Symbolic computation of solitary wave solutions and solitons through homogenization of degree

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Abstract. A simplified Hirota method for the computation of solitary waves and solitons of nonlinear partial differential equations (PDEs) is presented. A change of dependent variable transforms the PDE into an equation that is homogeneous of degree. Solitons are then computed using a perturbation-like scheme involving linear and nonlinear operators in a finite number of steps. The method is applied to fifth-order Korteweg-de Vries (KdV) equations due to Lax, Sawada-Kotera, and Kaup-Kupershmidt.
The method works for non-quadratic homogeneous equations for which the bilinear form might be unknown. Furthermore, homogenization of degree allows one to compute solitary wave solutions of nonlinear PDEs that do not have solitons. Examples include the Fisher and FitzHugh-Nagumo equations, and a combined KdV-Burgers equation. When applied to a wave equation with a cubic source term, one gets a "bi-soliton" solution describing the coalescence of two wavefronts.
The method is largely algorithmic and implemented in *Mathematica*.

Keywords: Hirota method, solitary waves, solitons, symbolic computation

In memory of Prof. R. Hirota (1932-2015)
Photograph courtesy of J. Hietarinta.

1 Introduction

In the 1970s, Hirota [42,43] started working on an algebraic method to compute solitons of completely integrable nonlinear partial differential equations (PDEs). His method has three major steps. Given a nonlinear PDE, (i) change the dependent variable (a.k.a. apply Hirota's transformation) so that the transformed PDE is homogeneous of degree in a new dependent variable (or variables), (ii) express that homogeneous equation into one or more bilinear equations using the Hirota operators, (iii) solve the bilinear equation(s) using a perturbation-like scheme that terminates after a finite number of steps.

Finding the Hirota transformation is quite challenging and often requires insight and ingenuity. Based on experience, Hietarinta [37] provides some useful tips for finding a suitable candidate thereby reducing the guesswork.

Next, finding the appropriate bilinear form for the homogeneous equation can also be a difficult task. In particular in cases where the homogeneous equation is cubic or quartic in the new dependent variable and would have to be decoupled into a pair of bilinear equations, either involving an extra independent variable or an additional function [40]. To circumvent this difficulty, we will not use the bilinear form of the homogeneous equation but include it for completeness.

To compute solitons, the type of solutions one seeks for the homogeneous equation is quite specific. They are a *finite* sums of polynomials in exponential functions with different traveling wave arguments. The terms in that sum are computed order-by-order, using a "tracking" or "bookkeeping" parameter (ϵ) which is set equal to one[3] after the exact solutions are computed.

Hirota's method [45,46,47,48,49] can be found in many books on solitons and complete integrability [2,3,16,77,82], books on differential equations (e.g., [108]), encyclopedia (e.g., [112]), and survey papers [9,70,71,78,92] most noteworthy those by Hietarinta [38,39,40].

Hietarinta's papers have a wealth of information about Hirota's method: how to use it to construct regular and oscillatory solitons (breathers), Bäcklund transformations and Lax pairs, and as a tool in a computer-aided search for possibly new completely integrable systems. His surveys have a plethora of examples including nonlinear Schrödinger (NLS) equations, the sine- and sinh-Gordon equations, shallow water wave equations, the Sasa-Satsuma equation, and systems of coupled equations such as the Hirota-Satsuma and Davey-Stewartson systems.

Hirota wrote a book [49] about his method. As far as we know, the only other book about the bilinear method is by Matsuno [74]. Several theses, for example, [89,115,126] have been written about Hirota's method and it is the subject of thousands of research papers.

Of course, there are several mathematically more rigorous methods to compute solitons, such as the Inverse Scattering Transform (IST), the Wronskian determinant methods, the Riemann-Hilbert approach, the dressing method, the Darboux and Bäcklund transformation methods, etc. In contrast to the more

[3] Unlike the *small* parameter ϵ used in perturbation methods where one seeks *approximate* solutions up to some order in ϵ.

advanced analytic methods that use complex analysis, such as IST and the Riemann-Hilbert method, Hirota's method can not solve the initial value problem for nonlinear PDEs. Regardless, Hirota's method is a direct, powerful, and effective method to quickly find the explicit form of solitons. Apart from soliton solutions, Hirota's method can be used to find rational (lump) solutions of PDEs and the method applies to various types of discrete equations as well. A discussion of those is beyond the scope of this paper.

A mathematical foundation for the Hirota method by Sato and other researchers at the Kyoto School of Mathematics can be found in, for example, [13,14,54,66,86,116]. There are deep connections of Hirota's method with infinite dimensional Lie algebras, transformation groups, Grassmanian manifolds, Wronskians, Gramians, Pfaffians, Bell polynomials, Plücker relations, etc. We refer the interested reader to the literature.

This survey paper is based on one (WH) of the authors' thirty years of experience with Hirota's method mainly from the perspective of applications and computer implementation. He argues that if one seeks solutions involving exponentials, replacing a nonlinear PDE (which usually consists of both linear and nonlinear terms) with an equation that is homogeneous in degree in a new dependent variable (or variables) is quite important, perhaps more so than working with Hirota's bilinear form(s) of the transformed equation. Therefore, "homogenization of degree" is at the core of what is now called[4] the *simplified* Hirota method in which Hirota's bilinear operators are no longer used. Instead, we use a perturbation-like scheme involving linear and nonlinear operators to solve the homogeneous equation without first recasting it into bilinear form.

Although the bilinear representation of the PDE is not used in our approach, dismissing it would be a mistake because it is a valuable tool in the search for completely integrable equations [36,37] and theoretical considerations (see, e.g., [116] and the references therein).

The concept of homogenization of degree is illustrated for the Burgers equation and the ubiquitous Korteweg-de Vries (KdV) equation. For the Burgers equation, a truncated Laurent series of its solution yields the Cole-Hopf transformation, which allows one to transform the Burgers equation into the heat equation. The latter is homogeneous of degree one (linear) and can be solved by separation of variables and other methods. Using Hirota's method, traveling wave solutions of the heat equation involving one or more exponentials readily lead to multiple kink solutions of the Burgers equation. Contrary to solitons, these do not collide elastically but coalesce into a single wavefront.

In the case of the KdV equation, a truncated Laurent series reveals the transformation that Hirota used to replace the KdV by a quadratic (bilinear) equation. The connection between Hirota's transformation and the truncated Laurent expansion, a.k.a. truncated Painlevé expansion or singular manifold expansion, has been long known [17,81,84]. As the examples will show, it is a crucial step in the application of any flavor of Hirota's method.

[4]Some authors [65,97,114,123] call it the Hereman or Hereman-Nuseir method.

The idea of homogenization is further illustrated on a class of completely integrable fifth-order KdV equations, including those of Lax [67], Sawada-Kotera (SK) and Caudrey-Dodd-Gibbon (CDG) [28,95], and Kaup-Kupershmidt (KK) [19,50,57]. Their solitons are computed with a straightforward algorithm involving linear and nonlinear operators which are not necessarily quadratic. Also, the cubic operators we introduce are not the same as the trilinear operators discussed in [25,40] because we split off the linear operator the same way as for quadratic equations.

The computations for the KK case are complicated, lengthy, and nearly impossible without using a symbolic manipulation program such as *Maple* or *Mathematica*. One reason is that the homogeneous equation is of fourth degree. Another reason is that the structure of the soliton solutions is quite different from those of the KdV, Lax, and SK equations. Although the soliton solutions of the KK equation were already presented in [30], and these for the Lax and SK equations have been computed long before that, from time to time their computation resurfaces in the literature, most recently in [56,63,64,104,107,113,114].

Homogenization of degree also allows one to find solitary wave solutions of nonlinear PDEs that are either not completely integrable or for which the bilinear form is unknown. A couple of such examples, mainly from mathematical biology, will be shown. We pay particular attention to a FitzHugh-Nagumo (FHN) equation with convection term for it has a so-called *bi-soliton* solution that describes the coalescence of wavefronts. The same happens for Burgers and wave equations with cubic source terms which are also discussed in detail.

The simplified Hirota method has been successfully used by many authors to find solitary wave and soliton solutions. Most notably, Wazwaz has extensively applied the method to find bi-soliton solutions [109,110] and soliton solutions of a large number of PDEs involving one or more space variables (see, e.g., [111,112,113] and many of his other papers). Additional applications to PDEs with multiple space variables can be found in, e.g., [65,97,114,123].

Before applying the (simplified) Hirota method, it is a good idea to test if the PDE has the Painlevé property [2,11] by running, e.g., the `PainleveTest.m` code [6]. The Laurent series used in the Painlevé test often provides insight in which homogenizing transformation to use.

We developed a *Mathematica* package, called `PDESolitonSolutions.m` [22]. It uses the homogenization method to solve several polynomial PDEs that are completely integrable as well as some that do not have soliton solutions. In this paper we focus on $(1+1)$-dimensional PDEs although our code already works for some PDEs involving up to three space variables (x, y, z) in addition to time (t). We cover only two examples of PDEs with multiple space variables. One of the examples is the well-studied Kadomtsev-Petviashvili (KP) equation.

The paper is organized as follows. In Section 2 we discuss the homogenization of the Burgers and KdV equations using logarithmic derivative transformations.

After a brief review of the original Hirota method, we describe the simplified version in Section 3 still using the KdV equation as the prime example.

In Section 4, we apply the simplified Hirota method to the Lax, SK, and KK equations. For each we compute the one-, two- and three-soliton solutions explicitly.

In Section 5 we show how the method needs to be adjusted to find solitons for the modified KdV (mKdV) equation.

To show how the simplified method can be applied to PDEs that are not "solitonic" in Section 6 we compute solitary wave solutions of the Fisher and FHN equations with and without convection terms. Additional examples include a combined KdV-Burgers equation, a Burgers and wave equation with cubic source terms, and an equation due to Calogero. For each of these equations we compute exact travelling wave solutions. None has soliton solutions although some have bi-soliton solutions.

Section 7 covers an equation in $(1+1)$ dimensions which has two-soliton but not three-soliton solutions.

In Section 8 we compute multi-soliton solutions for the KP equation and an equation in $(3+1)$ dimensions studied by Geng and Ma [21].

Section 9 covers software to automate Hirota's method. In particular, we discuss the implementation and limitations of `PDESolitonSolutions.m` and review related software packages.

Finally, some conclusions are drawn in Section 10 followed by a brief discussion of future work.

2 Homogenization of Nonlinear PDEs

2.1 The Burgers equation

Our initial example is the Burgers (a.k.a. Burgers-Bateman) equation,

$$u_t + 2uu_x - u_{xx} = 0, \tag{1}$$

named after Harry Bateman (1882-1946) and Johannes Burgers (1895-1981). The subscripts denote partial derivatives, e.g., $u_{xx} = \frac{\partial^2 u}{\partial x^2}$ and later on $u_{3x} = \frac{\partial^3 u}{\partial x^3}$, etc. Note that the coefficient of the diffusion term (u_{xx}) has been normalized. Equation (1) can be linearized with a logarithmic derivative transformation due to Cole and Hopf. First integrate[5] the Burgers equation with respect to x, yielding

$$\partial_t \left(\int^x u \, dx \right) + u^2 - u_x = 0. \tag{2}$$

Then substitute

$$u = c \, (\ln f)_x = c \left(\frac{f_x}{f} \right), \tag{3}$$

[5] Alternatively, set $u = v_x$ and integrate with respect to x to get $v_t + v_x^2 - v_{xx} = 0$. Substitution of $v = c \ln f$ yields (4). The same can be done for other equations in this paper.

where c is a constant, to get

$$f(f_t - f_{xx}) + (c+1)f_x^2 = 0. \tag{4}$$

Setting $c = -1$ yields the heat equation

$$f_t - f_{xx} = 0. \tag{5}$$

Then,

$$u(x,t) = -(\ln f)_x = -\frac{f_x}{f} \tag{6}$$

is the well-known Cole-Hopf transformation[6]. We now show where this mysterious transformation comes from. As in the Painlevé test [6], substitute a Laurent series

$$u(x,t) = f^{\alpha}(x,t)\sum_{k=0}^{\infty} u_k(x,t) f^k(x,t) \tag{7}$$

into (1). Note that $f(x,t)$ is the manifold of the poles since α is a negative integer. The most singular terms $f^{2\alpha-1}$ and $f^{\alpha-2}$ will balance when $\alpha = -1$ and vanish for $u_0(x,t) = -f_x$. Truncating (7) at the constant level term in f yields an auto-Bäcklund transformation,

$$u(x,t) = -\frac{f_x}{f} + u_1(x,t) = -(\ln f)_x + u_1(x,t), \tag{8}$$

provided $u_1(x,t)$ is also a solution of the Burgers equation. For the zero solution $(u_1 = 0)$ (8) becomes the Cole-Hopf transformation (6). The transformation allows us to replace the Burgers equation which has a mismatch of linear and quadratic terms in u by an equation that is homogeneous in degree in the new field variable f. The fact that the resulting equation happens to be of *first* degree (linear) is advantageous for it can be solved by separation of variables eventually resulting in a large class of solutions of (1).

Setting the stage for what follows, we consider a couple of simple solutions of (5). Substituting $f(x,t) = 1 + \mathrm{e}^{\theta} = 1 + \mathrm{e}^{k\,x - \omega\,t + \delta}$, where k is the wave number, ω the angular frequency, and δ a phase constant, into (5) yields the dispersion law $\omega = -k^2$. Hence,

$$\begin{aligned} u(x,t) = -(\ln f)_x = -\frac{f_x}{f} &= -k\left(\frac{\mathrm{e}^{\theta}}{1+\mathrm{e}^{\theta}}\right) = -k\left(\frac{\mathrm{e}^{\theta}\mathrm{e}^{-\frac{\theta}{2}}}{(1+\mathrm{e}^{\theta})\mathrm{e}^{-\frac{\theta}{2}}}\right) \\ &= -k\left(\frac{\mathrm{e}^{\frac{\theta}{2}}}{\mathrm{e}^{\frac{\theta}{2}}+\mathrm{e}^{-\frac{\theta}{2}}}\right) = -\tfrac{1}{2}k\left(\frac{2\mathrm{e}^{\frac{\theta}{2}}}{\mathrm{e}^{\frac{\theta}{2}}+\mathrm{e}^{-\frac{\theta}{2}}}\right) \\ &= -\tfrac{1}{2}k\left(\frac{\mathrm{e}^{\frac{\theta}{2}}+\mathrm{e}^{-\frac{\theta}{2}}+\mathrm{e}^{\frac{\theta}{2}}-\mathrm{e}^{-\frac{\theta}{2}}}{\mathrm{e}^{\frac{\theta}{2}}+\mathrm{e}^{-\frac{\theta}{2}}}\right) = -\tfrac{1}{2}k\left(1+\tanh\tfrac{\theta}{2}\right) \end{aligned} \tag{9}$$

[6]This transformation is consistent with the scaling symmetry [29] of the Burgers equation which is invariant under $x \to \lambda^{-1}x$, $t \to \lambda^{-2}t$, $u \to \lambda u$ with an arbitrary constant λ. Hence, one would expect a *first* derivative of $\ln f$.

with $\theta = kx + k^2 t + \delta$, or, simply

$$u(x,t) = K\left(1 - \tanh \Theta\right), \tag{10}$$

with $\Theta = Kx - 2K^2 t + \Delta$, $K = -\frac{k}{2}$, and $\Delta = -\frac{\delta}{2}$. This kink-shaped solution (shock wave) of the Burgers equation is pictured in Fig. 1.

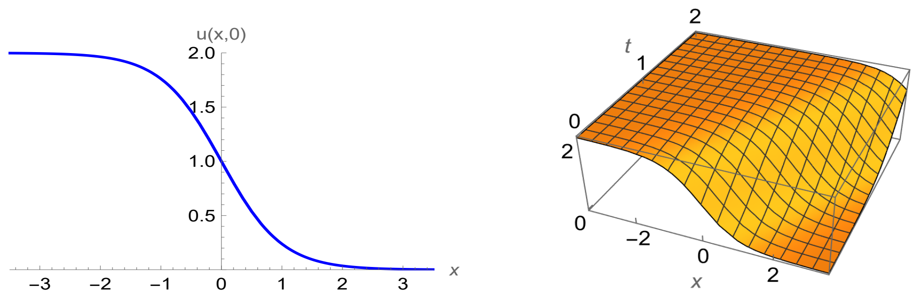

Fig. 1. 2D and 3D graphs of the one-kink solution (10) for $K = 1$ and $\Delta = 0$.

Due to its linearity, $f(x,t) = 1 + \sum_{i=1}^{N} \mathrm{e}^{\theta_i}$ where $\mathrm{e}^{\theta_i} = \mathrm{e}^{k_i\,x + k_i^2\,t + \delta_i}$ with k_i and δ_i arbitrary constants, also solves (5) yielding a N-kink solution

$$u(x,t) = -\frac{k_i \sum_{i=1}^{N} \mathrm{e}^{\theta_i}}{1 + \sum_{i=1}^{N} \mathrm{e}^{\theta_i}} \tag{11}$$

for any integer $N \geq 1$. Fig. 2 shows solution (11) for the case where two wavefronts ($N = 2$) coalesce into a single kink-shaped wavefront as time progresses. For a more detailed analysis of solutions of type (11) we refer to [106].

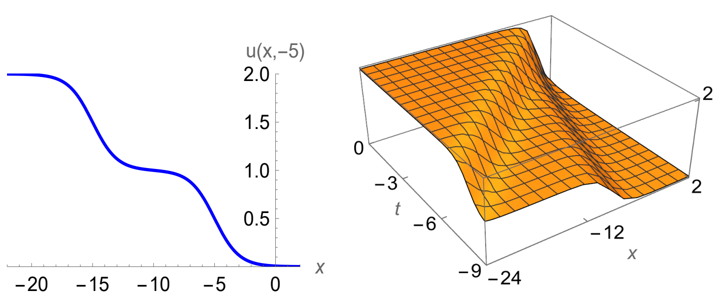

Fig. 2. 2D and 3D graphs of the two-kink solution (11) for $k_1 = -1, k_2 = -2$, and $\delta_1 = \delta_2 = 0$.

2.2 The Korteweg-de Vries equation

Next we explore the homogenization of the ubiquitous KdV equation,

$$u_t + 6uu_x + u_{3x} = 0, \tag{12}$$

named after Diederik Korteweg (1848-1941) and Gustav de Vries (1866-1934).

In [61] they derived the equation and its solitary wave and cnoidal wave solutions:

$$u(x,t) = 2k^2 \operatorname{sech}^2(kx - 4k^3 t + \delta), \tag{13}$$

$$u(x,t) = \tfrac{4}{3}k^2(1-m) + 2k^2 m \operatorname{cn}^2(kx - 4k^3 t + \delta; m), \tag{14}$$

where $m \in (0,1)$ is the modulus of the Jacobi elliptic cosine (cn) function. Both solutions are shown in Fig. 3. As m approaches 1, the peaks of the periodic solution get a little taller, the valleys become lower and flatter before they eventually spread out horizontally to become the pulse-type hyperbolic secant solution.

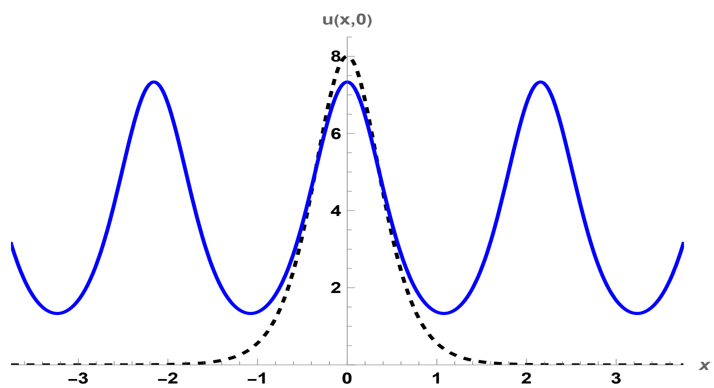

Fig. 3. Graphs of the solitary wave (dashed line) and cnoidal wave (solid line) solutions for $k = 2$, $m = \frac{9}{10}$, and $\delta = 0$.

The interaction of the more complicated soliton solutions (to be discussed later in this paper) were first observed in numerical simulations by Norman Zabusky and Martin Kruskal [122] in 1965.

