

Symbolic Computation of Conservation Laws of Nonlinear PDEs in Multi-dimensions

Willy Hereman

Department of Mathematical and Computer Sciences
Colorado School of Mines
Golden, Colorado, U.S.A.
whereman@mines.edu

http://www.mines.edu/fs_home/whereman/

Centre de Recherches Mathématiques
Université de Montréal, Montréal, Canada

Tuesday, May 23, 2006, 3:30p.m.

Acknowledgements

Collaborators:

Mark Hickman (Christchurch, NZ), Bernard Deconinck (Seattle), and Michael Colagrosso (CSM)

Ph.D. student Douglas Poole

Several undergraduate and Master students

Research supported in part by NSF
under Grant CCR-9901929

This presentation was made in **TeXpower**

Outline

- Conservation Laws of PDEs in multi-dimensions
- Example: Shallow water wave equations (Dellar)
- Algorithmic Methods for conservation laws
- Computer Demonstration
- Tools:
 - Euler operators (testing exactness)
 - Calculus-based formulas for homotopy operator
 - ★ symbolic integration by parts
 - ★ inversion of the total divergence operator
- Application to shallow water wave equations
- Conclusions and Future Work
- Software and Publications

Notations – Computations on the Jet Space

- Independent variables $\mathbf{x} = (x, y, z)$
- Dependent variables $\mathbf{u} = (u^{(1)}, u^{(2)}, \dots, u^{(j)}, \dots, u^{(N)})$
In examples: $\mathbf{u} = (u, v, \theta, h, \dots)$

- Partial derivatives $u_{kx} = \frac{\partial^k u}{\partial x^k}$, $u_{kxly} = \frac{\partial^{k+l} u}{\partial x^k \partial y^l}$, etc.

- *Differential functions*

Example: $f = uvv_x + x^2 u_x^3 v_x + u_x v_{xx}$ for $u(x), v(x)$

- Total derivative (with respect to x)

$$D_x = \frac{\partial}{\partial x} + \sum_{k=0}^{M_x^{(1)}} u^{(k+1)x} \frac{\partial}{\partial u_{kx}} + \sum_{k=0}^{M_x^{(2)}} v^{(k+1)x} \frac{\partial}{\partial v_{kx}}$$

$M_x^{(1)}$ is the order of f in u (with respect to x), etc.

- **Example:** $f = uvv_x + x^2 u_x^3 v_x + u_x v_{xx}$
 $M_x^{(1)} = 1$ and $M_x^{(2)} = 2$
- Total derivative with respect to x :

$$\begin{aligned}
D_x f &= \frac{\partial f}{\partial x} + \sum_{k=0}^1 u_{(k+1)x} \frac{\partial f}{\partial u_{kx}} + \sum_{k=0}^2 v_{(k+1)x} \frac{\partial f}{\partial v_{kx}} \\
&= \frac{\partial f}{\partial x} + u_x \frac{\partial f}{\partial u} + u_{2x} \frac{\partial f}{\partial u_x} \\
&\quad v_x \frac{\partial f}{\partial v} + v_{2x} \frac{\partial f}{\partial v_x} + v_{3x} \frac{\partial f}{\partial v_{2x}} \\
&= 2xu_x^3 v_x + u_x (vv_x) + u_{xx} (3x^2 u_x^2 v_x + v_{xx}) \\
&\quad + v_x (uv_x) + v_{xx} (uv + x^2 u_x^3) + v_{xxx} (u_x)
\end{aligned}$$

Conservation Laws

- Conservation law in $(1 + 1)$ dimensions

$$\boxed{D_t \rho + D_x J = 0} \quad (\text{on PDE})$$

conserved density ρ and flux J

- **Example:** Korteweg-de Vries (KdV) equation

$$u_t + uu_x + u_{3x} = 0$$

- Sample conservation law

$$D_t \left(u^3 - 3u_x^2 \right) + D_x \left(\frac{3}{4}u^4 - 6uu_x^2 + 3u^2u_{2x} + 3u_{2x}^2 - 6u_xu_{3x} \right) = 0$$

- **Key property:** Dilation invariance
- **Example:** KdV equation and its density-flux pairs are invariant under the scaling symmetry

$$(x, t, u) \rightarrow \left(\frac{x}{\lambda}, \frac{t}{\lambda^3}, \lambda^2 u \right),$$

λ is arbitrary parameter.

