1 \cdot \gamma_1 = R^\dagger \cdot \gamma_1. \quad (33)$$

Suppose $R^\dagger$ actually is the formal adjoint of a recursion operator for the MVDNLS equation, then it will map γ_2 into yet another gradient function. If the conserved quantity corresponding to this gradient has to be polynomial in u, v and their derivatives, then the values of the parameters in eq. (32) have to be such that $\Theta_1 \cdot \gamma_2$ is a total x-derivative:

$$\gamma_3 = \Theta_2^{-1} \cdot \Theta_1 \cdot \gamma_2 = -\begin{pmatrix} \partial_x^{-1} & 0 \\ 0 & \partial_x^{-1} \end{pmatrix} \cdot \Theta_1 \cdot \gamma_2. \quad (34)$$

This requirement uniquely determines all five parameters in eq. (32), namely $c_1 = c_2 = 0$, $c_3 = c_4 = -c_5 = 2$, thus giving us a single candidate for a second Hamiltonian operator:

$$\Theta_1 \equiv -\begin{pmatrix} \beta\partial_x + 2u\partial_x u - 2u_x\partial_x^{-1}u_x & -\partial_x^2 + 2v\partial_x u - 2u_x\partial_x^{-1}v_x \\ \partial_x^2 + 2u\partial_x v - 2v_x\partial_x^{-1}u_x & 2v\partial_x v - 2v_x\partial_x^{-1}v_x \end{pmatrix}, \quad (35)$$

together with the gradient

$$\gamma_3 = \beta\gamma_2 + \begin{pmatrix} \beta u^3 - u_{xx} + \frac{3}{2}u^5 + \frac{3}{2}uv^4 + 3u^3v^2 - 3v^2v_x - 3u^2v_x \\ -\beta v^3 - v_{xx} + \frac{3}{2}v^5 + \frac{3}{2}u^4 v + 3u^2v^3 + 3u_x v^2 + 3u^2 u_x \end{pmatrix} \quad (36)$$

Redefining γ_3 as $\gamma_3 - \beta\gamma_2$, it is straightforward to show that it satisfies eq. (17) and corresponds to the Hamiltonian functional

$$H_3 = \int_{-\infty}^{+\infty} \left[\frac{(u^2 + v^2)^3}{4} + u_x v^3 - u^3 v_x - \tfrac{1}{2}(u u_{xx} + v v_{xx}) + \frac{\beta}{4}(u^4 - v^4) - C_3 \right] \mathrm{d}x. \tag{37}$$

The density function associated with this functional is, up to partial integration, the conserved density ρ_3 given in eq. (11). Hence, it follows that $R^\dagger$ maps γ_2 into the gradient of a conserved quantity (i.e., $H_3 + \beta H_2$), suggesting that R indeed is a hereditary recursion operator for the MVDNLS equation.

In the Appendix it is proven that Θ_1 is a Hamiltonian operator and that it is compatible with Θ_2. Thus, we have shown that

$$R = \begin{pmatrix} \beta + 2u^2 + 2u_x \partial_x^{-1} u & -\partial_x + 2uv + 2u_x \partial_x^{-1} v \\ \partial_x + 2uv + 2v_x \partial_x^{-1} u & 2v^2 + 2v_x \partial_x^{-1} v \end{pmatrix} \tag{38}$$

which follows from eqs (28), (30) and (35), is a hereditary recursion operator for the MVDNLS equation:

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \Theta_1 \cdot \Theta_2^{-1} \cdot \begin{pmatrix} u \\ v \end{pmatrix}_x = R \cdot \begin{pmatrix} u \\ v \end{pmatrix}_x. \tag{39}$$

The recursion operator therefore defines a hierarchy of integrable evolution equations

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = R^n \cdot \begin{pmatrix} u \\ v \end{pmatrix}_x, \tag{40}$$

all sharing an infinite sequence of conserved quantities, the gradients of which are generated by the formal adjoint of R:

$$R^\dagger = \begin{pmatrix} \beta + 2u\partial_x^{-1} u \partial_x & -\partial_x + 2u\partial_x^{-1} v \partial_x \\ \partial_x + 2v\partial_x^{-1} u \partial_x & 2v\partial_x^{-1} v \partial_x \end{pmatrix}. \tag{41}$$

For $\beta = 0$ we recover from the MVDNLS the usual DNLS itself. Hence $R|_{\beta=0}$ or

$$R = \begin{pmatrix} 2u^2 + 2u_x \partial_x^{-1} u & -\partial_x + 2uv + 2u_x \partial_x^{-1} v \\ \partial_x + 2uv + 2v_x \partial_x^{-1} u & 2v^2 + 2v_x \partial_x^{-1} v \end{pmatrix} \tag{42}$$

is the recursion operator for DNLS, since it can be seen that the β term in Θ_1 does not alter the fact that it is a Hamiltonian operator (see Appendix). The Hamiltonian formulation (23) reduces to the one given by Kaup and Newell [7] when $\beta = 0$.