To compute soliton solutions with Hirota's method the original KdV equation needs to be replaced by an equation (in a new field variable) that is homogeneous of degree. To get a candidate transformation, again substitute a Laurent series (7) into (12). The most singular terms $f^{2\alpha-1}$ and $f^{\alpha-3}$ will balance when $\alpha = -2$. The terms f^{-5} and f^{-4} vanish when $u_0(x,t) = -f_x$ and $u_1(x,t) = 2f_{xx}$. Hence, we obtain an auto-Bäcklund transformation for the KdV equation

$$u(x,t) = -\frac{2{f_x}^2}{f^2} + \frac{2f_{xx}}{f} + u_2(x,t) = 2(\ln f)_{xx} + u_2(x,t), \tag{15}$$

where $u_2(x,t)$ is also a solution of the KdV equation. Taking $u_2 = 0$ yields the Hirota transformation[7] that "bilinearizes" the KdV equation. To see the effect of a logarithmic derivative transformation substitute

$$u = c\,(\ln f)_{xx} = c\left(\frac{f f_{xx} - {f_x}^2}{f^2}\right), \tag{16}$$

where c is an undetermined constant, into the integrated version of (12):

$$\partial_t\left(\int^x u\,dx\right) + 3u^2 + u_{xx} = 0. \tag{17}$$

This yields

$$f^3(f_{xt}+f_{4x}) - f^2(f_x f_t - 3(c-1)f_{xx}^2 + 4f_x f_{3x}) + 3(c-2)f_x^2(f_x^2 - 2f f_{xx}) = 0. \tag{18}$$

Setting $c = 2$ (confirming what we learned from the truncated Laurent series), (18) simplifies into[8]

$$f(f_{xt} + f_{4x}) - f_x f_t + 3f_{xx}^2 - 4f_x f_{3x} = 0, \tag{19}$$

which is homogeneous of *second* degree in f. Hirota [49] introduced the transformation $u = 2\,(\ln f)_{xx}$ in the early 1970s and realized that (19) can be written in bilinear form

$$\left(D_x D_t + D_x^4\right)(f{\cdot}f) = 0, \tag{20}$$

with operators D_x and D_t defined (see, e.g., [46,49]) as

$$D_x^m(f{\cdot}g) = (\partial_x - \partial_{x'})^m f(x,t)g(x',t)\Big|_{x'=x}, \tag{21}$$

$$D_t^n(f{\cdot}g) = (\partial_t - \partial_{t'})^n f(x,t)g(x,t')\Big|_{t'=t}, \tag{22}$$

with m and n positive integers.

Working with these Hirota operators is easy because it amounts to applying Leibniz rule for derivatives of products of functions with every other sign flipped. Thus,

$$D_x^m(f{\cdot}g) = \sum_{j=0}^{m} \frac{(-1)^{m-j} m!}{j!(m-j)!}\left(\frac{\partial^j f}{\partial x^j}\right)\left(\frac{\partial^{m-j} g}{\partial x^{m-j}}\right), \tag{23}$$

and, more general,

$$D_x^m D_t^n(f{\cdot}g) = (\partial_x - \partial_{x'})^m(\partial_t - \partial_{t'})^n f(x,t)g(x',t')\Big|_{x'=x,t'=t} \tag{24}$$

$$= \sum_{j=0}^{m}\sum_{i=0}^{n} \frac{(-1)^{n+m-i-j} m!n!}{j!(m-j)!i!(n-i)!}\left(\frac{\partial^{i+j} f}{\partial t^i \partial x^j}\right)\left(\frac{\partial^{n+m-i-j} g}{\partial t^{n-i}\partial x^{m-j}}\right). \tag{25}$$

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[7] Note that the KdV equation is invariant [29] when $x \to \lambda^{-1}x$, $t \to \lambda^{-3}t$, $u \to \lambda^2 u$. Therefore, a *second* derivative of $\ln f$ makes sense.

[8] Many authors, in particular those working on the mathematical foundation of Hirota's method, use τ instead of f and investigate the rich mathematical properties of the "tau" function.

For example,

$$D_x^4(f{\cdot}g) = f_{4x}g - 4f_{3x}g_x + 6f_{xx}g_{xx} - 4f_x g_{3x} + f g_{4x}, \tag{26}$$

and

$$D_x D_t(f{\cdot}g) = f_{xt}g - f_t g_x - f_x g_t + f g_{xt}. \tag{27}$$

With the above one can readily verify that $\left(D_x D_t + D_x^4\right)(f{\cdot}f) = 0$ yields (19).

3 Solving the Homogeneous PDE

3.1 Hirota's method

We now show how Hirota computed soliton solutions of (20). He sought a solution of the form

$$f(x,t) = 1 + \sum_{n=1}^{\infty} \epsilon^n f^{(n)}(x,t) = 1 + \epsilon f^{(1)} + \epsilon^2 f^{(2)} + \ldots, \tag{28}$$

where ϵ is a formal parameter. The building blocks of solitons are exponentials with different plane-wave arguments. Actually, $f^{(1)}$ will be the sum of a chosen but fixed number (N) of exponentials $\mathrm{e}^{\theta_i} = \mathrm{e}^{k_i\, x - \omega_i\, t + \delta_i}$ $(i = 1, \ldots, N)$. Then, $f^{(2)}$ will have products of just two of these exponentials such as $\mathrm{e}^{2\theta_i}$ and $\mathrm{e}^{\theta_i + \theta_j}$ $(i, j = 1, \ldots, N)$. In turn, $f^{(3)}$ will have products of three exponentials, for example, $\mathrm{e}^{3\theta_i}$, $\mathrm{e}^{2\theta_i + \theta_j}$, $\mathrm{e}^{\theta_i + 2\theta_k}$, and $\mathrm{e}^{\theta_i + \theta_j + \theta_k}$ $(i, j, k = 1, \ldots, N)$. The role of ϵ is to keep track of how many exponentials are in the mix because terms involving products of two exponentials can never be equated to terms with products of three exponentials, etc. In other words, ϵ serves as a *bookkeeping* parameter which can be set to one once the computations are done. As we will see in all the examples that follow, when solitons exist (28) will truncate and therefore be a *finite* sum of exponentials.

Substituting (28) into (20) and splitting order-by-order in ϵ gives

$$\begin{aligned}
&O(\epsilon^0): \ B(1{\cdot}1) = 0,\\
&O(\epsilon^1): \ B(1{\cdot}f^{(1)} + f^{(1)}{\cdot}1) = 0,\\
&O(\epsilon^2): \ B(1{\cdot}f^{(2)} + f^{(1)}{\cdot}f^{(1)} + f^{(2)}{\cdot}1) = 0,\\
&O(\epsilon^3): \ B(1{\cdot}f^{(3)} + f^{(1)}{\cdot}f^{(2)} + f^{(2)}{\cdot}f^{(1)} + f^{(3)}{\cdot}1) = 0,\\
&O(\epsilon^4): \ B(1{\cdot}f^{(4)} + f^{(1)}{\cdot}f^{(3)} + f^{(2)}{\cdot}f^{(2)} + f^{(3)}{\cdot}f^{(1)} + f^{(4)}{\cdot}1) = 0,\\
&\quad\ \vdots \qquad\qquad \vdots\\
&O(\epsilon^n): \ B\left(\sum_{j=0}^{n} f^{(j)}{\cdot}f^{(n-j)}\right) = 0, \ \ n \geq 0, \quad \text{with } f^{(0)} = 1,
\end{aligned} \tag{29}$$

where for the present example $B = D_x D_t + D_x^4$.

To illustrate, we compute the one- and two-soliton solutions of (12). Note that the first equation in (29) is trivially satisfied. Using (26) and (27), the second equation reduces[9] to $f^{(1)}_{xt} + f^{(1)}_{4x} = 0$.

One-soliton solution of the KdV equation

If we take $f^{(1)} = \mathrm{e}^{\theta} \equiv \mathrm{e}^{k\,x-\omega\,t+\delta}$, that second equation yields the dispersion law $\omega = k^3$. Next, one can readily verify that $B(f^{(1)}{\cdot}f^{(1)})$ is zero. Consequently, $f^{(2)}$ is zero and so are $f^{(3)}$, $f^{(4)}$, etc. Therefore, there are only two terms in (28). Explicitly,

$$f = 1 + \mathrm{e}^{\theta} = 1 + \mathrm{e}^{k\,x-k^3\,t+\delta} \tag{30}$$

after setting $\epsilon = 1$. Hence,

$$\begin{aligned} u(x,t) &= 2\left(\frac{f f_{xx} - {f_x}^2}{f^2}\right) = \frac{2k^2\,\mathrm{e}^{\theta}}{(1+\mathrm{e}^{\theta})^2} = \frac{2k^2\,\mathrm{e}^{\theta}\mathrm{e}^{-\theta}}{\left[\mathrm{e}^{-\frac{\theta}{2}}\left(1+\mathrm{e}^{\theta}\right)\right]^2} \\ &= \tfrac{1}{2}k^2\mathrm{sech}^2\left[\tfrac{1}{2}(kx - k^3t+\delta)\right] = 2K^2\mathrm{sech}^2\left(Kx - 4K^3t + \mathit{\Delta}\right), \end{aligned} \tag{31}$$

where $K = \frac{k}{2}$ and $\mathit{\Delta} = \frac{\delta}{2}$. Fig. 4 shows a 3D graph of this so-called *solitary wave solution* or *one-soliton solution* for $K = 2$ and $\mathit{\Delta} = 0$.

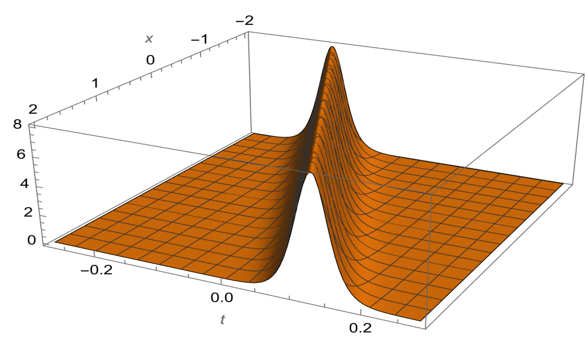


Fig. 4. 3D graph of the hump-shaped solution (31) for $K = 2$ and $\mathit{\Delta} = 0$.

Two-soliton solution of the KdV equation

Starting with $f^{(1)} = \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2}$, where $\mathrm{e}^{\theta_i} = \mathrm{e}^{k_i\,x-\omega_i\,t+\delta_i}$, the first nontrivial equation in (29) yields $\omega_i = k_i^3$. Then, $B(f^{(1)}{\cdot}f^{(1)}) = -6k_1k_2(k_1-k_2)^2\mathrm{e}^{\theta_1+\theta_2}$ which determines the form of $f^{(2)}$, namely, $f^{(2)} = a_{12}\mathrm{e}^{\theta_1+\theta_2}$, with some constant coefficient a_{12} to be computed. Then, $B(1{\cdot}f^{(2)}) = B(f^{(2)}{\cdot}1) = f^{(2)}_{xt} + f^{(2)}_{4x} =$

[9] With $B = D_xD_t + D_x^4$, one has $B(1.f) = B(f.1) = f_{xt} + f_{4x}$ for any function f.

$3a_{12}k_1k_2(k_1+k_2)^2\mathrm{e}^{\theta_1+\theta_2}$. Substitution of the pieces into the third equation of (29) then gives

$$a_{12} = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2. \tag{32}$$

One can show that from $O(\epsilon^3)$ onward one can set $f^{(3)}$, $f^{(4)}$, etc., equal to zero. Thus, f contains only four terms. With $\epsilon = 1$, using

$$f = 1 + \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2} + a_{12}\mathrm{e}^{\theta_1+\theta_2}, \tag{33}$$

and $u = 2(\ln f)_{xx}$, this yields

$$u(x,t) = \frac{2\left[k_1^2\mathrm{e}^{\theta_1} + k_2^2\mathrm{e}^{\theta_2} + 2(k_1-k_2)^2\mathrm{e}^{\theta_1+\theta_2} + a_{12}(k_2^2\mathrm{e}^{\theta_1} + k_1^2\mathrm{e}^{\theta_2})\mathrm{e}^{\theta_1+\theta_2}\right]}{\left(1 + \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2} + a_{12}\,\mathrm{e}^{\theta_1+\theta_2}\right)^2}. \tag{34}$$

Setting $k_i = 2K_i$, $\delta_i = 2\varDelta_i + \ln\left(\frac{K_2+K_1}{K_2-K_1}\right)$, the above can be written as

$$\begin{aligned} u(x,t) &= \frac{4\left(K_2^2 - K_1^2\right)\left[(K_2^2 - K_1^2) + K_1^2\cosh(2\varTheta_2) + K_2^2\cosh(2\varTheta_1)\right]}{\left[(K_2 - K_1)\cosh(\varTheta_2 + \varTheta_1) + (K_2 + K_1)\cosh(\varTheta_2 - \varTheta_1)\right]^2} \\ &= 2\left(K_2^2 - K_1^2\right)\left(\frac{K_1^2\mathrm{sech}^2(\varTheta_1) + K_2^2\mathrm{csch}^2(\varTheta_2)}{\left[K_1\tanh(\varTheta_1) - K_2\coth(\varTheta_2)\right]^2}\right), \end{aligned} \tag{35}$$

where $\varTheta_i = K_i x - 4K_i^3 t + \varDelta_i$ $(i = 1, 2)$. The elastic scattering of two solitons for the KdV equation is shown in Figs. 5 and 6 for $k_1 = 2$, $k_2 = \frac{3}{2}$, and $\delta_1 = \delta_2 = 0$.

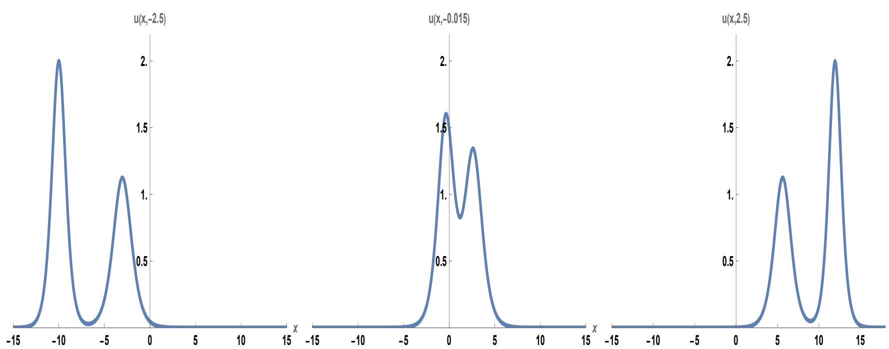


Fig. 5. Graph of the two-soliton solution (35) of the KdV equation at three different moments in time.

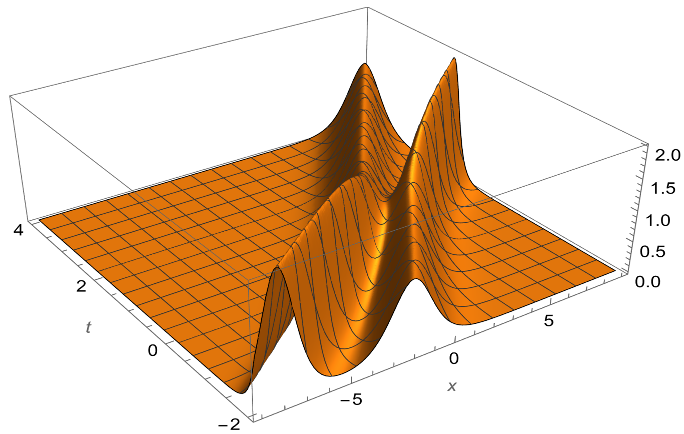


Fig. 6. Bird's eye view of a two-soliton collision for the KdV equation. Notice the phase shift after collision: the taller (faster) soliton is shifted forward and the shorter (slower) soliton backward relative to where they would have been if they had not collided.

3.2 Simplified Hirota method

In this Section we use a simplified version of Hirota's method which does not use the bilinear representation (20). Instead, we write (19) in the form

$$f\,\mathcal{L}f + \mathcal{N}(f,f) = 0, \tag{36}$$

where

$$\mathcal{L}f = f_{xt} + f_{4x} \tag{37}$$

and

$$\mathcal{N}(f,g) = -f_x g_t + 3 f_{xx} g_{xx} - 4 f_x g_{3x} \tag{38}$$

define a linear differential operator $\mathcal{L}$ and a quadratic differential operator $\mathcal{N}$. Note that the latter is linear in each of the auxiliary functions $f(x,t)$ and $g(x,t)$. So, we could also call it "bilinear" but, of course, $\mathcal{N}$ differs from Hirota's bilinear operator B. Substituting (28) into (36), and setting the coefficients of powers of ϵ to zero yields[10]

$$\begin{aligned}
O(\epsilon^1):&\ \mathcal{L}f^{(1)} = 0,\\
O(\epsilon^2):&\ \mathcal{L}f^{(2)} = -\mathcal{N}(f^{(1)},f^{(1)}),\\
O(\epsilon^3):&\ \mathcal{L}f^{(3)} = -\left(f^{(1)}\mathcal{L}f^{(2)} + \mathcal{N}(f^{(1)},f^{(2)}) + \mathcal{N}(f^{(2)},f^{(1)})\right),\\
\vdots\qquad&\qquad\vdots\\
O(\epsilon^n):&\ \mathcal{L}f^{(n)} = -\sum_{j=1}^{n-1}\left(f^{(j)}\mathcal{L}f^{(n-j)} + \mathcal{N}(f^{(j)},f^{(n-j)})\right),\quad n \geq 2.
\end{aligned} \tag{39}$$

[10]Details of the derivation are given in the Appendix.

The N-soliton solution of the KdV is then generated from

$$f^{(1)} = \sum_{i=1}^{N} \mathrm{e}^{\theta_i} \equiv \sum_{i=1}^{N} \mathrm{e}^{k_i\,x - \omega_i\,t + \delta_i}, \tag{40}$$

where N is a natural number, by solving the equations (39) successively to determine $f^{(2)}, f^{(3)}$, etc. The first equation, $\mathcal{L}f^{(1)} = 0$, yields the dispersion relation $\omega_i = k_i^3$. With (40) one readily computes

$$-\mathcal{N}(f^{(1)}, f^{(1)}) = -\sum_{i,j=1}^{N} 3k_i k_j^2 (k_i - k_j)\mathrm{e}^{\theta_i+\theta_j} = \sum_{1\le i<j\le N} 3k_i k_j (k_i - k_j)^2 \mathrm{e}^{\theta_i+\theta_j}. \tag{41}$$

Note that there are no terms $\mathrm{e}^{2\theta_i}$. Hence, $f^{(2)}$ must be of the from

$$f^{(2)} = \sum_{1\le i<j\le N} a_{ij}\mathrm{e}^{\theta_i+\theta_j}, \tag{42}$$

with constants[11] a_{ij} to be determined. Next, compute

$$\mathcal{L}f^{(2)} = \sum_{1\le i<j\le N} 3k_i k_j (k_i + k_j)^2\, a_{ij}\, \mathrm{e}^{\theta_i+\theta_j}, \tag{43}$$

and equate (41) with (43) to get

$$a_{ij} = \left(\frac{k_i - k_j}{k_i + k_j}\right)^2, \qquad 1 \le i < j \le N. \tag{44}$$

To keep matters transparent we show some details of the computation of the three-soliton solution and the result for the four-soliton solution.

Three-soliton solution of the KdV equation

Proceeding in a similar way with the third equation in (39) leads to the explicit form of $f^{(3)}$. For $N = 3$, we find

$$f^{(3)} = b_{123}\mathrm{e}^{\theta_1+\theta_2+\theta_3} \tag{45}$$

with

$$b_{123} = a_{12}\, a_{13}\, a_{23} = \left[\frac{(k_1 - k_2)\,(k_1 - k_3)\,(k_2 - k_3)}{(k_1 + k_2)\,(k_1 + k_3)\,(k_2 + k_3)}\right]^2. \tag{46}$$

For $N = 3$, one can verify that $f^{(n)} = 0$ for $n > 3$. Thus,

$$\begin{aligned} f = 1 &+ \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2} + \mathrm{e}^{\theta_3} + a_{12}\,\mathrm{e}^{\theta_1+\theta_2} + a_{13}\,\mathrm{e}^{\theta_1+\theta_3} + a_{23}\,\mathrm{e}^{\theta_2+\theta_3} \\ &+ a_{12}\,a_{13}\,a_{23}\,\mathrm{e}^{\theta_1+\theta_2+\theta_3} \end{aligned} \tag{47}$$

---

[11] The a_{ij} are often called phase factors because they can be absorbed in the exponents via $a_{ij} = \mathrm{e}^{A_{ij}}$.

after setting $\epsilon = 1$. Notice that (47) has no terms in $e^{2\theta_1}$, $e^{2\theta_2}$, $e^{2\theta_1+\theta_2}$, $e^{\theta_1+2\theta_2}$, etc. The explicit expression of $u(x,t)$ (not shown due to length) then follows from $u(x,t) = 2(\ln f)_{xx}$.

The elastic collision of three solitons for the KdV equation is shown in Figs. 7 and 8 for $k_1 = 2$, $k_2 = \frac{3}{2}$, $k_3 = 1$, and $\delta_1 = \delta_2 = \delta_3 = 0$.