- Some density-flux pairs for the KdV equation:

$$\rho^{(1)} = u \quad J^{(1)} = \frac{u^2}{2} + u_{2x}$$

$$\rho^{(2)} = u^2 \quad J^{(2)} = \frac{2u^3}{3} + 2uu_{2x} - u_x^2$$

$$\rho^{(3)} = u^3 - 3u_x^2$$

$$J^{(3)} = \frac{3}{4}u^4 - 6uu_x^2 + 3u^2u_{2x} + 3u_{2x}^2 - 6u_xu_{3x}$$

⋮

$$\rho^{(6)} = u^6 - 60u^3u_x^2 - 30u_x^4 + 108u^2u_{2x}^2$$

$$+ \frac{720}{7}u_{2x}^3 - \frac{648}{7}uu_{3x}^2 + \frac{216}{7}u_{4x}^2$$

⋮

- Conservation law in $(3 + 1)$ dimensions

$$\boxed{D_t \rho + \nabla \cdot \mathbf{J} = D_t \rho + D_x J_1 + D_y J_2 + D_z J_3 = 0} \quad (\text{on PDE})$$

conserved density ρ and flux $\mathbf{J} = (J_1, J_2, J_3)$

- **Example:** Shallow water wave (SWW) equations

[P. Dellar, Phys. Fluids **15** (2003) 292-297]

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u} + 2 \boldsymbol{\Omega} \times \mathbf{u} + \nabla(\theta h) - \frac{1}{2} h \nabla \theta = \mathbf{0}$$

$$\theta_t + \mathbf{u} \cdot (\nabla \theta) = 0$$

$$h_t + \nabla \cdot (\mathbf{u} h) = 0$$

where $\mathbf{u}(x, y, t)$, $\theta(x, y, t)$ and $h(x, y, t)$

- In components:

$$u_t + uu_x + vu_y - 2\Omega v + \frac{1}{2}h\theta_x + \theta h_x = 0$$

$$v_t + uv_x + vv_y + 2\Omega u + \frac{1}{2}h\theta_y + \theta h_y = 0$$

$$\theta_t + u\theta_x + v\theta_y = 0$$

$$h_t + hu_x + uh_x + hv_y + vh_y = 0$$

- SWW equations are invariant under

$$(x, y, t, u, v, h, \theta, \Omega) \rightarrow$$

$$(\lambda^{-1}x, \lambda^{-1}y, \lambda^{-b}t, \lambda^{b-1}u, \lambda^{b-1}v, \lambda^a h, \lambda^{2b-a-2}\theta, \lambda^b\Omega)$$

where $W(h) = a$ and $W(\Omega) = b$ ($a, b \in \mathbb{Q}$)

- First few densities-flux pairs of SWW system:

$$\begin{aligned}
 \rho^{(1)} &= h & \mathbf{J}^{(1)} &= \begin{pmatrix} uh \\ vh \end{pmatrix} \\
 \rho^{(2)} &= h\theta & \mathbf{J}^{(2)} &= \begin{pmatrix} uh\theta \\ vh\theta \end{pmatrix} \\
 \rho^{(3)} &= h\theta^2 & \mathbf{J}^{(3)} &= \begin{pmatrix} uh\theta^2 \\ vh\theta^2 \end{pmatrix} \\
 \rho^{(4)} &= (u^2 + v^2)h + h^2\theta & \mathbf{J}^{(4)} &= \begin{pmatrix} u^3h + uv^2h + 2uh^2\theta \\ v^3h + u^2vh + 2vh^2\theta \end{pmatrix} \\
 \rho^{(5)} &= v_x\theta - u_y\theta + 2\Omega\theta & \mathbf{J}^{(5)} &= \frac{1}{2} \begin{pmatrix} 4\Omega u\theta - 2uu_y\theta + 2uv_x\theta - h\theta\theta_y \\ 4\Omega v\theta + 2vv_x\theta - 2vu_y\theta + h\theta\theta_x \end{pmatrix}
 \end{aligned}$$

Computer Demonstration

Algorithmic Methods for Conservation Laws

- Use Noether's Theorem (Lagrangian formulation)
- Direct methods (Anderson, Bluman, Anco, Wolf, etc.) based on solving ODEs (or PDEs)
- **Strategy** (linear algebra and variational calculus)
 - Density is linear combination of scaling invariant terms with undetermined coefficients
 - Use variational derivative (Euler operator) to compute the undetermined coefficients
 - Use the homotopy operator to compute the flux (invert D_x or Div)
 - Work with linearly independent pieces in finite dimensional spaces