6. Conclusions

Slightly oblique propagation of Alfvén waves in strongly magnetized plasmas is described by a nonlinear vector evolution equation which differs from the vector form of the DNLS by an extra linear term, if one retains both the deviation from charge neutrality between the different plasma species and the influence of the displacement current in Ampère's law. The resulting modified vector derivative nonlinear Schrödinger equation cannot be transformed into the DNLS itself, and this has implications for the discussion of integrability and the possibility of finding conserved densities and solitary wave solutions. Of course, the reverse is true: the MVDNLS includes the DNLS as a special case.

While in a previous paper the applicability of the prolongation method, adapted to a vector nonlinear equation, and the existence of some invariants were indicative of complete integrability, in the present paper we have shown that the MVDNLS indeed possesses a bi-Hamiltonian structure, and hence, through the resulting recursion operator, an infinite sequence of conserved densities. We were guided in this by the explicit symbolic computation of the first seven conserved densities.

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Appendix

A skew-symmetric operator Θ is a Hamiltonian operator if and only if it satisfies the identity [16]

$$\langle a, \Theta'[\Theta \cdot c] \cdot b \rangle + \langle b, \Theta'[\Theta \cdot a] \cdot c \rangle + \langle c, \Theta'[\Theta \cdot b] \cdot a \rangle \equiv 0, \tag{A.1}$$

where the scalar product was defined in eq. (16). Since this identity involves taking the Fréchet derivative of Θ in a certain direction, it is trivially satisfied if Θ does not depend upon u or v, i.e. if it is a "constant" operator. Hence, every constant skew-symmetric operator is a Hamiltonian operator.

Verifying that Θ_1 satisfies eq. (A.1) is a very cumbersome task. A method which makes such a verification a lot more tractable relies on defining a functional multivector [20]

$$\mathscr{X} = \frac{1}{2} \int \left\{ (\xi \quad \eta) \wedge \Theta \cdot \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right\} \mathrm{d}x. \tag{A.2}$$

Using this formalism, it can be shown [20] that the identity (A.1) can be recast into the form

$$\mathscr{X}'\left[\Theta \cdot \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right] \equiv 0, \tag{A.3}$$

assuming, by definition, that ξ and η are independent of u, v or their derivatives.

In our case we define the multivector

$$\begin{aligned} \mathscr{X}_1 = \frac{1}{2} \int \{ & (\beta + 2u^2)\xi \wedge \xi_x - 2u_x \xi \wedge \partial_x^{-1} u_x \xi - 2\xi \wedge \eta_{xx} \\ & + 4uv\xi \wedge \eta_x + 4u_x v \xi \wedge \eta - 4u_x \xi \wedge \partial_x^{-1} v_x \eta \\ & + 2v^2 \eta \wedge \eta_x - 2v_x \eta \wedge \partial_x^{-1} v_x \eta \} \,\mathrm{d}x. \end{aligned} \tag{A.4}$$

It is a straightforward, albeit tedious, calculation to show that

$$\mathscr{X}_1'\left[\Theta_1 \cdot \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right] = 0, \tag{A.5}$$

thus proving that Θ_1 is a Hamiltonian operator.

In the same manner it can be shown that

$$\mathscr{X}_1'\left[\Theta_2 \cdot \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right] = 0, \tag{A.6}$$

and hence that $\Theta_1 + \Theta_2$ is a Hamiltonian operator as well, i.e. thus Θ_1 and Θ_2 are compatible Hamiltonian operators. Note that since the only term in $\mathscr{X}_1$ depending on β is independent of u or v, its presence (or absence) does not alter the properties (A.5) or (A.6). Consequently, the above considerations also hold in the special case of the standard DNLS equation.

Concerning the injective nature of the recursion operator R, which is needed to guarantee that $R^\dagger$ maps gradients to gradients (see Proposition 2 in [16]), it suffices to consider the action of R on the space $\mathscr{S}_P$ of (2-component) vector functions with polynomial components in u, v and their derivatives, which are all $\mathscr{C}^\infty$ functions fast decreasing at infinity to symmetric but in the case of u non-zero boundary conditions. It is easily seen that the kernel of R in this space consists of the elements which are of zero polynomial degree. Since R either raises the polynomial degree of the different terms by two or leaves it invariant, R can be made injective by restricting the domain to the space $\mathscr{S}_{P_0} \subset \mathscr{S}_P$ without zero degree elements.

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