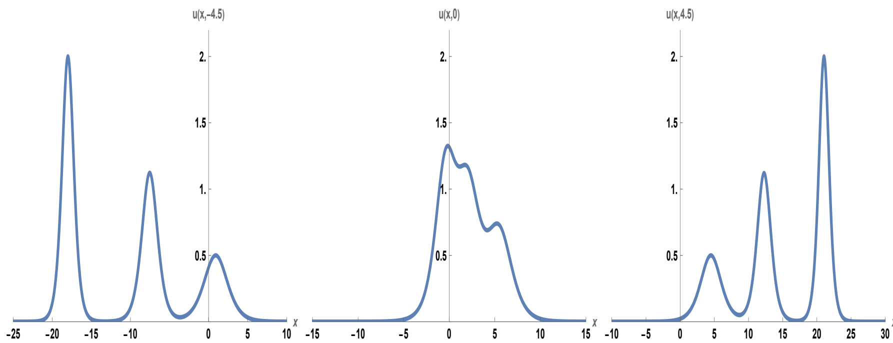


Fig. 7. Graph of the three-soliton solution of the KdV equation at three different moments in time.

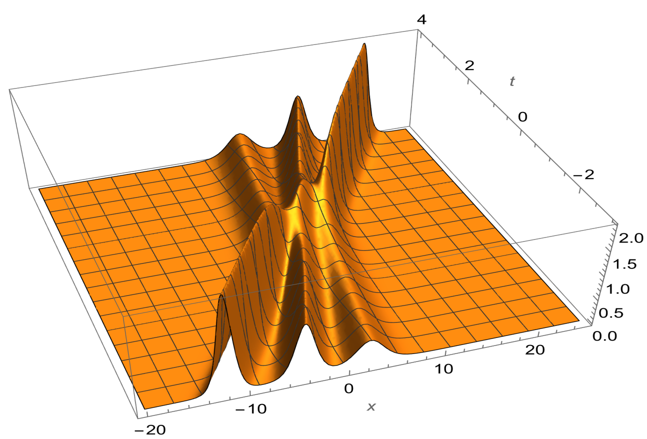


Fig. 8. Bird's eye view of three solitons colliding for the KdV equation. Notice the phase shift after collision: the faster soliton has advanced and the slower ones are behind. The shortest of the three solitons is shifted the most.

Four-soliton solution of the KdV equation

The computation of the four-soliton solution proceeds along the same lines. After setting $\epsilon = 1$,

$$
\begin{aligned}
f = {} & 1 + \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2} + \mathrm{e}^{\theta_3} + \mathrm{e}^{\theta_4} + a_{12}\,\mathrm{e}^{\theta_1+\theta_2} + a_{13}\,\mathrm{e}^{\theta_1+\theta_3} + a_{14}\,\mathrm{e}^{\theta_1+\theta_4} + a_{23}\,\mathrm{e}^{\theta_2+\theta_3} \\
& + a_{24}\,\mathrm{e}^{\theta_2+\theta_4} + a_{34}\,\mathrm{e}^{\theta_3+\theta_4} + a_{12}a_{13}a_{23}\,\mathrm{e}^{\theta_1+\theta_2+\theta_3} + a_{12}a_{14}a_{24}\,\mathrm{e}^{\theta_1+\theta_2+\theta_4} \\
& + a_{13}a_{14}a_{34}\mathrm{e}^{\theta_1+\theta_3+\theta_4} + a_{23}a_{24}a_{34}\mathrm{e}^{\theta_2+\theta_3+\theta_4} + a_{12}a_{13}a_{14}a_{23}a_{24}a_{34}\mathrm{e}^{\theta_1+\theta_2+\theta_3+\theta_4}, \quad (48)
\end{aligned}
$$

with a_{ij} as defined in (44).

The four-soliton solution $u(x,t)$ of the KdV equation follows from $u(x,t) = 2(\ln f)_{xx}$. Its analytic expression is not shown for it would fill pages.

***N*-soliton solution of the KdV equation**

Hirota introduced [46, Eq. (5.38)] a concise formula for the function f leading to the N-soliton solution of the KdV equation,

$$f = \sum_{\mu=0,1} \mathrm{e}^{\left[\sum_{i<j}^{(N)} \mu_i\mu_j A_{ij} + \sum_{i=1}^{N} \mu_i\theta_i\right]}, \tag{49}$$

where $\sum_{\mu=0,1}$ denotes the sum over the 2^N combinations of $\mu_1 = 0,1$, $\mu_2 = 0,1$, $\ldots$, $\mu_N = 0,1$. Furthermore, $\sum_{i<j}^{(N)}$ indicates summation over all possible pairs (i,j) with i and j chosen from the N elements $\{1,2,...,N\}$ but $i < j$, and $a_{ij} = \mathrm{e}^{A_{ij}}$.

Inspired by the result obtained by the IST, the N-soliton solution can be written in a compact form [20,42,101,103] as

$$u(x,t) = 2\left(\ln \det(I+M)\right)_{xx} \tag{50}$$

where I is the $N \times N$ identity matrix and

$$M_{\ell m} = \frac{\mathrm{e}^{\Theta_\ell + \Theta_m}}{K_\ell + K_m} \quad \text{with } \Theta_\ell = K_\ell x - 4K_\ell^3 t + \Delta_\ell. \tag{51}$$

Note that $\det(I+M)$ will match f in (30), (33), (47), and (48) when $k_i = 2K_i$ and $\delta_i = 2\Delta_i - \ln(2K_i)$ with $K_i > 0$.

4 Application to a Class of Fifth-order Evolution Equations

In this section we investigate the soliton solutions of a three-parameter family of fifth-order KdV equations,

$$u_t + \alpha u^2 u_x + \beta u_x u_{xx} + \gamma u u_{3x} + u_{5x} = 0, \tag{52}$$

where α, β, and γ are nonzero real parameters. With $u = \frac{1}{\gamma}\tilde{u}$ one gets

$$\tilde{u}_t + \tfrac{\alpha}{\gamma^2}\tilde{u}^2\tilde{u}_x + \tfrac{\beta}{\gamma}\tilde{u}_x\tilde{u}_{xx} + \tilde{u}\tilde{u}_{3x} + \tilde{u}_{5x} = 0, \tag{53}$$

showing that the individual values of the parameters are less important than the ratios $\frac{\alpha}{\gamma^2}$ and $\frac{\beta}{\gamma}$. Table 1 shows the values of these ratios for which (52) is known to be completely integrable together with values of (α, β, γ) used in the literature. The names of the equations are also listed together with a couple of references. Using scales on u, x, and t, the named equations cannot be transformed into one another; they are fundamentally different[12].

$\frac{\alpha}{\gamma^2}$	$\frac{\beta}{\gamma}$	(α, β, γ)	Name	References
$\frac{3}{10}$	2	$(30, 20, 10)$		[67]
		$(120, 40, 20)$	Lax	[94]
		$(5, 5, 5)$		
$\frac{1}{5}$	1	$(45, 15, 15)$	Sawada-Kotera	[95]
		$(180, 30, 30)$	Caudrey-Dodd-Gibbon	[10,15]
$\frac{1}{5}$	$\frac{5}{2}$	$(20, 25, 10)$	Kaup-Kupershmidt	[19,50,57]

Table 1. Completely integrable fifth-order evolutions equations of type (52).

Integrate (52),

$$\partial_t \left(\int^x u \, dx \right) + \tfrac{1}{3}\alpha u^3 + \tfrac{1}{2}(\beta - \gamma) u_x^2 + \gamma u u_{xx} + u_{4x} = 0, \tag{54}$$

and substitute (16) where c is a constant, to get

$$\begin{aligned} &6f^5(f_{xt} + f_{6x}) - 3f^4(2f_x f_t + \ldots + 12 f_x f_{5x}) + 2f^3\left((\ldots)f_{xx}^3 + \ldots + (\ldots)f_x^2 f_{4x}\right) \\ &+3f^2 f_x^2\left((\ldots)f_{xx}^2 + (\ldots)f_x f_{3x}\right) + 2f_x^4(360 - 6\beta c + \alpha c^2 - 12\gamma c)(3 f f_{xx} - f_x^2) = 0, \end{aligned} \tag{55}$$

which is of sixth degree. In the next subsections we investigate the integrable cases listed in Table 1. For each case the constant c can be obtained from substituting a Laurent series into (52).

4.1 The Lax equation

Using $\alpha = \frac{3}{10}\gamma^2$, $\beta = 2\gamma$, and $c = \frac{20}{\gamma}$, (55) reduces to a homogeneous *trilinear* equation

$$f^2(f_{xt} + f_{6x}) - f(f_x f_t - 5 f_{xx} f_{4x} + 6 f_x f_{5x}) + 10(f_{xx}^3 - 2 f_x f_{xx} f_{3x} + f_x^2 f_{4x}) = 0, \tag{56}$$

[12] After a trivial scaling the CDG equation becomes the SK equation. They are the *same* equations which often goes unnoticed in the literature (see, e.g., [64,91]).

which can be written in bilinear form consisting of two coupled equations [49, p. 56], [46,94]:

$$\begin{aligned}
&\left(D_x D_s + D_x^4\right)(f \cdot f) = 0,\\
&\left(D_x D_t + D_x^6\right)(f \cdot f) - \tfrac{5}{3}\left(D_s^2 + D_s D_x^3\right)(f \cdot f) = 0,
\end{aligned} \tag{57}$$

for only one function f but with an extra independent variable s which corresponds to the time variable in the KdV equation. This comes as no surprise because the Lax equation belongs to the family of KdV flows [82, p. 114] each with its own time variable. Upon elimination of s via suitable cross differentiations one obtains (56).

Note that (56) can also be recast in terms of Hirota trilinear operators [40, Eq. (8.113)]. Completely integrable trilinear equations have been studied [25,40,41,69] but are less common than their bilinear counterparts. Specific examples can be found in, for example, [72,93,96].

We will not use (57) in the subsequent computation of solitons. Instead, we write the cubic equation (56) as

$$f^2 \mathcal{L} f + f \mathcal{N}_1(f,f) + \mathcal{N}_2(f,f,f) = 0, \tag{58}$$

with operators

$$\begin{aligned}
\mathcal{L} f &= f_{xt} + f_{6x},\\
\mathcal{N}_1(f,g) &= -(f_t g_x - 5 f_{xx} g_{4x} + 6 f_x g_{5x}),\\
\mathcal{N}_2(f,g,h) &= 10(f_{xx} g_{xx} h_{xx} - 2 f_x g_{xx} h_{3x} + f_x g_x h_{4x}),
\end{aligned} \tag{59}$$

where f, g, and h are auxiliary functions.

Upon substitution of (28) into (58) the first four equations of the perturbation scheme become[13]

$$\begin{aligned}
O(\epsilon^1) : \mathcal{L} f^{(1)} &= 0,\\
O(\epsilon^2) : \mathcal{L} f^{(2)} &= -\mathcal{N}_1(f^{(1)}, f^{(1)}),\\
O(\epsilon^3) : \mathcal{L} f^{(3)} &= -\Big(2 f^{(1)} \mathcal{L} f^{(2)} + \mathcal{N}_1(f^{(1)}, f^{(2)}) + \mathcal{N}_1(f^{(2)}, f^{(1)})\\
&\quad + f^{(1)} \mathcal{N}_1(f^{(1)}, f^{(1)}) + \mathcal{N}_2(f^{(1)}, f^{(1)}, f^{(1)})\Big),\\
O(\epsilon^4) : \mathcal{L} f^{(4)} &= -\Big(2 f^{(1)} \mathcal{L} f^{(3)} + \Big(2 f^{(2)} + {f^{(1)}}^2\Big) \mathcal{L} f^{(2)} + \mathcal{N}_1(f^{(1)}, f^{(3)})\\
&\quad + \mathcal{N}_1(f^{(3)}, f^{(1)}) + \mathcal{N}_1(f^{(2)}, f^{(2)}) + f^{(1)} \Big(\mathcal{N}_1(f^{(1)}, f^{(2)})\\
&\quad + \mathcal{N}_1(f^{(2)}, f^{(1)})\Big) + f^{(2)} \mathcal{N}_1(f^{(1)}, f^{(1)}) + \mathcal{N}_2(f^{(1)}, f^{(1)}, f^{(2)})\\
&\quad + \mathcal{N}_2(f^{(1)}, f^{(2)}, f^{(1)}) + \mathcal{N}_2(f^{(2)}, f^{(1)}, f^{(1)})\Big),
\end{aligned} \tag{60}$$

where we used the first equation to simplify the other ones. Starting from (40), one can proceed as in KdV case to construct soliton solutions of any order N.

[13] The derivation is given in the Appendix.

The only difference is that for the Lax equation $\omega_i = k_i^5$ instead of $\omega_i = k_i^3$. For example, the one-soliton solution

$$u(x,t) = \tfrac{5}{\gamma} k^2 \mathrm{sech}^2 \left[\tfrac{1}{2}(kx - k^5 t + \delta) \right] = \tfrac{20}{\gamma} K^2 \mathrm{sech}^2 \left(Kx - 16K^5 t + \Delta \right), \quad (61)$$

where $K = \frac{k}{2}$ and $\Delta = \frac{\delta}{2}$, solves

$$u_t + \tfrac{3}{10}\gamma^2 u^2 u_x + 2\gamma u_x u_{xx} + \gamma u u_{3x} + u_{5x} = 0. \quad (62)$$

4.2 The Sawada-Kotera equation

Using $\alpha = \frac{1}{5}\gamma^2,\ \beta = \gamma$, and $c = \frac{30}{\gamma}$ one gets a quadratic equation,

$$f(f_{xt} + f_{6x}) - f_x f_t - 10 f_{3x}^2 + 15 f_{xx} f_{4x} - 6 f_x f_{5x} = 0, \quad (63)$$

which can be written in bilinear form [46] as

$$\left(D_x D_t + D_x^6 \right)(f \cdot f) = 0. \quad (64)$$

Ignoring the bilinear representation, we write (63) in the form (36) with

$$\mathcal{L} f = f_{xt} + f_{6x}, \quad (65)$$

$$\mathcal{N}(f,g) = -f_x g_t - 10 f_{3x} g_{3x} + 15 f_{xx} g_{4x} - 6 f_x g_{5x}, \quad (66)$$

and proceed as in the KdV case, leading to the following soliton solutions.

One-soliton solution of the SK equation

The solitary wave solution

$$\begin{aligned} u(x,t) &= \tfrac{15}{2\gamma} k^2 \mathrm{sech}^2 \left[\tfrac{1}{2}(kx - k^5 t + \delta) \right] \\ &= \tfrac{30}{\gamma} K^2 \mathrm{sech}^2 \left(Kx - 16K^5 t + \Delta \right), \end{aligned} \quad (67)$$

where $K = \frac{k}{2}$ and $\Delta = \frac{\delta}{2}$, solves

$$u_t + \tfrac{1}{5}\gamma^2 u^2 u_x + \gamma u_x u_{xx} + \gamma u u_{3x} + u_{5x} = 0. \quad (68)$$

Higher-order soliton solutions of the SK equation

The computation of higher-order soliton solutions is analogous to the KdV equation; see (33), (47), and (48). Except that the dispersion relation is now quintic, $\omega_i = k_i^5$, and the a_{ij} must be replaced by

$$a_{ij} = \frac{(k_i - k_j)^2 (k_i^2 - k_i k_j + k_j^2)}{(k_i + k_j)^2 (k_i^2 + k_i k_j + k_j^2)} = \frac{(k_i - k_j)^3 (k_i^3 + k_j^3)}{(k_i + k_j)^3 (k_i^3 - k_j^3)}. \quad (69)$$

The actual two- and three-soliton solutions $u(x,t)$ of the SK equation are very long expressions (not shown).

4.3 The Kaup-Kupershmidt equation

Using $\alpha = \frac{1}{5}\gamma^2,\ \beta = \frac{5}{2}\gamma$, and $c = \frac{15}{\gamma}$, (55) becomes a *quartic* equation,

$$\begin{aligned}&4f^3(f_{xt} + f_{6x}) - f^2(4f_t f_x - 5{f_{3x}}^2 + 24 f_x f_{5x})\\&-30 f f_x (f_{xx} f_{3x} - 2 f_x f_{4x}) + 15 {f_x}^2 (3{f_{xx}}^2 - 4 f_x f_{3x}) = 0,\end{aligned} \tag{70}$$

which can be written as a coupled system of bilinear equations [49, p. 36], [88,104],

$$\left(D_x D_t + \tfrac{1}{16} D_x^6\right)(f\cdot f) + \tfrac{15}{4} D_x^2 (f\cdot g) = 0, \tag{71}$$

$$D_x^4 (f\cdot f) - 4fg = 0, \tag{72}$$

for two unknown functions f and g. One can verify that upon elimination of g in (71) and (72) indeed yields (70).

In what follow, we will ignore the bilinear system and write (70) in operator form as

$$f^3 \mathcal{L} f + f^2 \mathcal{N}_1(f,f) + f\mathcal{N}_2(f,f,f) + \mathcal{N}_3(f,f,f,f) = 0. \tag{73}$$

This homogeneous equation involves one linear operator and three nonlinear operators defined as

$$\mathcal{L}f = 4(f_{xt} + f_{6x}), \tag{74}$$

$$\mathcal{N}_1(f,g) = -(4 f_t g_x - 5 f_{3x} g_{3x} + 24 f_x g_{5x}), \tag{75}$$

$$\mathcal{N}_2(f,g,h) = -30 f_x (g_{xx} h_{3x} - 2 g_x h_{4x}), \tag{76}$$

$$\mathcal{N}_3(f,g,h,j) = 15 f_x g_x (3 h_{xx} j_{xx} - 4 h_x j_{3x}), \tag{77}$$

for auxiliary functions $f(x,t), g(x,t), h(x,t)$, and $j(x,t)$. The nonlinear operators are bilinear, trilinear, and quadrilinear, respectively.

Substituting (28) into (73) and equating the coefficients of powers of ϵ to zero yields[14] the perturbation scheme of which the first four equations read

$$\begin{aligned}
O(\epsilon^1) : \mathcal{L}f^{(1)} &= 0,\\
O(\epsilon^2) : \mathcal{L}f^{(2)} &= -\mathcal{N}_1(f^{(1)}, f^{(1)}),\\
O(\epsilon^3) : \mathcal{L}f^{(3)} &= -\Big(3f^{(1)}\mathcal{L}f^{(2)} + 2f^{(1)}\mathcal{N}_1(f^{(1)}, f^{(1)}) + \mathcal{N}_1(f^{(2)}, f^{(1)})\\
&\quad + \mathcal{N}_1(f^{(1)}, f^{(2)}) + \mathcal{N}_2(f^{(1)}, f^{(1)}, f^{(1)})\Big),\\
O(\epsilon^4) : \mathcal{L}f^{(4)} &= -\Big(3f^{(1)}\mathcal{L}f^{(3)} + 3\left(f^{(2)} + {f^{(1)}}^2\right)\mathcal{L}f^{(2)} + \mathcal{N}_1(f^{(1)}, f^{(3)})\\
&\quad + \mathcal{N}_1(f^{(3)}, f^{(1)}) + \mathcal{N}_1(f^{(2)}, f^{(2)}) + 2f^{(1)}\Big(\mathcal{N}_1(f^{(1)}, f^{(2)})\\
&\quad + \mathcal{N}_1(f^{(2)}, f^{(1)})\Big) + \left(2f^{(2)} + {f^{(1)}}^2\right)\mathcal{N}_1(f^{(1)}, f^{(1)})\\
&\quad + \mathcal{N}_2(f^{(1)}, f^{(1)}, f^{(2)}) + \mathcal{N}_2(f^{(1)}, f^{(2)}, f^{(1)}) + \mathcal{N}_2(f^{(2)}, f^{(1)}, f^{(1)})\\
&\quad + f^{(1)}\mathcal{N}_2(f^{(1)}, f^{(1)}, f^{(1)}) + \mathcal{N}_3(f^{(1)}, f^{(1)}, f^{(1)}, f^{(1)})\Big),
\end{aligned} \tag{78}$$

[14] Details of the derivation are given in the Appendix.

where we used the first equation to simplify the subsequent ones. Clearly, the number of terms grows at each order in ϵ and the computational complexity increases accordingly. Full details of the step-by-step solution of the perturbation scheme for the KK equation with coefficients $\alpha = 20, \beta = 25$, and $\gamma = 10$, can be found [30,85] where we used *Macsyma* to perform the lengthy computations. Here we summarize the results for general α, β, and γ subject to the conditions $\alpha = \frac{1}{5}\gamma^2$, and $\beta = \frac{5}{2}\gamma$.

One-soliton solution of the KK equation

Taking $f^{(1)} = \mathrm{e}^{\theta} = \mathrm{e}^{kx-\omega t+\delta}$, $\mathcal{L}f^{(1)} = 0$ yields $\omega = k^5$. In contrast to the KdV case, the right hand side of the second equation,

$$-\mathcal{N}(f^{(1)}, f^{(1)}) = 15k^6\,\mathrm{e}^{2\theta}, \tag{79}$$

does not vanish but has a term in $\mathrm{e}^{2\theta}$. Thus, $f^{(2)}$ must be of the form

$$f^{(2)} = a\,\mathrm{e}^{2\theta}, \tag{80}$$

with undetermined constant coefficient a. Then,

$$\mathcal{L}f^{(2)} = 240ak^6\,\mathrm{e}^{2\theta} \tag{81}$$

and equating the right hand sides of (79) and (81) yields $a = \frac{1}{16}$. Next, we check that we can set $f^{(n)} = 0$ for $n \geq 3$ by verifying that the right hand sides of the subsequent equations in (78) are all zero. This is indeed the case and the perturbation scheme terminates after two steps. Setting $\epsilon = 1$,

$$f = 1 + \mathrm{e}^{\theta} + \tfrac{1}{16}\,\mathrm{e}^{2\theta}, \tag{82}$$

and $u = \frac{15}{\gamma}(\ln f)_{xx}$ yields

$$u = \tfrac{240}{\gamma}k^2\,\left[\frac{\mathrm{e}^{\theta}(16 + 4\mathrm{e}^{\theta} + \mathrm{e}^{2\theta})}{(16 + 16\mathrm{e}^{\theta} + \mathrm{e}^{2\theta})^2}\right] \tag{83}$$

which solves

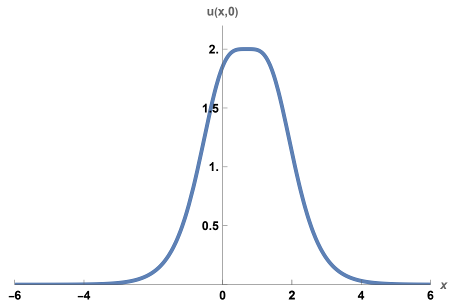
$$u_t + \tfrac{1}{5}\gamma^2u^2u_x + \tfrac{5}{2}\gamma u_xu_{xx} + \gamma uu_{3x} + u_{5x} = 0. \tag{84}$$

The one-soliton solution can also be written as

$$u = \tfrac{240}{\gamma}k^2\,\left(\frac{\left[1 - \tanh^2(\frac{\theta}{2})\right]\left[21 - 30\tanh\frac{\theta}{2} + 13\tanh^2(\frac{\theta}{2})\right]}{\left[33 - 30\tanh\frac{\theta}{2} + \tanh^2(\frac{\theta}{2})\right]^2}\right) \tag{85}$$

$$= \tfrac{240}{\gamma}k^2\,\left(\frac{4 + 17\cosh\theta - 15\sinh\theta}{\left[16 + 17\cosh\theta - 15\sinh\theta\right]^2}\right), \tag{86}$$

where $\theta = kx - k^5t + \delta$. Fig. 9 shows the 2D and 3D graphs of the one-soliton solution for $\gamma = 10, k = 2$, and $\delta = 0$. In comparison with the solitary wave solution of the KdV equation shown in Figs. 3 and 4, the solution of the KK equation is wider and flatter at the top.



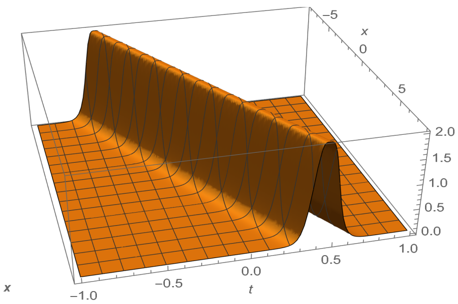


Fig. 9. 2D and 3D graphs of solution (85) with $\gamma = 10, k = 2$, and $\delta = 0$.

Two-soliton solution of the KK equation

Starting from

$$f^{(1)} = \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2}, \tag{87}$$

where $\theta_i = k_i x - k_i^5 t + \delta_i$ $(i = 1, 2)$, we compute

$$-\mathcal{N}_1(f^{(1)}, f^{(1)}) = 15k_1^6\,\mathrm{e}^{2\theta_1} + 15k_2^6\,\mathrm{e}^{2\theta_2} + 10k_1k_2(2k_1^4 - k_1^2k_2^2 + 2k_2^4)\,\mathrm{e}^{\theta_1+\theta_2}. \tag{88}$$

Thus $f^{(2)}$ must be of the form

$$f^{(2)} = a_1\,\mathrm{e}^{2\theta_1} + a_2\,\mathrm{e}^{2\theta_2} + a_{12}\,\mathrm{e}^{\theta_1+\theta_2}, \tag{89}$$

with the (constant) coefficients a_1, a_2, and a_{12} to be determined. Then,

$$\begin{aligned}\mathcal{L}f^{(2)} &= 240a_1k_1^6\,\mathrm{e}^{2\theta_1} + 240a_2k_2^6\,\mathrm{e}^{2\theta_2}\\ &\quad + 20a_{12}k_1k_2(k_1+k_2)^2(k_1^2 + k_1k_2 + k_2^2)\,\mathrm{e}^{\theta_1+\theta_2}.\end{aligned} \tag{90}$$

Equating (88) with (90) determines $a_1 = a_2 = \frac{1}{16}$, as expected, and

$$a_{12} = \frac{2k_1^4 - k_1^2k_2^2 + 2k_2^4}{2(k_1+k_2)^2(k_1^2 + k_1k_2 + k_2^2)}. \tag{91}$$

Therefore,

$$f^{(2)} = \tfrac{1}{16}\,\mathrm{e}^{2\theta_1} + \tfrac{1}{16}\,\mathrm{e}^{2\theta_2} + \frac{(2k_1^4 - k_1^2k_2^2 + 2k_2^4)}{2(k_1+k_2)^2(k_1^2 + k_1k_2 + k_2^2)}\,\mathrm{e}^{\theta_1+\theta_2}. \tag{92}$$

The main difference with the the KdV, Lax, and SK equations is that the terms $\mathrm{e}^{2\theta_1}$ and $\mathrm{e}^{2\theta_2}$ in $f^{(2)}$ no longer drop out. At $O(\epsilon^3)$ one gets

$$f^{(3)} = b_{12}\left(\mathrm{e}^{\theta_1+2\theta_2} + \mathrm{e}^{2\theta_1+\theta_2}\right), \tag{93}$$

with

$$b_{12} = \frac{(k_1-k_2)^2(k_1^2 - k_1k_2 + k_2^2)}{16(k_1+k_2)^2(k_1^2 + k_1k_2 + k_2^2)}. \tag{94}$$

At the next order

$$f^{(4)} = b_{12}^2\, \mathrm{e}^{2(\theta_1+\theta_2)} = \frac{(k_1-k_2)^4(k_1^2-k_1k_2+k_2^2)^2}{256(k_1+k_2)^4(k_1^2+k_1k_2+k_2^2)^2}\, \mathrm{e}^{2(\theta_1+\theta_2)}. \qquad (95)$$

After verification that all $f^{(n)}$ are zero for $n \geq 5$ and setting $\epsilon = 1$,

$$\begin{aligned} f = {} & 1 + \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2} + \tfrac{1}{16}\,\mathrm{e}^{2\theta_1} + \tfrac{1}{16}\,\mathrm{e}^{2\theta_2} + a_{12}\,\mathrm{e}^{\theta_1+\theta_2} \\ & + b_{12}\left(\mathrm{e}^{2\theta_1+\theta_2} + \mathrm{e}^{\theta_1+2\theta_2}\right) + b_{12}^2\,\mathrm{e}^{2(\theta_1+\theta_2)}. \end{aligned} \qquad (96)$$

The explicit expression of $u(x,t)$ (not shown due to length) then follows from $u = \frac{15}{\gamma}(\ln f)_{xx}$. The collision of two solitons for the KK equation is shown in Figs. 10 and 11 for $k_1 = 2$, $k_2 = 1$, and $\delta_1 = \delta_2 = 0$.

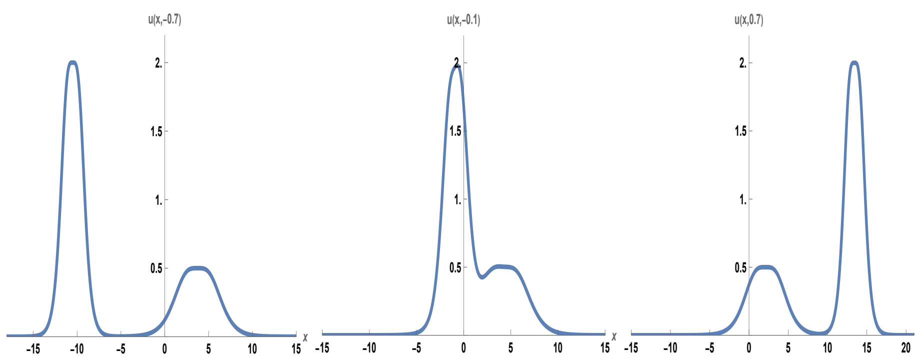


Fig. 10. Graph of the two-soliton solution of the KK equation at three different moments in time.

Three-soliton solution of the KK equation

Starting with

$$f^{(1)} = \sum_{i=1}^{3} \mathrm{e}^{\theta_i} = \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2} + \mathrm{e}^{\theta_3}, \qquad (97)$$

where $\theta_i = k_i x - k_i^5 t + \delta_i$, the equations of the perturbation scheme are solved order-by-order yielding expressions for $f^{(2)}, f^{(3)}, \ldots, f^{(6)}$ because, as it turns out, $f^{(n)} = 0$ for $n \geq 7$. The latter requires verification that the right hand sides at $O(\epsilon^7)$ and beyond all vanish in order for the perturbation scheme to terminate. The computations are very lengthy, time consuming, and currently

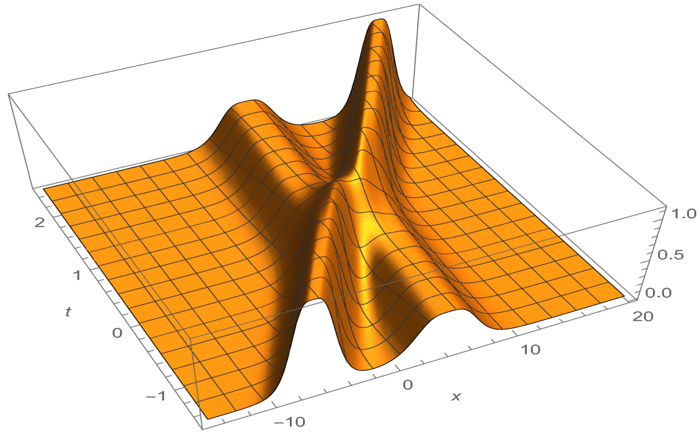


Fig. 11. Bird's eye view of the collision of two solitons for the KK equation. Notice the phase shift after the collision.

at the limit of what *Mathematica* can handle[15]. Summarizing the results:

$$f^{(2)} = \tfrac{1}{16}\sum_{i=1}^{3} \mathrm{e}^{2\theta_i} + \sum_{1\le i<j\le 3} a_{ij}\,\mathrm{e}^{\theta_i+\theta_j}, \tag{98}$$

with phase factors

$$a_{ij} = \frac{2k_i^4 - k_i^2 k_j^2 + 2k_j^4}{2(k_i+k_j)^2(k_i^2+k_ik_j+k_j^2)}, \qquad 1\le i<j\le 3. \tag{99}$$

Next,

$$f^{(3)} = \sum_{1\le i<j\le 3} b_{ij}\,\left(\mathrm{e}^{2\theta_i+\theta_j} + \mathrm{e}^{\theta_i+2\theta_j}\right) + c_{123}\,\mathrm{e}^{\theta_1+\theta_2+\theta_3}, \tag{100}$$

where

$$b_{ij} = \frac{(k_i-k_j)^2(k_i^2-k_ik_j+k_j^2)}{16(k_i+k_j)^2(k_i^2+k_ik_j+k_j^2)}, \qquad 1\le i<j\le 3, \tag{101}$$

and

$$\begin{aligned} c_{123} = \tfrac{1}{D}\big[&(2k_1^4-k_1^2k_2^2+2k_2^4)(k_3^8+k_1^4k_2^4)+(2k_1^4-k_1^2k_3^2+2k_3^4)(k_2^8+k_1^4k_3^4)\\ &+(2k_2^4-k_2^2k_3^2+2k_3^4)(k_1^8+k_2^4k_3^4)\big]-\tfrac{1}{2D}\big[(k_1^2+k_2^2)(k_1^4+k_2^4)(k_3^6+k_1^2k_2^2k_3^2)\\ &+(k_1^2+k_3^2)(k_1^4+k_3^4)(k_2^6+k_1^2k_2^2k_3^2)+(k_2^2+k_3^2)(k_2^4+k_3^4)(k_1^6+k_1^2k_2^2k_3^2)\\ &+12k_1^4k_2^4k_3^4\big] \end{aligned} \tag{102}$$

[15] With the code PDESolitonSolutions.m discussed in Section 9, the computation of the three-soliton solution takes about 4 minutes on a Dell XPS-15 laptop with Intel Core i7 processor at 4.7GHz and 32 GB of memory.

with

$$D = 4 \prod_{1 \le i < j \le 3} (k_i + k_j)^2 \, (k_i^2 + k_i k_j + k_j^2). \tag{103}$$

Carrying on,

$$\begin{aligned} f^{(4)} &= \sum_{1 \le i < j \le 3} b_{ij}^2 \, \mathrm{e}^{2(\theta_i + \theta_j)} + 16 \left(a_{23} b_{12} b_{13} \, \mathrm{e}^{2\theta_1 + \theta_2 + \theta_3} \right. \\ &\qquad \left. + \, a_{13} b_{12} b_{23} \mathrm{e}^{\theta_1 + 2\theta_2 + \theta_3} + a_{12} b_{13} b_{23} \, \mathrm{e}^{\theta_1 + \theta_2 + 2\theta_3} \right), \end{aligned} \tag{104}$$

$$\begin{aligned} f^{(5)} &= 256 \, b_{12} b_{13} b_{23} \left(b_{12} \, \mathrm{e}^{2\theta_1 + 2\theta_2 + \theta_3} + b_{13} \mathrm{e}^{2\theta_1 + \theta_2 + 2\theta_3} \right. \\ &\qquad \left. + \, b_{23} \, \mathrm{e}^{\theta_1 + 2\theta_2 + 2\theta_3} \right), \end{aligned} \tag{105}$$

$$f^{(6)} = 16 \, (16 b_{12} b_{13} b_{23})^2 \, \mathrm{e}^{2(\theta_1 + \theta_2 + \theta_3)}. \tag{106}$$

Finally, after setting $\epsilon = 1$,

$$f = 1 + f^{(1)} + f^{(2)} + f^{(3)} + f^{(4)} + f^{(5)} + f^{(6)}, \tag{107}$$

and $u(x,t) = \frac{15}{\gamma} (\ln f)_{xx}$ (not shown due to length) then solves (84).

The collision of three solitons for the KK equation is shown in Figs. 12 and 13 for $k_1 = 2, k_2 = \frac{3}{2}, k_3 = 1$, and $\delta_1 = \delta_2 = \delta_3 = 0$.

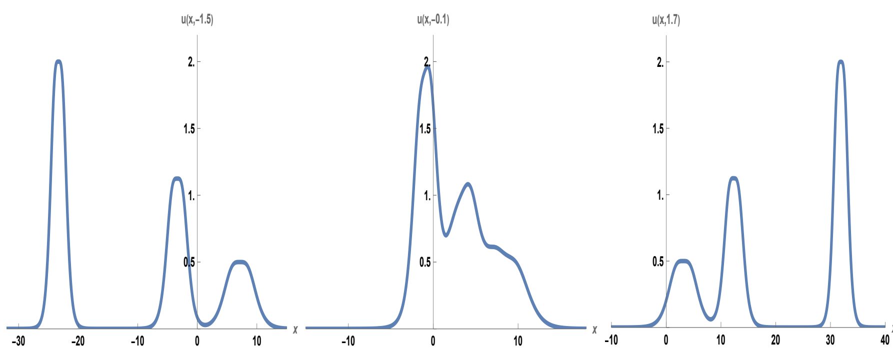


Fig. 12. Graph of the three-soliton solution of the KK equation at three different moments in time.

5 The Modified KdV Equation

Of course, not every polynomial soliton equation in $(1+1)$ dimensions can be solved with a solution of type (28). Consider, for example, the mKdV equation,

$$u_t + 24 u^2 u_x + u_{3x} = 0, \tag{108}$$

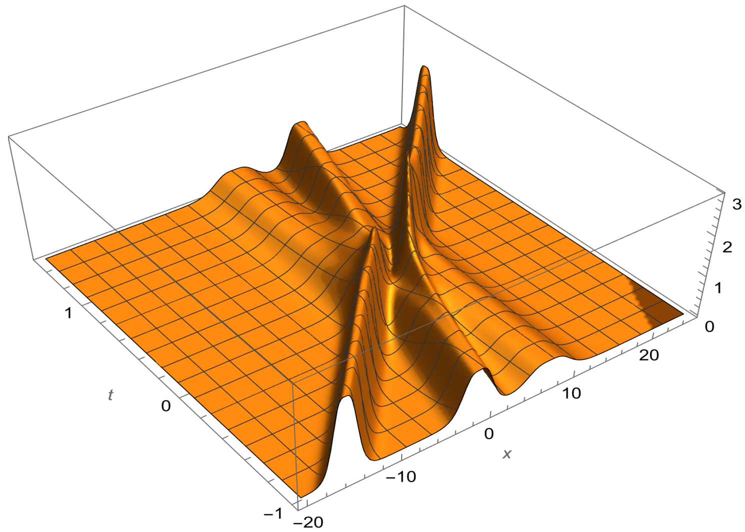


Fig. 13. Bird's eye view of the collision of three solitons for the KK equation. Notice the phase shift after the collision.

which after integration becomes

$$\partial_t \left(\int^x u\, dx \right) + 8u^3 + u_{xx} = 0. \tag{109}$$

The Laurent series for (108) suggests the transformation

$$u = \pm\tfrac{1}{2} i\, (\ln \tilde{f})_x = \pm\tfrac{1}{2} i \left(\frac{\tilde{f}_x}{\tilde{f}} \right). \tag{110}$$

Substitution of either of these branches into (109) yields

$$\tilde{f}(\tilde{f}_t + \tilde{f}_{3x}) - 3\tilde{f}_x \tilde{f}_{xx} = 0. \tag{111}$$

Although homogeneous of second degree and deceptively simple, it has no solution of the form $\tilde{f} = 1 + \mathrm{e}^\theta$ where $\theta = k\,x - \omega\,t + \delta$. Indeed, the term in e^θ vanishes for $\omega = k^3$ but the term $-3k^3 \mathrm{e}^{2\theta}$ is only zero when $k = 0$. It is clear from (110) that to obtain a real-valued solution, i.e., $u^\star = u$, $\tilde{f}$ must be a complex function. One can readily verify that $u = \pm\frac{1}{2} i\, (\ln(f + ig))_x$, for real functions f and g does not work either. So, $\tilde{f}$ must be a ratio of complex functions. Hence,

$$u = \pm\tfrac{1}{2} i \left(\ln \left(\frac{f + i\,g}{h + i\,j} \right) \right)_x, \tag{112}$$

where f, g, h, and j are real functions. From $u^\star = u$ it follows that $h = f$ and $j = -g$. Observe that (108) remains invariant when u is replaced by its negative.

Therefore, without loss of generality, we continue with the plus sign,

$$u = \tfrac{1}{2} i \left(\ln \left(\frac{f + i\, g}{f - i\, g} \right) \right)_x = \left(\arctan \left(\tfrac{f}{g} \right) \right)_x = \frac{f_x g - f g_x}{f^2 + g^2}, \tag{113}$$

which is Hirota's transformation for the mKdV equation [43]. Note that the roles of f and g can thus be interchanged in the computations below.

Goldstein [23] gave a different argument[16] to arrive at (113). Accounting for the $\pm$ signs in (110), he argued that the solution may have two families of singularities and therefore assumed[17]

$$u = \tfrac{1}{2} i \left(\frac{F_x}{F} - \frac{G_x}{G} \right) = \tfrac{1}{2} i \left(\ln \left(\frac{F}{G} \right) \right)_x . \tag{114}$$

Note that the two terms (in the first equality above) indeed account for the two branches in (110). Setting $F = f + i\, g$ and $G = f - i\, g$ then gives (113).

Applying Hirota's transformation (113) to (109) yields

$$\begin{aligned} & f^3 (g_t + g_{3x}) - g^3 (f_t + f_{3x}) - f^2 (f_t g + 3 f_x g_{xx} + 3 f_{xx} g_x + f_{3x} g) \\ & + g^2 (f g_t + f g_{3x} + 3 f_x g_{xx} + 3 f_{xx} g_x) + 6 f g_x (f_x^2 + g_x^2) \\ & - 6 f_x g (f_x^2 + g_x^2) + 6 f g (f_x f_{xx} - g_x g_{xx}) = 0, \end{aligned} \tag{115}$$

which is clearly not of the usual form the simplified Hirota method applies to. The terms in (115) can be regrouped as

$$\begin{aligned} & (f^2 + g^2)(f_t g - f g_t - f g_{3x} + 3 f_x g_{xx} - 3 f_{xx} g_x + f_{3x} g) \\ & - 6 (f_x g - f g_x)(f f_{xx} - f_x^2 + g g_{xx} - g_x^2) = 0. \end{aligned} \tag{116}$$

Taking advantage of the fact that there are two free functions in play, Hirota [43,45] then set the factors multiplying $f^2 + g^2$ and $f_x g - f g_x$ separately equal to zero, to get the coupled system

$$f (g_t + g_{3x}) - g (f_t + f_{3x}) - 3 (f_x g_{xx} - f_{xx} g_x) = 0, \tag{117}$$

$$f f_{xx} - f_x^2 + g g_{xx} - g_x^2 = 0, \tag{118}$$

which can be written in bilinear form as

$$(D_t + D_x^3)(f \cdot g) = 0, \tag{119}$$

$$D_x^2 (f \cdot f + g \cdot g) = 0. \tag{120}$$

Ignoring the bilinear form, one could write (117) and (118) as

$$f\, \mathcal{L} g - g\, \mathcal{L} f + \mathcal{N}_1 (f, g) = 0, \tag{121}$$

$$\mathcal{N}_2 (f, f) + \mathcal{N}_2 (g, g) = 0, \tag{122}$$

---

[16] The argument is based on modified singular manifold expansion methods [17,24,81].

[17] With $u = \pm \frac{1}{2}\, i\, (\ln (F/G))_x$, (108) can be replaced by $(D_t + D_x^3)(F \cdot G) = 0$ and $D_x^2 (F \cdot G) = 0$ where $G = F^{\star}$. See, e.g., [28] for explicit expressions of F and G for the two- and three-soliton cases.

with

$$\mathcal{L}f = f_t + f_{3x} \tag{123}$$

and

$$\mathcal{N}_1(f,g) = -3(f_x g_{xx} - f_{xx} g_x), \tag{124}$$
$$\mathcal{N}_2(f,g) = f g_{xx} - f_x g_x. \tag{125}$$

With a suitable adaptation of the method in Section 3.2, one could then seek a solution of (121) and (122) using

$$f = f^{(0)} + \epsilon f^{(1)} + \epsilon^2 f^{(2)} + \ldots, \tag{126}$$
$$g = g^{(0)} + \epsilon g^{(1)} + \epsilon^2 g^{(2)} + \ldots. \tag{127}$$

Based on the interchangeability of f and g, one can either take $f^{(0)} = g^{(1)} = 0$ and $g^{(0)} = 1$, or equivalently, $g^{(0)} = f^{(1)} = 0$ and $f^{(0)} = 1$. In either case, $\mathcal{L}\mathrm{e}^{\theta_i} = \mathcal{L}\mathrm{e}^{k_i x - \omega_i t + \delta_i} = 0$ determines the dispersion relation $\omega_i = k_i^3$. Proceeding with the former case but skipping the details of the computations we summarize the results.

One-soliton solution of the mKdV equation

With

$$f = \mathrm{e}^{\theta} = \mathrm{e}^{kx - k^3 t + \delta} \quad \text{and} \quad g = 1, \tag{128}$$

one gets

$$\begin{aligned} u &= \frac{f_x}{1+f^2} = k\frac{\mathrm{e}^{\theta}}{1+\mathrm{e}^{2\theta}} = \tfrac{1}{2}k\,\mathrm{sech}\,\theta \\ &= \tfrac{1}{2}k\,\mathrm{sech}\,(kx - k^3 t + \delta) = K\mathrm{sech}\left(2Kx - 8K^3 t + \delta\right) \end{aligned} \tag{129}$$

with $K = \frac{k}{2}$.

Two-soliton solution of the mKdV equation

Now

$$\begin{aligned} f &= \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2}, \\ g &= 1 - a_{12}\mathrm{e}^{\theta_1+\theta_2}, \end{aligned} \tag{130}$$

with $\theta_i = k_i x - k_i^3 t + \delta_i$ and $a_{12} = \left(\frac{k_1-k_2}{k_1+k_2}\right)^2$. Then,

$$u = \frac{k_1\mathrm{e}^{\theta_1} + k_2\mathrm{e}^{\theta_2} + a_{12}\,(k_1\mathrm{e}^{\theta_2} + k_2\mathrm{e}^{\theta_1})\mathrm{e}^{\theta_1+\theta_2}}{1 + \mathrm{e}^{2\theta_1} + \mathrm{e}^{2\theta_2} + \dfrac{8k_1k_2}{(k_1+k_2)^2}\,\mathrm{e}^{\theta_1+\theta_2} + a_{12}^2\,\mathrm{e}^{2\theta_1+2\theta_2}}. \tag{131}$$

Three-soliton solution of the mKdV equation

After some computations one finds that

$$\begin{aligned} f &= \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2} + \mathrm{e}^{\theta_3} - b_{123}\mathrm{e}^{\theta_1+\theta_2+\theta_3}, \\ g &= 1 - a_{12}\,\mathrm{e}^{\theta_1+\theta_2} - a_{13}\,\mathrm{e}^{\theta_1+\theta_3} - a_{23}\,\mathrm{e}^{\theta_2+\theta_3}, \end{aligned} \tag{132}$$

with $\theta_i = k_i x - k_i^3 t + \delta_i,\ a_{ij} = \left(\frac{k_i - k_j}{k_i + k_j}\right)^2$, and $b_{123} = a_{12}a_{13}a_{23}$.

***N*-soliton solution of the mKdV equation**

A concise formula [35,52,73] for the function $\tilde{F} = g + i\, f$ leading to the N-soliton solution $u = \frac{1}{2} i \left(\ln\left(\frac{\tilde{F}}{\tilde{F}^\star}\right)\right)_x$ of the mKdV equation is given[18] by

$$\tilde{F} = \sum_{\mu=0,1} \mathrm{e}^{\left[\sum_{i<j}^{(N)} \mu_i \mu_j A_{ij} + \sum_{j=1}^{N} \mu_j (\theta_j + i\frac{\pi}{2})\right]}, \tag{133}$$

where the summations have the same meaning as in (49) and again $a_{ij} = \mathrm{e}^{A_{ij}}$. The extra $i\frac{\pi}{2}$ takes care of the complex coefficients and sign reversals.

The N-soliton solution can be written [43,100,102] as

$$u(x,t) = \frac{1}{2i}\left(\ln \frac{\det(I + iM)}{\det(I - iM)}\right)_x, \tag{134}$$

where I is the $N \times N$ identity matrix and

$$M_{\ell m} = \frac{\mathrm{e}^{\Theta_\ell + \Theta_m}}{K_\ell + K_m} \quad \text{with } \Theta_\ell = K_\ell x - 4K_\ell^3 t + \Delta_\ell. \tag{135}$$

Note that $\frac{\det(I+iM)}{\det(I-iM)}$ matches $\frac{\tilde{F}}{\tilde{F}^\star} = \frac{g + i\, f}{g - i\, f}$ with f and g in (128), (130), and (132) when $k_i = 2K_i$ and $\delta_i = 2\Delta_i - \ln(2K_i)$ with $K_i > 0$.

6 Application to Non-solitonic PDEs

6.1 The Fisher equation with convection

One of the examples discussed in [26] is the Fisher equation with convection term [76,79],

$$u_t + \alpha u u_x - u_{xx} - u(1-u) = 0, \quad \alpha \neq 0, \tag{136}$$

where α is the convection coefficient. This equation can also be viewed as a Burgers equation with quadratic source term. Motivated by a truncated Laurent series, use

$$u = -\tfrac{2}{\alpha}(\ln f)_x = -\tfrac{2}{\alpha}\left(\frac{f_x}{f}\right) \tag{137}$$

to replace (136) with a homogeneous equation of second degree

$$f(f_{3x} + f_x - f_{xt}) + f_x(f_t - f_{xx} + \tfrac{2}{\alpha} f_x) \equiv f\, \mathcal{L} f + \mathcal{N}(f,f) = 0, \tag{138}$$

where

$$\mathcal{L} f = f_{3x} + f_x - f_{xt}, \tag{139}$$

$$\mathcal{N}(f,g) = f_x(g_t - g_{xx} + \tfrac{2}{\alpha} g_x). \tag{140}$$

[18] Recall that the roles of f and g can be interchanged because $-u$ solves (108) whenever u does. $\tilde{F}$ is F with the roles of f and g reversed.

Seeking a solution of type (28), $\mathcal{L}f^{(1)} = \mathcal{L}(\sum_{i=1}^{N} \mathrm{e}^{\theta_i})$ yields $\omega_i = -(1+k_i^2)$. The second equation in (39) then becomes

$$\mathcal{L}f^{(2)} = -\sum_{i=1}^{N} k_i \left(1 + \tfrac{2}{\alpha} k_i\right) \mathrm{e}^{2\theta_i} - \sum_{1 \le i < j \le N} \left(k_i + k_j + \tfrac{4}{\alpha} k_i k_j\right) \mathrm{e}^{\theta_i + \theta_j}. \tag{141}$$

If we were to include the terms $\mathrm{e}^{2\theta_i}$ in $f^{(2)}$ the perturbation scheme would not terminate. Hence, we are forced to set all wave numbers equal to $k_i = -\frac{\alpha}{2}$ $(i = 1, 2, \ldots, N)$. Thus, $N = 1$ and only a solitary solution can be obtained. Note that both sums in (141) vanish when $k_i = -\frac{\alpha}{2}$. Hence, $f^{(2)} = 0$ and

$$f(x,t) = 1 + \mathrm{e}^{\theta} = 1 + \mathrm{e}^{-\frac{\alpha}{2}x + \frac{1}{4}(4+\alpha^2)t + \delta}. \tag{142}$$

Finally, from (137)

$$u(x,t) = \frac{\mathrm{e}^{\theta}}{1+\mathrm{e}^{\theta}} = \tfrac{1}{2}\left(1 - \tanh\left[\tfrac{1}{2}\left(\tfrac{\alpha}{2}x - \tfrac{1}{4}(4+\alpha^2)t - \delta\right)\right]\right), \tag{143}$$

since $k = -\frac{\alpha}{2}$. The graphs of the kink solution (143) in 2D and 3D are similar to those in Fig. 1.

6.2 The Fisher equation

A transformation to homogenize the Fisher equation [18,80] without convection,

$$u_t - u_{xx} - u(1-u) = 0, \tag{144}$$

is remarkably different from (137). Indeed, a truncated Laurent series suggests

$$u = -6(\ln f)_{xx} + \tfrac{6}{5}(\ln f)_t, \tag{145}$$

which yields

$$\begin{aligned} &f(f_{4x} + f_{xx} - \tfrac{6}{5} f_{xxt} + \tfrac{1}{5} f_{tt} - \tfrac{1}{5} f_t) - 4 f_x f_{3x} + 3 f_{xx}^2 - f_x^2 \\ &- \tfrac{6}{5} f_t f_{xx} + \tfrac{12}{5} f_x f_{xt} + \tfrac{1}{25} f_t^2 \equiv f\,\mathcal{L}f + \mathcal{N}(f,f) = 0. \end{aligned} \tag{146}$$

Here,

$$\mathcal{L}f = f_{4x} + f_{xx} - \tfrac{6}{5} f_{xxt} + \tfrac{1}{5} f_{tt} - \tfrac{1}{5} f_t, \tag{147}$$

$$\mathcal{N}(f,g) = -4 f_x g_{3x} + 3 f_{xx} g_{xx} - f_x g_x - \tfrac{6}{5} f_t g_{xx} + \tfrac{12}{5} f_{xt} g_x + \tfrac{1}{25} f_t g_t. \tag{148}$$

Solving (39) with $f^{(1)} = \sum_{i=1}^{N} \mathrm{e}^{\theta_i} = \sum_{i=1}^{N} \mathrm{e}^{k_i x - \omega_i t + \delta_i}$ as a starting point, one gets $\omega_i = -5k_i^2$ or $\omega_i = -(1+k_i^2)$.

Case 1: For $\omega_i = -5k_i^2$ the second equation in (39) reads

$$\mathcal{L}f^{(2)} = \sum_{i=1}^{N} k_i^2 (1 - 6k_i^2)\, \mathrm{e}^{2\theta_i} + 2 \sum_{1 \le i < j \le N} c_{ij}\, \mathrm{e}^{\theta_i + \theta_j}, \tag{149}$$

where $c_{ij} = k_i k_j \left[1 + 2k_i k_j - 4(k_i^2 + k_j^2)\right]$. If we put terms $\mathrm{e}^{2\theta_i}$ in $f^{(2)}$ the perturbation scheme does not terminate. Hence, $k_i = \pm\frac{1}{\sqrt{6}}$ $(i = 1, 2, \ldots, N)$ which also makes $c_{ij} = 0$. This leads us to conclude that a multi-soliton solution does not exist and $N = 1$. With $k = \pm\frac{1}{\sqrt{6}}$ we have $\omega = -\frac{5}{6}$. Using (145) with $f = 1 + \mathrm{e}^{\theta}$ gives

$$u(x,t) = \frac{\mathrm{e}^{2\theta}}{(1+\mathrm{e}^{\theta})^2} = \frac{1}{(1+e^{-\theta})^2} = \tfrac{1}{4}\left(1 + \tanh\tfrac{\theta}{2}\right)^2, \tag{150}$$

where each of these forms of the solution appears in the literature (see, e.g., [4]). Explicitly, for $k = -\frac{1}{\sqrt{6}}$,

$$u(x,t) = \tfrac{1}{4}\left(1 - \tanh\left[\tfrac{1}{2}\left(\tfrac{1}{\sqrt{6}}x - \tfrac{5}{6}t - \delta\right)\right]\right)^2, \tag{151}$$

which is a wave traveling to the right. The graph of this kink solution is the same as in Fig. 1 but with a steeper slope due to the square in (151). For $k = \frac{1}{\sqrt{6}}$,

$$u(x,t) = \tfrac{1}{4}\left(1 + \tanh\left[\tfrac{1}{2}\left(\tfrac{1}{\sqrt{6}}x + \tfrac{5}{6}t + \delta\right)\right]\right)^2, \tag{152}$$

which is a left-traveling wave, a bit steeper than the one shown in Fig. 1 after a vertical flip. Note that (152) does not follow from (143) in the limit for $\alpha \to 0$.

Case 2: For $\omega_i = -(1 + k_i^2)$ the second equation in (39) becomes

$$\mathcal{L}f^{(2)} = -\tfrac{1}{25}\left(\sum_{i=1}^{N}(1+k_i^2)(1+6k_i^2)\,\mathrm{e}^{2\theta_i} + 2\sum_{1\le i<j\le N} c_{ij}\,\mathrm{e}^{\theta_i+\theta_j}\right), \tag{153}$$

where $c_{ij} = \left[1 + 35k_ik_j + 46k_i^2k_j^2 - 2(k_i^2+k_j^2)(7+10k_ik_j)\right]$. So, for real wave numbers k_i the terms $\mathrm{e}^{2\theta_i}$ do not vanish. No solitary wave solutions or solitons can be obtained in this case.

6.3 The FitzHugh-Nagumo equation with convection

The FHN equation with convection term [58],

$$u_t + \alpha u u_x - u_{xx} + u(1-u)(a-u) = 0, \tag{154}$$

where α denotes the convection coefficient and a is an arbitrary constant, is also called the Burgers-Huxley equation [87].

A truncated Laurent series suggests two possible transformations, namely,

$$u = \sqrt{m}\,(\ln f)_x = \sqrt{m}\left(\frac{f_x}{f}\right), \quad m > 0. \tag{155}$$

and

$$u = -\frac{2}{\sqrt{m}}\,(\ln f)_x = -\frac{2}{\sqrt{m}}\left(\frac{f_x}{f}\right), \quad m > 0, \tag{156}$$

where we have replaced α by $\frac{m-2}{\sqrt{m}}$ in (154) to simplify their forms and the computations below. Using (155), (154) transforms into

$$\begin{aligned}&f(f_{3x}-af_x-f_{xt})+f_x\left(f_t-(m+1)f_{xx}+\sqrt{m}(1+a)f_x\right)\\&\equiv f\,\mathcal{L}f+\mathcal{N}(f,f)=0,\end{aligned}\tag{157}$$

where

$$\mathcal{L}f=f_{3x}-af_x-f_{xt},\tag{158}$$

$$\mathcal{N}(f,g)=f_x\left(g_t-(m+1)g_{xx}+\sqrt{m}(1+a)g_x\right).\tag{159}$$

To compute a single solitary wave solution we take $f=1+\mathrm{e}^{\theta}$. Then, $\mathcal{L}\mathrm{e}^{\theta}=0$ yields $\omega=a-k^2$. Next, $\mathcal{N}(\mathrm{e}^{\theta},\mathrm{e}^{\theta})=0$ determines $k=\frac{1}{\sqrt{m}}$ or $k=\frac{a}{\sqrt{m}}$. Thus, $\omega=\frac{am-1}{m}$ or $\omega=\frac{a(m-a)}{m}$, respectively. Returning to u, we obtain the solitary wave solutions

$$u(x,t)=\tfrac{1}{2}\left(1+\tanh\left[\tfrac{1}{2}\left(\frac{1}{\sqrt{m}}x-\frac{(am-1)}{m}t+\delta\right)\right]\right)\tag{160}$$

and

$$u(x,t)=\tfrac{1}{2}a\left(1+\tanh\left[\tfrac{1}{2}\left(\frac{a}{\sqrt{m}}x-\frac{a(m-a)}{m}t+\delta\right)\right]\right).\tag{161}$$

Although it is impossible to find a two-soliton solution, a so-called *bi-soliton* solution can be computed which describes coalescent wave fronts. Indeed, taking $f=1+\mathrm{e}^{\theta_1}+\mathrm{e}^{\theta_2}$, with $\omega_i=a-k_i^2$ $(i=1,2)$, after some computations one gets

$$u(x,t)=\frac{\mathrm{e}^{\theta_1}+a\,\mathrm{e}^{\theta_2}}{1+\mathrm{e}^{\theta_1}+\mathrm{e}^{\theta_2}},\tag{162}$$

where

$$\theta_1=\tfrac{1}{\sqrt{m}}x-\left(\tfrac{am-1}{m}\right)t+\delta_1,\quad \theta_2=\tfrac{a}{\sqrt{m}}x-\left(\tfrac{a(m-a)}{m}\right)t+\delta_2.\tag{163}$$

Since $\alpha=\frac{m-2}{\sqrt{m}}$, possible[19] values for m are

$$m=\tfrac{1}{2}\left(4+\alpha^2\pm\alpha\sqrt{8+\alpha^2}\right),\quad m>0.\tag{164}$$

This solution can be found in [30] and [58] where it was obtained with a different method.

Skipping the details, with (156) one obtains the following solutions

$$u(x,t)=\tfrac{1}{2}\left(1-\tanh\left[\tfrac{1}{2}\left(\frac{\sqrt{m}}{2}x+\frac{(4a-m)}{4}t-\delta\right)\right]\right)\tag{165}$$

and

$$u(x,t)=\tfrac{1}{2}a\left(1-\tanh\left[\tfrac{1}{2}\left(\frac{a\sqrt{m}}{2}x+\frac{a(4-am)}{4}t-\delta\right)\right]\right),\tag{166}$$

[19] For any positive value of m, the pair (α,m) must still satisfy $\alpha=\frac{m-2}{\sqrt{m}}$.

with m given in (164). The bi-soliton solution corresponding to (156) is (162) with

$$\theta_1 = -\tfrac{\sqrt{m}}{2}x - \left(\tfrac{4a-m}{4}\right)t + \delta_1, \quad \theta_2 = -\tfrac{a\sqrt{m}}{2}x - \left(\tfrac{a(4-am)}{4}\right)t + \delta_2. \tag{167}$$

Solution (162) with either (163) or (167) describes the coalescence of two wave fronts pictured in Fig. 14.

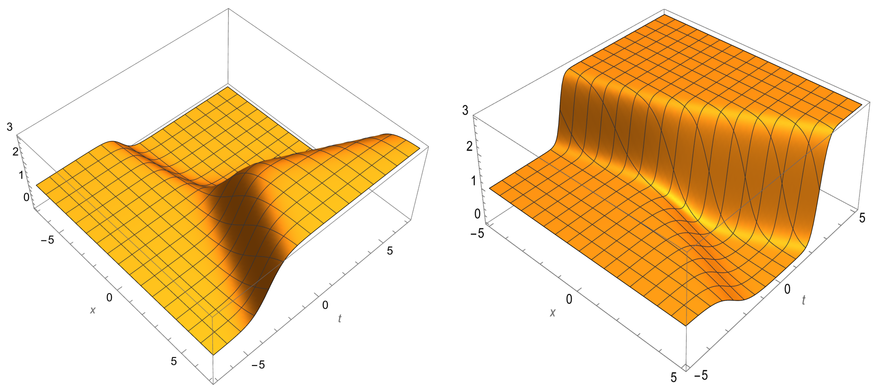


Fig. 14. 3D graphs of solution (162) with (163) (left) and (167) (right) both for $a = 3, \alpha = 1$ (i.e., $m = 4$), and $\delta_1 = \delta_2 = 0$.

Finally, for $m = 2$ (i.e., $\alpha = 0$), one gets a solitary wave solution of the FHN equation without convection [5].

6.4 A Burgers equation with a cubic source term

Consider the Burgers equation with a polynomial source term of third degree,

$$u_t + \alpha u u_x - u_{xx} = 3u(2-u)(u+1), \tag{168}$$

which is of the kind treated in [99, Eq. (26)]. Eq. (168) can also be viewed as an equation of FitzHugh-Naguma-type with convection term[20]. Such equations are known to have coalescent wave fronts [27,58]. Based on a truncated Laurent series, there are potentially two homogenizing transformations:

$$u = \sqrt{m}\,(\ln f)_x = \sqrt{m}\left(\frac{f_x}{f}\right), \quad m > 0. \tag{169}$$

and

$$u = -\frac{2}{3\sqrt{m}}\,(\ln f)_x = -\frac{2}{3\sqrt{m}}\left(\frac{f_x}{f}\right), \quad m > 0, \tag{170}$$

[20] Except that $u - 1$ is now replaced by $u + 1$.

where we used $\alpha = \frac{3m-2}{\sqrt{m}}$ in (168) to simplify their forms. Starting with (169), substitution into (168) yields

$$f\left(6f_x - f_{xt} + f_{3x}\right) + f_x\left(f_t + 3\sqrt{m}f_x - (1+3m)\,f_{xx}\right) \tag{171}$$
$$\equiv f\,\mathcal{L}f + \mathcal{N}(f,f) = 0. \tag{172}$$

Here, $\mathcal{L}$ and $\mathcal{N}$ are defined by

$$\mathcal{L}f = 6f_x - f_{xt} + f_{3x}, \tag{173}$$
$$\mathcal{N}(f,g) = f_x\left(g_t + 3\sqrt{m}g_x - (1+3m)\,g_{xx}\right). \tag{174}$$

For the single solitary wave solution, $\mathcal{L}\mathrm{e}^{\theta} = 0$ yields $\omega = -(k^2+6)$. Next, $\mathcal{N}(\mathrm{e}^{\theta}, \mathrm{e}^{\theta}) = 0$ determines $k = -\frac{1}{\sqrt{m}}$ or $k = \frac{2}{\sqrt{m}}$. Thus, $\omega = -\frac{6m+1}{m}$ or $\omega = -\frac{2(3m+2)}{m}$, respectively. So, with $f = 1+\mathrm{e}^{\theta}$ we obtain the solitary wave solutions

$$u(x,t) = -\tfrac{1}{2}\left(1 - \tanh\left[\tfrac{1}{2\sqrt{m}}x - \frac{(6m+1)}{2m}t - \frac{\delta}{2}\right]\right) \tag{175}$$

and

$$u(x,t) = 1 + \tanh\left[\frac{1}{\sqrt{m}}x + \frac{(3m+2)}{m}t + \frac{\delta}{2}\right], \tag{176}$$

where, with regard to $\alpha = \frac{3m-2}{\sqrt{m}}$, possible values[21] for m are

$$m = \tfrac{1}{18}\left(12 + \alpha^2 \pm \alpha\sqrt{24+\alpha^2}\right), \quad m > 0. \tag{177}$$

As with the FHN equation with convection term, no two-soliton solution exists but a bi-soliton solution can be found which describes coalescent wave fronts. Indeed, taking $f = 1 + \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2}$ where $\omega_i = -(k_i^2+6)$ $(i=1,2)$ one gets

$$\begin{aligned} u(x,t) &= \frac{\sqrt{m}(k_1\mathrm{e}^{\theta_1} + k_2\mathrm{e}^{\theta_2})}{1+\mathrm{e}^{\theta_1}+\mathrm{e}^{\theta_2}} \\ &= \frac{2\,\mathrm{e}^{\frac{2}{\sqrt{m}}x + \frac{2(3m+2)}{m}t + \delta_1} - \mathrm{e}^{-\frac{1}{\sqrt{m}}x + \frac{(6m+1)}{m}t + \delta_2}}{1 + \mathrm{e}^{\frac{2}{\sqrt{m}}x + \frac{2(3m+2)}{m}t + \delta_1} + \mathrm{e}^{-\frac{1}{\sqrt{m}}x + \frac{(6m+1)}{m}t + \delta_2}}, \end{aligned} \tag{178}$$

because $k_1 = \frac{2}{\sqrt{m}}$ and $k_2 = -\frac{1}{\sqrt{m}}$ with m in (177). For $m = 1$, a solution of (168) with $\alpha = 1$ then reads

$$u(x,t) = \frac{2\mathrm{e}^{2x+10t+\delta_1} - \mathrm{e}^{-x+7t+\delta_2}}{1+\mathrm{e}^{2x+10t+\delta_1}+\mathrm{e}^{-x+7t+\delta_2}}. \tag{179}$$

The solution procedure using (170) is similar and leads to

$$u(x,t) = -\tfrac{1}{2}\left(1 + \tanh\left[\tfrac{3\sqrt{m}}{4}x + \frac{3(3m+8)}{8}t + \frac{\delta}{2}\right]\right) \tag{180}$$

[21] For any positive value of m, the pair (α, m) must still satisfy $\alpha = \frac{3m-2}{\sqrt{m}}$.

and

$$u(x,t) = 1 - \tanh\left[\frac{3\sqrt{m}}{2}x - \frac{3(3m+2)}{2}t - \frac{\delta}{2}\right], \tag{181}$$

with m given in (177). The bi-soliton solution corresponding to (170) reads

$$\begin{aligned} u(x,t) &= -\frac{2}{3\sqrt{m}}\frac{(k_1 \mathrm{e}^{\theta_1} + k_2 \mathrm{e}^{\theta_2})}{(1 + \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2})} \\ &= -\frac{\mathrm{e}^{\frac{3\sqrt{m}}{2}x + \frac{3(3m+8)}{4}t + \delta_1} - 2\,\mathrm{e}^{-3\sqrt{m}x + 3(3m+2)t + \delta_2}}{1 + \mathrm{e}^{\frac{3\sqrt{m}}{2}x + \frac{3(3m+8)}{4}t + \delta_1} + \mathrm{e}^{-3\sqrt{m}x + 3(3m+2)t + \delta_2}}, \end{aligned} \tag{182}$$

because $k_1 = \frac{3\sqrt{m}}{2}$ and $k_2 = -3\sqrt{m}$ with m in (177). For $m = 1$, a bi-soliton solution of (168) with $\alpha = 1$ then becomes

$$u(x,t) = -\frac{\mathrm{e}^{\frac{3}{2}x + \frac{33}{4}t + \delta_1} - 2\mathrm{e}^{-3x + 15t + \delta_2}}{1 + \mathrm{e}^{\frac{3}{2}x + \frac{33}{4}t + \delta_1} + \mathrm{e}^{-3x + 15t + \delta_2}}. \tag{183}$$

Solutions (179) and (183), describing two coalescent wave fronts, are shown in Fig. 15.

Returning to α via (177) also allows one to consider the case $\alpha = 0$ (i.e., $m = \frac{2}{3}$), leading to solutions of (168) with a cubic source but without convection.

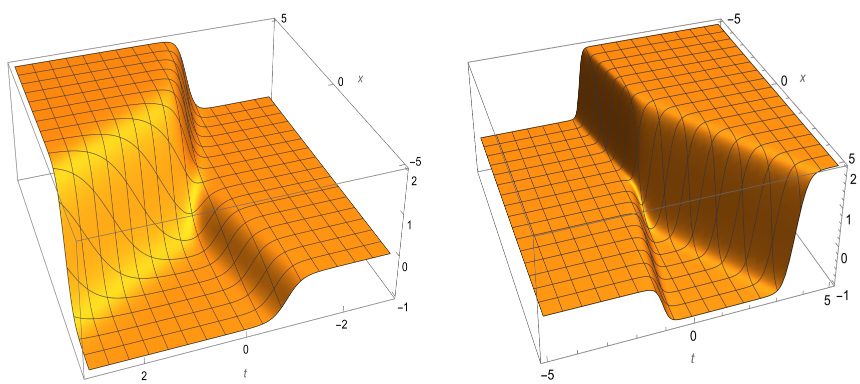


Fig. 15. 3D graphs of (179) (left) and (183) (right) for $\delta_1 = -\delta_2 = 1$.

6.5 A wave equation with cubic source term

Consider the wave equation,

$$\tfrac{1}{8}u_{tt} + u_t + uu_x - \tfrac{1}{8}u_{xx} = u(u-1)(u+2), \tag{184}$$

which is a special case of an equation investigated in [99, Eq. (2)]. The Laurent series solution suggests the transformation

$$u = \tfrac{1}{2}\kappa\left[(\ln f)_t - \kappa(\ln f)_x\right] = \tfrac{1}{2}\kappa\left(\frac{f_t - \kappa f_x}{f}\right), \tag{185}$$

with $\kappa = \pm 1$. We first consider the case where $\kappa = 1$. Using

$$u = \tfrac{1}{2}\left(\frac{f_t - f_x}{f}\right), \tag{186}$$

allows one to replace (184) by

$$\begin{aligned}&f\left(16f_t + 8f_{tt} + f_{3t} - 16f_x - 8f_{xt} - f_{xtt} - f_{xxt} + f_{3x}\right)\\ &-\left(3f_t - f_x\right)\left(4f_t + f_{tt} - 4f_x - 2f_{xt} + f_{xx}\right)\\ &\equiv f\,\mathcal{L}f + \mathcal{N}(f,f) = 0,\end{aligned} \tag{187}$$

with

$$\mathcal{L}f = 16f_t + 8f_{tt} + f_{3t} - 16f_x - 8f_{xt} - f_{xtt} - f_{xxt} + f_{3x}, \tag{188}$$

$$\mathcal{N}(f,g) = -\left(3f_t - f_x\right)\left(4g_t + g_{tt} - 4g_x - 2g_{xt} + g_{xx}\right). \tag{189}$$

Then, $\mathcal{L}\mathrm{e}^{\theta} = 0$ yields $\omega = -k$, $\omega = 4 - k$, or $\omega = 4 + k$. With $f = 1 + \mathrm{e}^{\theta}$, one readily obtains

$$u(x,t) = -\tfrac{1}{2}(\omega + k)\left(\frac{\mathrm{e}^{\theta}}{1 + \mathrm{e}^{\theta}}\right) = -\tfrac{1}{4}(\omega + k)\left(1 + \tanh\tfrac{\theta}{2}\right). \tag{190}$$

Obviously, the choice $\omega = -k$ must be rejected. For $\omega = 4 - k$, one finds that $\mathcal{N}(\mathrm{e}^{\theta}, \mathrm{e}^{\theta}) \equiv 0$ and

$$u(x,t) = -\left(1 + \tanh\left[\tfrac{1}{2}\left(kx - (4-k)t + \delta\right)\right]\right) \tag{191}$$

with arbitrary k and δ. For $\omega = 4 + k$, $\mathcal{N}(\mathrm{e}^{\theta}, \mathrm{e}^{\theta}) = 0$ determines $k = -2$ or $k = -3$ resulting in $\omega = 2$ or $\omega = 1$, respectively. The case $k = -2$ (i.e., $\omega = 2$) is rejected for it leads to $u(x,t) \equiv 0$. For $k = -3$ (i.e., $\omega = 1$) one gets

$$u(x,t) = \tfrac{1}{2}\Big(1 - \tanh\left[\tfrac{1}{2}\left(3x + t - \delta\right)\right]\Big), \tag{192}$$

which is different from (191) when $k = -3$.

Attempting to find a solution of type (28), $\mathcal{L}f^{(1)} = \mathcal{L}(\sum_{i=1}^{N} \mathrm{e}^{\theta_i})$ determines $\omega_i = -k_i, 4 - k_i$, or $4 + k_i$. As with the solitary wave solution, to avoid trivial solutions we continue with $\omega_i = 4 - k_i$ and $\omega_i = 4 + k_i$.

Here again, it is impossible to find a two-soliton solution but a bi-soliton solution can be computed. Indeed, taking $f = 1 + \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2} + a_{12}\mathrm{e}^{\theta_1 + \theta_2}$, leads to $a_{12} = 0$. Then, for $\omega_i = 4 - k_i$ $(i = 1, 2)$, after some computation one gets

$$u(x,t) = -2\left(\frac{\mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2}}{1 + \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2}}\right), \tag{193}$$

where $\theta_1 = k_1 x - (4 - k_1)t + \delta_1$ and $\theta_2 = k_2 x - (4 - k_2)t + \delta_2$. Solution (193) agrees with the result in [99, Eq. (37)]. As shown in Fig. 16, (193) describes two coalescent wave fronts. For $\omega_i = 4 + k_i$, after some computations one gets $k_1 = -2, k_2 = -3$, and $a_{12} = 1$, resulting in (192) with δ replaced by δ_2.

The computations for $\kappa = -1$ in (185) are similar but only lead to

$$u(x,t) = \tfrac{1}{2}\Big(1 + \tanh\left[\tfrac{1}{2}\left(x - 3t + \delta\right)\right]\Big) \tag{194}$$

which does not follow from (191) when $k = 1$.

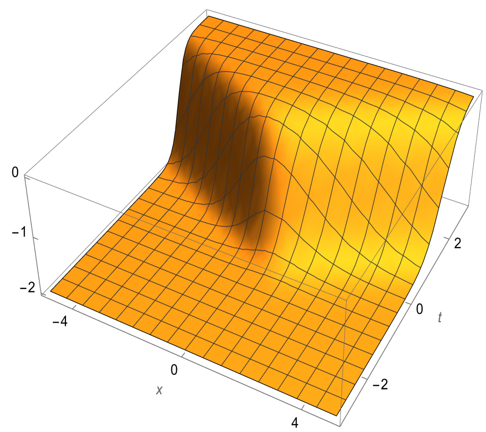


Fig. 16. 3D graph of (193) with $k_1 = 1, k_2 = -2$, and $\delta_1 = \delta_2 = 0$.

6.6 A combined KdV-Burgers equation

A combined KdV-Burgers equation [98],

$$u_t + 6uu_x + u_{3x} - 5\beta u_{xx} = 0, \tag{195}$$

where $\beta > 0$, is used in models where both dispersive and dissipative effects are relevant. A Laurent series of the solution of (195) suggests the transformation

$$u = 2(\ln f)_{xx} - 2\beta(\ln f)_x, \tag{196}$$

which we substitute into the integrated KdV-Burgers equation,

$$\partial_t \left(\int^x u \, dx \right) + 3u^2 + u_{xx} - 5\beta u_x = 0 \tag{197}$$

to get

$$
\begin{aligned}
&f(f_{xt} - \beta f_t + 5\beta^2 f_{xx} - 6\beta f_{3x} + f_{4x}) \\
&-f_x f_t + \beta^2 f_x^2 + 6\beta f_x f_{xx} + 3 f_{xx}^2 - 4 f_x f_{3x} = 0.
\end{aligned} \tag{198}
$$

This homogeneous equation is of the form $f\,\mathcal{L}f + \mathcal{N}(f,f) = 0$. Therefore, we can proceed as in the KdV case and solve (39) step-by-step with

$$\mathcal{L}f = f_{xt} - \beta f_t + 5\beta^2 f_{xx} - 6\beta f_{3x} + f_{4x}, \tag{199}$$

$$\mathcal{N}(f,g) = -f_x g_t + \beta^2 f_x g_x + 6\beta f_x g_{xx} + 3 f_{xx} g_{xx} - 4 f_x g_{3x}. \tag{200}$$

We summarize the results. First, $\mathcal{L}\mathrm{e}^{\theta} = \mathcal{L}\mathrm{e}^{kx-\omega t+\delta} = 0$ yields $(\beta - k)(\omega - k^3 + 5\beta k^2) = 0$. Thus, two cases have to be considered.

Case 1: When $\omega = k^2(k - 5\beta)$ and $k \neq \beta$, $\mathcal{N}(\mathrm{e}^{\theta}, \mathrm{e}^{\theta}) = 0$ determines $k = -\beta$. So, $\omega = -6\beta^3$. Inserting $f = 1 + \mathrm{e}^{\theta}$ into (196) yields

$$u(x,t) = 2\beta^2 \left(\frac{\mathrm{e}^{\theta}(2 + \mathrm{e}^{\theta})}{(1 + \mathrm{e}^{\theta})^2} \right) = \tfrac{1}{2}\beta^2 \left(3 - \tanh \tfrac{\theta}{2}\right)\left(1 + \tanh \tfrac{\theta}{2}\right), \tag{201}$$

with $\theta = -\beta x + 6\beta^3 t + \delta$.

Case 2: When $k = \beta$, $\mathcal{N}(\mathrm{e}^{\theta}, \mathrm{e}^{\theta}) = 0$ determines $\omega = -6\beta^3$, yielding

$$u(x,t) = -2\beta^2 \left(\frac{\mathrm{e}^{2\theta}}{(1 + \mathrm{e}^{\theta})^2} \right) = -\tfrac{1}{2}\beta^2 \left(1 + \tanh \tfrac{\theta}{2}\right)^2, \tag{202}$$

with $\theta = \beta x + 6\beta^3 t + \delta$. Solutions (201) and (202) are shown in Fig. 17 for $\beta = 2$ and $\delta = 0$.

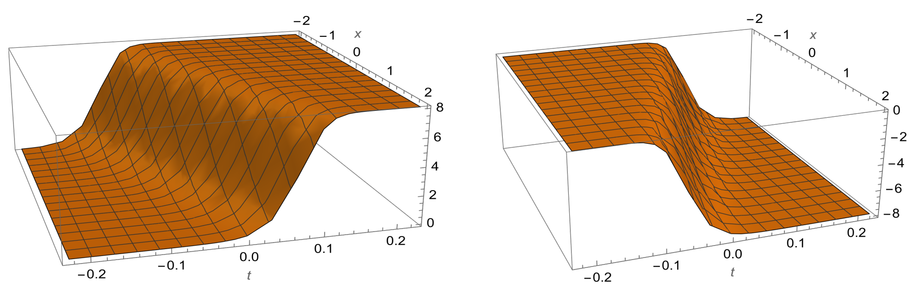


Fig. 17. 3D graphs of (201) (left) and (202) (right) for $\beta = 2$ and $\delta = 0$.

An attempt to find a multi-soliton or bi-soliton solutions based on (28) failed. Assuming $k_i \neq \beta$ (discussed in Case 2) and working with (28), $\mathcal{L}f^{(1)} = \mathcal{L}(\sum_{i=1}^{N} \mathrm{e}^{\theta_i})$

determines $\omega_i = k_i^2(k_i - 5\beta)$. The second equation in (39) then becomes

$$\mathcal{L} f^{(2)} = -\beta \sum_{i=1}^{N} k_i^2 (k_i + \beta)\, \mathrm{e}^{2\theta_i} - \sum_{1 \le i < j \le N} c_{ij}\, \mathrm{e}^{\theta_i + \theta_j}, \tag{203}$$

where $c_{ij} = k_i k_j \left[2\beta^2 + \beta(k_i + k_j) + 6 k_i k_j - 3(k_i^2 + k_j^2) \right]$. Putting terms $\mathrm{e}^{2\theta_i}$ in $f^{(2)}$ prevents the perturbation scheme from terminating. Hence, $k_i = -\beta$ $(i = 1, 2, \ldots, N)$ which also makes $c_{ij} = 0$. But if the wave numbers have to be equal then $N = 1$ and that brings us back to Case 1 and (201).

6.7 An equation due to Calogero

For the equation

$$u_t - 3(3 u u_x^2 + u^4 u_x + u^2 u_{xx}) - u_{3x} = 0 \tag{204}$$

due to Calogero [8], the Laurent series (7) has $\alpha = -\frac{1}{2}$. Therefore, to apply the simplified Hirota method we first change the dependent variable. Setting $u = \sqrt{v}$ with $v > 0$ gives

$$4v^2 v_t - 3v_x^3 - 12 v^2 v_x^2 - 12 v^4 v_x + 6 v v_x v_{xx} - 12 v^3 v_{xx} - 4 v^2 v_{3x} = 0, \tag{205}$$

which looks more complicated than (204) but has a truncated Laurent series with $\alpha = -1$. Then, with the transformation

$$v = \tfrac{1}{2}(\ln f)_x = \tfrac{1}{2}\left(\frac{f_x}{f}\right) \tag{206}$$

(205) can be replaced by an equation of fourth degree,

$$\begin{aligned} & f(4 f_x^2 f_{xt} - 3 f_{xx}^3 + 6 f_x f_{xx} f_{3x} - 4 f_x^2 f_{4x}) - f_x^2 (4 f_x f_t + 3 f_{xx}^2 - 4 f_x f_{3x}) \\ & \equiv f\, \mathcal{N}_1(f, f, f) + \mathcal{N}_2(f, f, f, f) = 0, \end{aligned} \tag{207}$$

with

$$\mathcal{N}_1(f, g, h) = 4 f_x g_x h_{xt} - 3 f_{xx} g_{xx} h_{xx} + 6 f_x g_{xx} h_{3x} - 4 f_x g_x h_{4x}, \tag{208}$$

$$\mathcal{N}_2(f, g, h, j) = -f_x g_x (4 h_x j_t + 3 h_{xx} j_{xx} - 4 h_x j_{3x}). \tag{209}$$

If one seeks a solution to (207) of type (28), then $\mathcal{N}_1(\mathrm{e}^\theta, \mathrm{e}^\theta, \mathrm{e}^\theta)$ with $\theta = kx - \omega t + \delta$ yields $\omega = -\frac{1}{4} k^3$. Fortuitously, if the dispersion law holds then $\mathcal{N}_2(\mathrm{e}^\theta, \mathrm{e}^\theta, \mathrm{e}^\theta, \mathrm{e}^\theta) = 0$ and, therefore, $f = 1 + \mathrm{e}^\theta$ solves (207). Using (206) and $u = \sqrt{v}$, after some algebra one gets

$$u = \tfrac{1}{2}\sqrt{k}\, \sqrt{1 + \tanh\left[\tfrac{1}{8}\left(4kx + k^3 t + 4\delta\right)\right]}, \tag{210}$$

where $k > 0$. This solution was computed in [30] with a different method. It is graphed in Fig. 18 for $k = 4$ and $\delta = 0$.

If one tries to find a multi-soliton solution with $f^{(1)} = \sum_{i=1}^{N} \mathrm{e}^{\theta_i}$ with $\theta_i = k_i x + \frac{1}{4} k_i^3 t + \delta_i$, then $\mathcal{N}_1(f^{(1)}, f^{(1)}, f^{(1)})$ only vanishes if the wave numbers are equal.

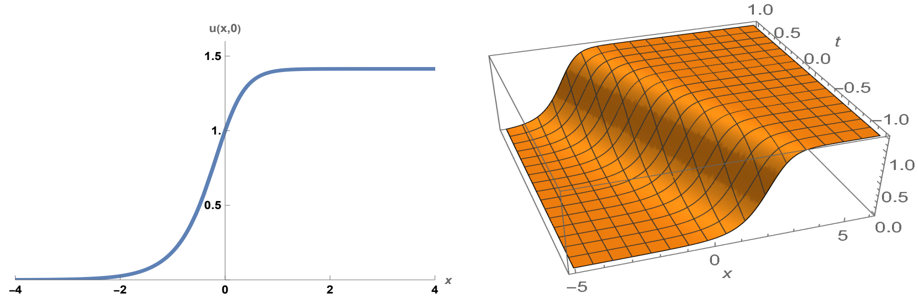


Fig. 18. 2D and 3D graphs of (210) for $k = 4$ and $\delta = 0$.

7 An Equation with Two but not Three Solitons

Equations that have two-soliton but not three-soliton solutions have been discovered. The best known example is the sine-Gordon equation in two space variables which already appears in early work by Hirota [44] and was later studied in greater generality in [59]. Another example is a $(3+1)$-dimensional eight-order equation due to Kac-Wakimoto [90,105].

With respect to equations in $(1+1)$ dimensions, Hietarinta (see [36,37] and references therein) did an extensive search of bilinear forms for which the necessary condition to have a three-soliton solution is violated. Although the appropriate bilinear forms are given explicitly, the equations in the original field variable u are not always available in his papers. Reversing the process, i.e., finding the nonlinear PDE (or a system thereof) that leads to a (known) bilinear form is not straightforward. Consult, e.g., [36,37,40,70] for strategies and explicit examples.

Taking a different example, we study the soliton solutions of a polynomial equation in $(1+1)$ dimensions,

$$u_t + \tfrac{15}{784}u^3u_x + \tfrac{15}{28}uu_xu_{xx} + \tfrac{15}{56}u^2u_{3x} + \tfrac{5}{2}u_{xx}u_{3x} + u_xu_{4x} + uu_{5x} + u_{7x} = 0, \quad (211)$$

which appears in [51, Eq. (19) for $K = 56$]. The authors claim that this equation has at most a two-soliton solution. However, they do not give the constraint on the wave numbers k_i that prevents the existence of, e.g., a three-soliton solution. We therefore investigate (211) in more detail.

Based on a truncated Laurent series, we substitute

$$u = 56\,(\ln f)_{xx} = 56\left(\frac{ff_{xx} - {f_x}^2}{f^2}\right) \quad (212)$$

into the integrated form of (211),

$$\partial_t\left(\int^x u\,dx\right) + \tfrac{15}{3136}u^4 + \tfrac{15}{56}u^2u_{xx} + \tfrac{5}{4}u_{xx}^2 + uu_{4x} + u_{6x} = 0, \quad (213)$$

yielding

$$f\left(f_{xt}+f_{8x}\right)-f_x f_t+35{f_{4x}}^2-56 f_{3x} f_{5x}+28 f_{xx} f_{6x}-8 f_x f_{7x}=0, \tag{214}$$

which is of the form $f\mathcal{L}f+\mathcal{N}(f,f)=0$ with

$$\mathcal{L}f=f_{xt}+f_{8x}, \tag{215}$$

$$\mathcal{N}(f,g)=-f_x g_t+35 f_{4x} g_{4x}-56 f_{3x} g_{5x}+28 f_{xx} g_{6x}-8 f_x g_{7x}. \tag{216}$$

As usual, $\mathcal{L}\mathrm{e}^{\theta}=0$ yields the dispersion relation $\omega=k^7$. So, with $f=1+\mathrm{e}^{\theta}$ we obtain the solitary wave solution

$$u(x,t)=14k^2\operatorname{sech}^2\left[\tfrac{1}{2}\left(kx-k^7t+\delta\right)\right]. \tag{217}$$

Seeking a solution of the form (28), as before $\mathcal{L}f^{(1)}=\mathcal{L}(\sum_{i=1}^{N}\mathrm{e}^{\theta_i})=0$ with $\theta_i=k_ix-\omega_it+\delta_i$ yields $\omega_i={k_i}^7$.

For the two-soliton solution, taking $f=1+\mathrm{e}^{\theta_1}+\mathrm{e}^{\theta_2}+a_{12}\mathrm{e}^{\theta_1+\theta_2}$, after some computations one gets

$$a_{12}=\left(\frac{(k_1-k_2)\left({k_1}^2-k_1k_2+{k_2}^2\right)}{(k_1+k_2)\left({k_1}^2+k_1k_2+{k_2}^2\right)}\right)^2 \tag{218}$$

and, then from (212),

$$\begin{aligned}u(x,t)=56\Bigg(&\frac{{k_1}^2\mathrm{e}^{\theta_1}+{k_2}^2\mathrm{e}^{\theta_2}+a_{12}\mathrm{e}^{\theta_1+\theta_2}(k_1+k_2)^2}{1+\mathrm{e}^{\theta_1}+\mathrm{e}^{\theta_2}+a_{12}\mathrm{e}^{\theta_1+\theta_2}}\\&-\frac{\left(k_1\mathrm{e}^{\theta_1}+k_2\mathrm{e}^{\theta_2}+a_{12}\mathrm{e}^{\theta_1+\theta_2}(k_1+k_2)\right)^2}{(1+\mathrm{e}^{\theta_1}+\mathrm{e}^{\theta_2}+a_{12}\mathrm{e}^{\theta_1+\theta_2})^2}\Bigg)\end{aligned} \tag{219}$$

with $\theta_i=k_ix-{k_i}^7t+\delta_i$.

The collision of two solitons for equation (211) is shown in Figs. 19 and 20 for $k_1=1$, $k_2=2$, and $\delta_1=\delta_2=0$.

The existence of a two-soliton solution comes as no surprise because (214) can be written in bilinear form as

$$\left(D_xD_t+D_x^8\right)(f\cdot f)=0, \tag{220}$$

which satisfies the conditions[22] for the existence of a two-soliton solution (see, for example, [34, Eq. (22)] and [46, Eq. (5.47)]).

In an attempt to find a three-soliton solution, one would take

$$f=1+\mathrm{e}^{\theta_1}+\mathrm{e}^{\theta_2}+\mathrm{e}^{\theta_3}+a_{12}\mathrm{e}^{\theta_1+\theta_2}+a_{13}\mathrm{e}^{\theta_1+\theta_3}+a_{23}\mathrm{e}^{\theta_2+\theta_3}+b_{123}\mathrm{e}^{\theta_1+\theta_2+\theta_3} \tag{221}$$

[22] For a derivation of such conditions see, e.g., [74].

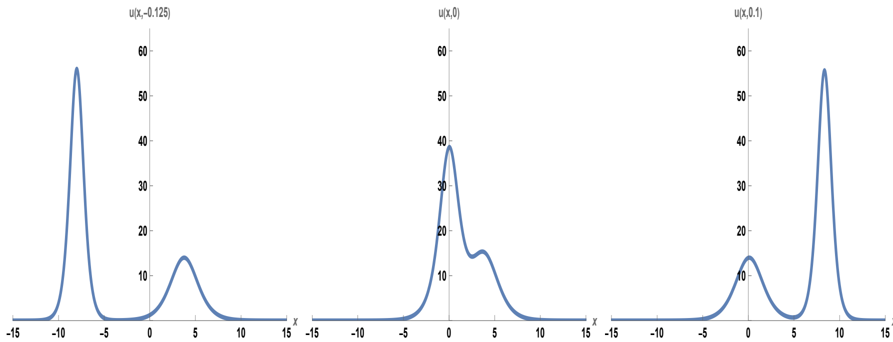


Fig. 19. Graph of the two-soliton solution (219) of (211) at three different moments in time.

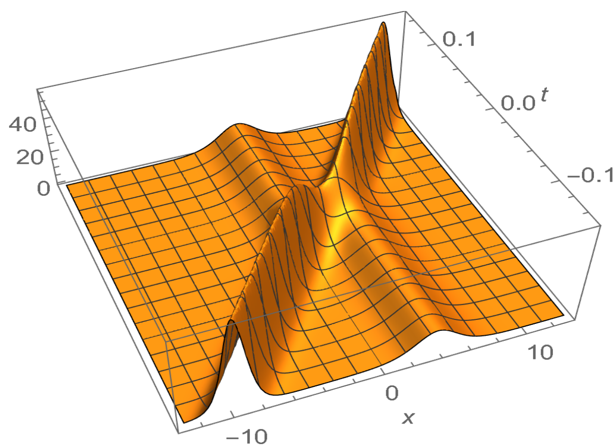


Fig. 20. Bird's eye view of the collision of two solitons for equation (211). Notice the phase shift after the collision.

with

$$a_{ij} = \left(\frac{(k_i - k_j)\left({k_i}^2 - k_i k_j + {k_j}^2\right)}{(k_i + k_j)\left({k_i}^2 + k_i k_j + {k_j}^2\right)} \right)^2 \tag{222}$$

and $b_{123} = a_{12}a_{13}a_{23}$ and substitute it into (214). A lengthy computation shows that the equation is only satisfied if the wave numbers are equal or zero. Actually, this agrees with Hietarinta's earlier studies of equations that have a bilinear form.

Indeed, for an N-soliton solution to exist, the condition [46,74]

$$S[P,n] = \sum_{\sigma=\pm 1} P\left(\sum_{i=1}^{n} \sigma_i k_i, -\sum_{i=1}^{n} \sigma_i \omega_i\right) \prod_{i<j}^{(n)} P(\sigma_i k_i - \sigma_j k_j, -\sigma_i \omega_i + \sigma_j \omega_j) \sigma_i \sigma_j = 0 \tag{223}$$

must hold for $n = 2, 3, \ldots, N$. In (223), P is the polynomial corresponding to the bilinear operator B, $\sum_{\sigma=\pm 1}$ indicates the summation over all possible combinations of $\sigma_1 = \pm 1,\ \sigma_2 = \pm 1,\ \ldots,\ \sigma_n = \pm 1$ and $\prod_{i<j}^{(n)}$ means the product of all possible combinations of the n elements with $i < j$, and all k_i, ω_i subject to the dispersion law $\omega_i(k_i)$. Note that (223) is a condition for P and not for the k_i. Also, all ω_i are replaced in terms of the k_i because (223) is evaluated on the dispersion law.

For (220), $P(D_x, D_t) = B = D_x D_t + D_x^8$ and the three-soliton condition $S[P,3] = 0$ (see, [34, Eq. (28)]) gives[23]

$$\begin{aligned} &(k_1 k_2 k_3)^4 \left[(k_1^2 - k_2^2)(k_1^2 - k_3^2)(k_2^2 - k_3^2)\right]^2 \\ &(k_1^2 k_2^2 + k_1^2 k_3^2 + k_2^2 k_3^2)(k_1^4 + k_2^4 + k_3^4 + k_1^2 k_2^2 + k_1^2 k_3^2 + k_2^2 k_3^2) = 0. \end{aligned} \tag{224}$$

Thus, the wave numbers must be either equal, each other's opposites, or zero. In conclusion, the non-existence of a three-soliton solution agrees with the claim in [51].

8 Soliton Solutions in Multiple Space Dimensions

8.1 The Kadomtsev-Petviashvili equation

Arguably, the KP equation [2,16,55],

$$(u_t + 6uu_x + u_{3x})_x + 3\sigma u_{yy} = 0 \tag{225}$$

for $u(x, y, t)$ and $\sigma = \pm 1$, is one of the most studied soliton equations involving more than one space variable. We only consider the so-called KPII equation [7,60] where $\sigma = 1$. A Laurent series of its solution suggests the transformation $u = 2\,(\ln f)_{xx}$. We therefore integrate (225) twice,

$$\partial_t \left(\int^x u\, dx\right) + 3u^2 + u_{xx} + 3\partial_y^2 \left(\int^x \left(\int^x u\, dx\right) dx\right) = 0, \tag{226}$$

before replacing u in terms of f. The resulting equation,

$$f(f_{xt} + f_{4x} + 3f_{yy}) - f_x f_t + 3f_{xx}^2 - 4f_x f_{3x} - 3f_y^2 = 0, \tag{227}$$

[23] Based on symmetry considerations, simplified expressions of (223) are given in [71, Eq. (2.9)] and [83, Eqs. (4.3) and (4.4)]. A computer implementation can be found in [126, pp. 27-29 and p. 82].

can be written in bilinear form

$$\left(D_x D_t + D_x^4 + 3D_y^2\right)(f \cdot f) = 0, \tag{228}$$

where D_y is the Hirota operator defined in a similar way as D_x and D_t in (21) and (22), respectively.

Continuing with (227), the computation of soliton solutions is similar to the KdV case in Section 2.2. Indeed, the forms of $f(x, y, t)$ for multi-soliton solutions remain the same except that $\theta_i = k_i x + l_i y - \omega_i t + \delta_i$ with $\omega_i = \frac{k_i^4 + 3l_i^2}{k_i}$. Setting $l_i = k_i P_i$ simplifies matters. Then $\theta_i = k_i\left(x + P_i y - (k_i^2 + 3P_i^2)t\right) + \delta_i$ and the phase factors are

$$a_{ij} = \frac{(k_i - k_j)^2 - (P_i - P_j)^2}{(k_i + k_j)^2 - (P_i - P_j)^2} \tag{229}$$

and $b_{123} = a_{12}\, a_{13}\, a_{23}$.

Setting $k = 2K$ and $\delta = 2\Delta$, we obtain the solitary wave solution

$$u(x, y, t) = 2K^2 \mathrm{sech}^2\left[K\left(x + Py - (4K^2 + 3P^2)t\right) + \Delta\right] \tag{230}$$

which is essentially one-dimensional.

The lengthy expressions for the two- and three-soliton solutions are not shown for brevity. A graph of the two-soliton solution of (225) at $t = 0.35$ for $K_1 = \frac{1}{2}, K_2 = 1, P_1 = -\frac{1}{8}, P_2 = \frac{3}{16}$ and $\Delta_1 = \Delta_2 = 0$ is shown in Fig. 21. Various types of soliton interactions have been reported in the literature and observed at flat beaches [1].

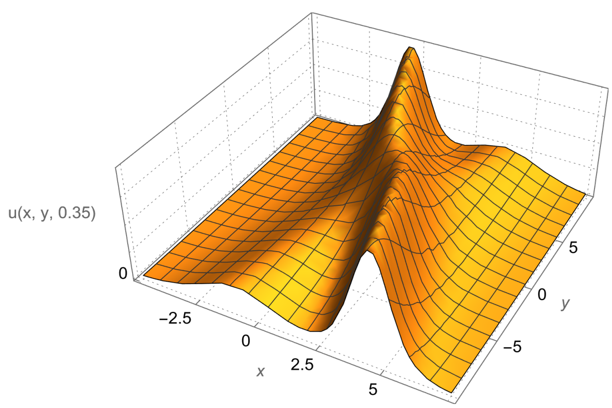


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Fig. 21. Snapshot of a two-soliton solution for the KP equation.

8.2 A $(3+1)$-dimensional evolution equation

Consider the $(3+1)$-dimensional evolution equation [21],

$$3W_{xz} - 2\left(2W_t + W_{3x} - 2WW_x\right)_y + 2\left(W_x \partial_x^{-1} W_y\right)_x = 0, \tag{231}$$

which can be written as

$$3u_{xxz} - \left(2u_{xt} + u_{4x} - 2u_x u_{xx}\right)_y + 2(u_{xx}u_y)_x = 0 \tag{232}$$

after substituting $W = u_x$. Integrating (232) twice with respect to x, yields

$$3u_z - 2\partial_t\left(\int^x u_y\,dx\right) - u_{xxy} + 2u_x u_y = 0. \tag{233}$$

A Laurent series solution of (232) suggests the transformation $u = -3\,(\ln f)_x$ which indeed allows one to replace (233) by a homogeneous equation,

$$f(-2f_{yt} + 3f_{xz} - f_{xxxy}) + 2f_y f_t - 3(f_x f_z + f_{xx} f_{xy} - f_x f_{xxy}) + f_{3x} f_y = 0, \tag{234}$$

of the form $f\mathcal{L}f + \mathcal{N}(f,f) = 0$ with

$$\mathcal{L}f = -2f_{yt} + 3f_{xz} - f_{xxxy}, \tag{235}$$

$$\mathcal{N}(f,g) = 2f_y g_t - 3(f_x g_z + f_{xx} g_{xy} - f_x g_{xxy}) + f_{3x} g_y. \tag{236}$$

To compute a single solitary wave solution we set $f = 1 + \mathrm{e}^{\theta}$, where $\theta = kx + ly + mz - \omega t + \delta$. Then, $\mathcal{L}\mathrm{e}^{\theta} = 0$ yields $\omega = \frac{k(k^2 l - 3m)}{2l}$. Since $\mathcal{N}(\mathrm{e}^{\theta}, \mathrm{e}^{\theta}) = 0$ we readily obtain a solitary wave solution

$$u(x,t) = -\frac{3}{2}k\left(1 + \tanh\left[\frac{1}{2}\left(kx + ly + mz - \frac{k(k^2 l - 3m)t}{2l} + \delta\right)\right]\right). \tag{237}$$

For a two-soliton solution we take $f = 1 + \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2} + a_{12}\mathrm{e}^{\theta_1+\theta_2}$, with $\theta_i = k_i x + l_i y + m_i z - \omega_i t + \delta_i$ and $\omega_i = \frac{k_i(k_i^2 l_i - 3m_i)}{2l_i}$. After some computations

$$a_{12} = \frac{k_1k_2l_1l_2(k_1-k_2)(l_1-l_2) - (k_1l_2 - k_2l_1)(l_1m_2 - l_2m_1)}{k_1k_2l_1l_2(k_1+k_2)(l_1+l_2) - (k_1l_2 - k_2l_1)(l_1m_2 - l_2m_1)}. \tag{238}$$

Thus, a two-soliton solution exists without having to impose any restrictions on the components (k_i, l_i, m_i) of the wave vector. In [21], the authors took $l_i = k_i$ and $m_i = k_i^3$ from the outset and therefore only computed a special two-soliton solution for which $\omega_i = -k_i^3$. Assuming a traveling frame $\theta_i = k_i(x+y) + k_i^3(t+z)$ from the start is too restrictive. Indeed, by a change of variables $(x,y,z,t) \to (\xi,\eta)$ with $\xi = x+y$ and $\eta = t+z$, one can readily show that after two integrations with respect to ξ equation (231) becomes the integrated KdV equation, that is, (17) with t replaced by η, x by ξ, and $u(x,t)$ by $u(\xi,\eta)$.

Moving on with our computations, a three-soliton solution does not exist for arbitrary wave numbers. Indeed,

$$f = 1 + \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2} + \mathrm{e}^{\theta_3} + a_{12}\mathrm{e}^{\theta_1+\theta_2} + a_{13}\mathrm{e}^{\theta_1+\theta_3} + a_{23}\mathrm{e}^{\theta_2+\theta_3} + b_{123}\mathrm{e}^{\theta_1+\theta_2+\theta_3} \tag{239}$$

only yields a three-soliton solution if $l_i = k_i$ with m_i still arbitrary (and a lengthy computation shows that the same holds for a four-soliton solution). The dispersion law and coefficients then simplify into $\omega_i = \frac{1}{2}(k_i^3 - 3m_i)$, $a_{ij} = ((k_i - k_j)/(k_i + k_j))^2$, and $b_{123} = a_{12}a_{13}a_{23}$, which are the same as for the KdV equation.

A graph of a two-soliton solution of (231) at $t = 0.05$ and $z = 1$ with $k_1 = 2, k_2 = \frac{3}{2}, l_1 = -\frac{1}{4}, l_2 = \frac{3}{4}, m_1 = 4, m_2 = \frac{9}{4}$, and $\delta_1 = \delta_2 = 0$ is shown in Fig. 22.

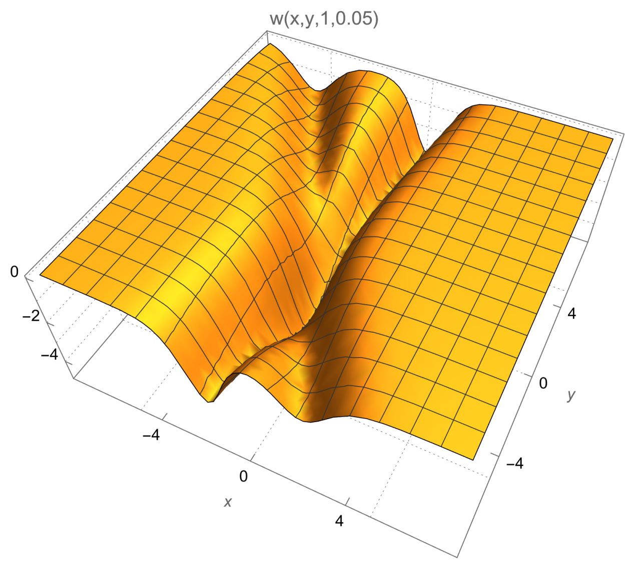

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Fig. 22. Plot of a two-soliton solution for (231) at $t = 0.05$ and $z = 1$ with $k_1 = 2, k_2 = \frac{3}{2}, l_1 = -\frac{1}{4}, l_2 = \frac{3}{4}, m_1 = 4, m_2 = \frac{9}{4}$, and $\delta_1 = \delta_2 = 0$.

9 Symbolic Software

Symbolic software for Hirota's method comes in two flavors: (i) code that aims at finding the bilinear form of a nonlinear PDE and (ii) code to compute soliton solutions with and without the use of the bilinear form.

9.1 Early developments of soliton software

As part of the design of symbolic software for soliton theory, in the early 1990s Hereman and Zhuang [28,31,32,33] implemented the Hirota method in *Macsyma*, a commercial computer algebra system now superseded by *Maxima*[24], a descendant of the original *DOE Macsyma* system. The code `HIROTA_SINGLE.MAX` is able to automatically compute up to three-soliton solutions of well-known nonlinear PDEs that can be transformed into a single bilinear equation of KdV-type [31,34], including the KdV, Boussinesq, KP, SK, and shallow water wave equations. To compute soliton solutions of these mostly (1+1)-dimensional PDEs, the bilinear form must be given explicitly. The code can also verify condition (223) for the existence of three- or four-soliton solutions ($n = 3$ or 4). To cover bilinear equations of mKdV-type [35], Hereman and Zhuang made `HIROTA_SYSTEM.MAX` [28,126] which was applied to various extensions of the mKdV equation taken from [52]. Codes for the sine-Gordon equation, NLS equations, and various other types of soliton equations which have quite complicated bilinear forms [37] were not developed. The code `HIROTA_SINGLE.MAX` was converted into *Mathematica* syntax and released under the name `hirota.m`. Further details about these open source codes[25] can be found in [32,126].

Although the simplified Hirota method (which does not use the bilinear form) was already published in [30,85], its implementation did not start until 2012 and is still ongoing. Cook *et al.* [12] developed the *Mathematica* code `Homogenize-And-Solve.m` to automate the computation of the soliton solutions discussed in Section 4 and other soliton equations in $(1 + 1)$-dimensions. That code is now superseded by `PDESolitonSolutions.m` [22].

9.2 Implementation and Limitations of PDESolitonSolutions.m

The current version of `PDESolitonSolutions.m` [22] computes up to three-soliton solutions for a given single PDE in one dependent variable (called u below) which is function of up to three space variables (x, y, z) and time (t). The PDE must have polynomial terms with constant coefficients. Presently, the code can not handle systems of PDEs. The algorithm largely follows the steps of Section 3.2:

(i) The PDE is integrated with respect to x as many times as possible.

(ii) The code first attempts to find a transformation to homogenize the given PDE based on the (truncated) Laurent series expansion of its solution. If unsuccessful, the code tries a transformation of type $u = c\,(\ln f)_{nx}$, with integer $1 \le n \le n_{\max}$ (with default value $n_{\max} = 4$) and constant c. Starting with $n = 1$, the code seeks the lowest value of n and matching c so that the PDE is transformed into an equation that is homogeneous in f.

(iii) A solution of type $f(x, y, z, t) = 1 + \sum_{n=1}^{p} \epsilon^n f^{(n)}(x, y, z, t)$ is sought where $1 \le p \le p_{\max}$ (with default value $p_{\max} = 8$). The bookkeeping parameter ϵ helps with splitting expressions into single exponentials, products of two exponentials, etc. Substituting the above sum for f into a homogeneous equation for f (of degree d) yields an expression of degree $\ell_{\max} = d\,p_{\max}$ in ϵ.

[24] *Maxima* is freely available from SourceForge at https://maxima.sourceforge.io/.

[25] The codes are still available at https://inside.mines.edu/~whereman.

(iv) Starting with $f^{(1)} = \sum_{i=1}^{N} \phi_i(x, y, z, t)$, where the natural number N refers to the N-soliton solution one aims to compute and $\phi_i(x, y, z, t) \equiv \mathrm{e}^{\theta_i} = \mathrm{e}^{k_i x + l_i y + m_i z - \omega_i t + \delta_i}$, at order ϵ the code balances the linear terms in ϕ_i to determine the dispersion relation $\omega_i(k_i, l_i, m_i)$.

(v) Next, based on the monomials in the functions ϕ_i that occur at order ϵ^2, the code builds $f^{(2)} = \sum_{i,j} a_{ij} \phi_i \phi_j$ and computes the coefficients a_{ij} (and possible constraints for k_i, l_i, and m_i) by balancing like products of two exponentials. Note that $i = j$ is allowed to account for terms in ϕ_i^2.

(vi) At the next orders in ϵ, expressions for $f^{(3)}, f^{(4)}$, etc., are computed the same way. If at some order $n < p_{\max}$ in ϵ the function $f^{(n)}$ becomes identically zero, the code verifies that $f^{(n+1)}, \ldots, f^{(p_{\max})}$ can be set to zero. It also verifies whether or not the coefficients of $\epsilon^{n+1}, \ldots, \epsilon^{\ell_{\max}}$ in the expression mentioned in (iii) all vanish. For non-solitonic equations this may lead to (additional) constraints on the wave numbers. If both verifications are successful, the code returns the solutions after explicitly verifying that the final f indeed satisfies the homogenized PDE. If none of the $f^{(n)}$ become zero, the code reports that a N-soliton solution could not be computed. The code will return a solitary wave solution for $N = 1$ and a bi-soliton solution for $N = 2$, provided such solutions exist.

Some remarks are warranted:

(i) The current code only considers integration with respect to x ignoring the possibility to integrate the given PDE with respect to y or z.

(ii) In addition to transformations based on a truncated Laurent series, currently only single-term logarithmic derivative transformations with respect to x up to fourth-order are used. At present only transformations involving one new dependent variable (f) are considered. Therefore, the current code can not find solutions of, for example, the mKdV equation.

(iii) With regard to the growing complexity of $f^{(n)}$ as n increases, $p_{\max} = 8$ has been set as default value.

(iv) The current code is limited to three space variables and time. To prevent long expressions and avoid *Mathematica*'s conversion of products of exponentials into a single exponential, the explicit form of $\phi_i(x, y, z, t)$ is never used. Instead, the code uses rules for derivatives of $\phi_i(x, y, z, t)$, such as $\phi_i(x, y, z, t)_{nx} = k_i^n \phi_i(x, y, z, t)$ and $\phi_i(x, y, z, t)_{mt} = (-\omega_i)^m \phi_i(x, y, z, t)$.

(v) For example for the two-soliton case, $f^{(2)} = a_{11}\phi_1^2 + a_{12}\phi_1\phi_2 + a_{22}\phi_2^2$ where some of these terms might not be included. Indeed, after substitution of $f = 1 + f^{(1)} = 1 + \phi_1 + \phi_2$ into the homogeneous equation, the code generates the list of monomials of type $\phi_i\phi_j$ (including ϕ_i^2) that occur at order ϵ^2 and makes a linear combination of those monomials with undetermined coefficients a_{ij} to create $f^{(2)}$ with the minimal number of terms. The coefficients a_{ij} are then computed by requiring that like terms in $\phi_i\phi_j$ vanish.
The same procedure is used to compute N-soliton solutions. Starting from $f = 1 + f^{(1)} = 1 + \phi_1 + \phi_2 + \ldots + \phi_N$, the code constructs the minimal expressions for all subsequent $f^{(n)}$ in which each term is a product of n (not necessarily distinct) functions taken from $\{\phi_1, \phi_2, \ldots, \phi_N\}$. The code

determines which of these products are actually needed and combines them with undetermined coefficients.

(vi) For homogeneous equations of high degree, some symbolic verifications can be quite slow. To speed things up, the code does no longer *symbolically* verify that coefficients of higher orders in ϵ in the perturbation scheme vanish as soon as two consecutive coefficients of lower orders terms already vanished identically. Once two consecutive expressions are determined to be zero, the code *numerically* tests if the expressions at higher order are also zero. This applies to both the computation of the $f^{(n)}$ as well as the coefficients of ϵ in the perturbation scheme.

Furthermore, verifying that the (lengthy) expressions of f indeed solve the homogeneous equation can be time consuming, especially for cubic and quartic equations. Indeed, checking that (107) satisfies (70) is computationally very expensive. Therefore, after the solution is substituted into the homogeneous equation, all independent variables, wave numbers (k_i, l_i, m_i), phase constants δ_i, and parameters in the PDE (if present) are repeatedly replaced by random real numbers in $[-2, 2]$. In each case it is checked if the resulting expression is zero within machine precision. Likewise, the solitary wave and one-soliton solutions for $u(x, y, z, t)$ are tested symbolically but the numerical approach is used to verify that the often long expressions of two- and three-soliton solutions $u(x, y, z, t)$ indeed solve the original PDE.

9.3 Other software packages for Hirota's method

As early as 1988, Ito [53] designed code in REDUCE to interactively investigate nonlinear PDEs with Hirota's bilinear and Wronskian operators.

In [124], Zhou *et al.* introduced the *Maple* package `Bilinearization` to convert (mainly) nonlinear evolution equations of KdV-type into their bilinear form using logarithmic-derivative transformations. To cover the mKdV and nonlinear Schrödinger equations, they later extended the algorithm to work for arctan and rational transformations. They also added the code `Multisoliton` to compute up to three-soliton solutions for single bilinear equations and simple systems of bilinear equations. Ye *et al.* [120,121] presented a more efficient method to do the same but only with logarithmic-derivative transformations. Their method is also implemented in *Maple*. When successful, these *Maple* codes return the bilinear form explicitly.

Yang and Ruan [117,118,119] have produced the *Maple* packages `HBFTrans`, `HBFTrans2`, and `HBFGen` to transform nonlinear PDEs into their bilinear forms, again based on logarithmic derivative transformations. In their newest algorithms, they take advantage of the properties of the Hirota operators and the scaling invariance[26] of the original equation. Doing so, makes their codes more efficient and faster.

[26] Dilation or scaling symmetry is a special Lie-point symmetry shared by many integrable PDEs [29].

Based on the Bell polynomial approach [68,69], Miao *et al.* [75] developed the *Maple* package `PDEBellII` to compute bilinear forms, bilinear Bäcklund transformations, Lax pairs, and conservation laws of KdV-type equations. In contrast to `PDEBell`, developed earlier by Yang and Chen, `PDEBellII` does no longer use scaling invariance to make it applicable to a broader class of nonlinear PDEs.

For completeness, we mention the new computational method of Kumar *et al.* [62] for the construction of bilinear forms which, as far as we know, has not been implemented yet.

10 Conclusions and Future Work

Hirota's bilinear method is an effective method to construct soliton solutions of completely integrable nonlinear PDEs. In this paper we discussed a simplified version of Hirota's method (which does not use Hirota's bilinear operators) and used it to construct solitary and soliton solutions of various soliton equations as well as some nonlinear polynomial equations that do not have solitons.

We showed that the Hirota transformation is crucial to obtain a PDE that is homogeneous of degree (in the new dependent variable). We focused on logarithmic derivative transformations but, as we saw with the mKdV equation, rational and arctan transformations might be required, or combinations thereof. To homogenize, e.g., the Davey-Stewartson system, one needs a mixture of rational and logarithmic derivative transformations. There is no systematic way for finding these transformations but the first few terms of a Laurent series solution and scaling invariance of the PDE can help determine a suitable candidate thereby reducing the guess work.

The actual recasting of the transformed PDE into bilinear form in terms of Hirota's operators, which assumes a quadratic equation or a tricky decoupling into quadratic equations, is not required to compute solitary wave solutions or solitons. Indeed, without bilinear forms, exact solutions of the transformed equation can still be constructed straightforwardly by solving a perturbation-like scheme on the computer using a symbolic manipulation package.

The simplified version of Hirota's method is largely algorithmic and now available as the *Mathematica* program `PDESolitonSolutions.m`. In future releases a broader class of transformations (likely involving two functions f and g) will be considered to make the code applicable to a large set of PDEs including mKdV-type equations. A future version of the code might follow the algorithm presented in this paper even closer. It will use the perturbation schemes involving the linear and nonlinear operators which will automatically be generated by splitting the homogeneous equations into linear and nonlinear pieces. This "divide-and-conquer" strategy is expected to make the computations faster. An extension of the algorithm and code to systems of PDEs is being investigated.

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Appendix

In the derivations below we use that $\mathcal{L}(f)$ is linear in f, $\mathcal{N}_1(f,g)$ is bilinear (i.e., linear in both f and g), $\mathcal{N}_2$ is trilinear, and $\mathcal{N}_3$ is quadrilinear.

Bilinear scheme

For the derivation of the perturbation scheme for an equation of type (36) we need Cauchy's product formula (to regroup terms in powers of ϵ),

$$\left(\sum_{p=1}^{\infty} \epsilon^p \, a_p\right) \left(\sum_{q=1}^{\infty} \epsilon^q \, b_q\right) = \sum_{n=2}^{\infty} \epsilon^n \sum_{j=1}^{n-1} a_j \, b_{n-j}. \tag{240}$$

Then,

$$\begin{aligned} f \, \mathcal{L} f &= \left(1 + \sum_{r=p}^{\infty} \epsilon^p \, f^{(p)}\right) \mathcal{L} \left(\sum_{q=1}^{\infty} \epsilon^q \, f^{(q)}\right) \\ &= \left(1 + \sum_{p=1}^{\infty} \epsilon^p \, f^{(p)}\right) \sum_{q=1}^{\infty} \epsilon^q \, \mathcal{L} f^{(q)} \\ &= \sum_{q=1}^{\infty} \epsilon^q \, \mathcal{L} f^{(q)} + \left(\sum_{p=1}^{\infty} \epsilon^p \, f^{(p)}\right) \left(\sum_{q=1}^{\infty} \epsilon^q \, \mathcal{L} f^{(q)}\right) \\ &= \sum_{n=1}^{\infty} \epsilon^n \, \mathcal{L} f^{(n)} + \sum_{n=2}^{\infty} \epsilon^n \sum_{j=1}^{n-1} f^{(j)} \mathcal{L} f^{(n-j)}, \end{aligned} \tag{241}$$

where we have applied (240) with $a_p = f^{(p)}$ and $b_q = \mathcal{L}f^{(q)}$. Similarly, we compute

$$
\begin{aligned}
\mathcal{N}(f,f) &= \mathcal{N}\left(1+\sum_{p=1}^{\infty}\epsilon^p\, f^{(p)},\, 1+\sum_{q=1}^{\infty}\epsilon^q\, f^{(q)}\right)\\
&= \mathcal{N}\left(\sum_{p=1}^{\infty}\epsilon^p\, f^{(p)},\, \sum_{q=1}^{\infty}\epsilon^q\, f^{(q)}\right)\\
&= \sum_{n=2}^{\infty}\epsilon^n\sum_{j=1}^{n-1}\mathcal{N}(f^{(j)}, f^{(n-j)}),
\end{aligned}
\tag{242}
$$

where again we applied (240) and used the bilinearity of $\mathcal{N}(f,g)$. Adding (241) and (242), the coefficient of ϵ^n is

$$
\mathcal{L}f^{(n)} + \sum_{j=1}^{n-1}\left(f^{(j)}\mathcal{L}f^{(n-j)} + \mathcal{N}(f^{(j)}, f^{(n-j)})\right) = 0, \quad n \geq 2. \tag{243}
$$

Trilinear scheme

For the derivation of the perturbation scheme for equations of type (58) we need Cauchy's product formula for three sums:

$$
\left(\sum_{p=1}^{\infty}\epsilon^p\, a_p\right)\left(\sum_{q=1}^{\infty}\epsilon^q\, b_q\right)\left(\sum_{r=1}^{\infty}\epsilon^r\, c_r\right) = \sum_{n=3}^{\infty}\epsilon^n\sum_{j=2}^{n-1}\sum_{\ell=1}^{j-1} a_\ell\, b_{n-j}\, c_{j-\ell}. \tag{244}
$$

Substituting (28) into (58) and applying (240) and (244) yields the following term in ϵ^n:

$$
\begin{aligned}
&\mathcal{L}f^{(n)} + \sum_{j=1}^{n-1}\left(2f^{(j)}\mathcal{L}f^{(n-j)} + \mathcal{N}_1(f^{(j)}, f^{(n-j)})\right) + \sum_{j=2}^{n-1}\sum_{\ell=1}^{j-1}\Big(f^{(\ell)}f^{(n-j)}\mathcal{L}f^{(j-\ell)}\\
&+ f^{(\ell)}\mathcal{N}_1(f^{(n-j)}, f^{(j-\ell)}) + \mathcal{N}_2(f^{(\ell)}, f^{(n-j)}, f^{(j-\ell)})\Big) = 0, \quad n \geq 3.
\end{aligned}
\tag{245}
$$

Quadrilinear scheme

Setting up the perturbation scheme for equations of type (73) requires the formula

$$
\begin{aligned}
&\left(\sum_{p=1}^{\infty}\epsilon^p\, a_p\right)\left(\sum_{q=1}^{\infty}\epsilon^q\, b_q\right)\left(\sum_{r=1}^{\infty}\epsilon^r\, c_r\right)\left(\sum_{s=1}^{\infty}\epsilon^s\, d_s\right)\\
&= \sum_{n=4}^{\infty}\epsilon^n\sum_{j=3}^{n-1}\sum_{\ell=2}^{j-1}\sum_{m=1}^{\ell-1} a_m\, b_{n-j}\, c_{j-\ell}\, d_{\ell-m}.
\end{aligned}
\tag{246}
$$

Substituting (28) into (73) and applying (240), (244), and (246) yields the following at $O(\epsilon^n)$:

$$\begin{aligned}
&\mathcal{L}f^{(n)} + \sum_{j=1}^{n-1} \Big(3f^{(j)}\mathcal{L}f^{(n-j)} + \mathcal{N}_1(f^{(j)}, f^{(n-j)})\Big) \\
&+ \sum_{j=2}^{n-1}\sum_{\ell=1}^{j-1} \Big(3f^{(\ell)}f^{(n-j)}\mathcal{L}f^{(j-\ell)} + 2f^{(\ell)}\mathcal{N}_1(f^{(n-j)}, f^{(j-\ell)}) + \mathcal{N}_2(f^{(\ell)}, f^{(n-j)}, f^{(j-\ell)})\Big) \\
&+ \sum_{j=3}^{n-1}\sum_{\ell=2}^{j-1}\sum_{m=1}^{\ell-1} \Big(f^{(m)}f^{(n-j)}f^{(j-\ell)}\mathcal{L}f^{(\ell-m)} + f^{(m)}f^{(n-j)}\mathcal{N}_1(f^{(j-\ell)}, f^{(\ell-m)}) \\
&+ f^{(m)}\mathcal{N}_2(f^{(n-j)}, f^{(j-\ell)}, f^{(\ell-m)}) + \mathcal{N}_3(f^{(m)}, f^{(n-j)}, f^{(j-\ell)}, f^{(\ell-m)})\Big) = 0, \\
&\qquad\qquad n \geq 4. \quad (247)
\end{aligned}$$

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