Review of Vector Calculus

- Definition: \mathbf{F} is *conservative* if $\mathbf{F} = \nabla f$
- Definition: \mathbf{F} is *irrotational* or *curl free* if $\nabla \times \mathbf{F} = \mathbf{0}$
- Theorem (gradient test): $\mathbf{F} = \nabla f$ iff $\nabla \times \mathbf{F} = \mathbf{0}$

Review of Vector Calculus

- Definition: \mathbf{F} is *conservative* if $\mathbf{F} = \nabla f$
- Definition: \mathbf{F} is *irrotational* or *curl free* if $\nabla \times \mathbf{F} = \mathbf{0}$
- Theorem (gradient test): $\mathbf{F} = \nabla f$ iff $\nabla \times \mathbf{F} = \mathbf{0}$
The curl annihilates gradients!

Review of Vector Calculus

- Definition: \mathbf{F} is *conservative* if $\mathbf{F} = \nabla f$
- Definition: \mathbf{F} is *irrotational* or *curl free* if $\nabla \times \mathbf{F} = \mathbf{0}$
- Theorem (gradient test): $\mathbf{F} = \nabla f$ iff $\nabla \times \mathbf{F} = \mathbf{0}$
The curl annihilates gradients!
- Definition: \mathbf{F} is *incompressible* or *divergence free* if $\nabla \cdot \mathbf{F} = 0$
- Theorem (curl test): $\mathbf{F} = \nabla \times \mathbf{G}$ iff $\nabla \cdot \mathbf{F} = 0$

Review of Vector Calculus

- Definition: \mathbf{F} is *conservative* if $\mathbf{F} = \nabla f$
- Definition: \mathbf{F} is *irrotational* or *curl free* if $\nabla \times \mathbf{F} = \mathbf{0}$
- Theorem (gradient test): $\mathbf{F} = \nabla f$ iff $\nabla \times \mathbf{F} = \mathbf{0}$
The curl annihilates gradients!
- Definition: \mathbf{F} is *incompressible* or *divergence free* if $\nabla \cdot \mathbf{F} = 0$
- Theorem (curl test): $\mathbf{F} = \nabla \times \mathbf{G}$ iff $\nabla \cdot \mathbf{F} = 0$
The divergence annihilates curls!

Review of Vector Calculus

- Definition: \mathbf{F} is *conservative* if $\mathbf{F} = \nabla f$
 - Definition: \mathbf{F} is *irrotational* or *curl free* if $\nabla \times \mathbf{F} = \mathbf{0}$
 - Theorem (gradient test): $\mathbf{F} = \nabla f$ iff $\nabla \times \mathbf{F} = \mathbf{0}$
The curl annihilates gradients!
 - Definition: \mathbf{F} is *incompressible* or *divergence free* if $\nabla \cdot \mathbf{F} = 0$
 - Theorem (curl test): $\mathbf{F} = \nabla \times \mathbf{G}$ iff $\nabla \cdot \mathbf{F} = 0$
The divergence annihilates curls!
- Question: How can one test that $f = \nabla \cdot \mathbf{F}$?

Review of Vector Calculus

- Definition: \mathbf{F} is *conservative* if $\mathbf{F} = \nabla f$
 - Definition: \mathbf{F} is *irrotational* or *curl free* if $\nabla \times \mathbf{F} = \mathbf{0}$
 - Theorem (gradient test): $\mathbf{F} = \nabla f$ iff $\nabla \times \mathbf{F} = \mathbf{0}$
The curl annihilates gradients!
 - Definition: \mathbf{F} is *incompressible* or *divergence free* if $\nabla \cdot \mathbf{F} = 0$
 - Theorem (curl test): $\mathbf{F} = \nabla \times \mathbf{G}$ iff $\nabla \cdot \mathbf{F} = 0$
The divergence annihilates curls!
- Question: How can one test that $f = \nabla \cdot \mathbf{F}$?
- No theorem from vector calculus!

Tools from the Calculus of Variations

- Definition: a differential function f is *exact* iff $f = D_x F$
- Theorem (exactness test): $f = D_x F$ iff $\mathcal{L}_{u^{(j)}(x)}^{(0)} f \equiv 0, \quad j = 1, 2, \dots, N$
- Definition: a differential function f is a *divergence* if $f = \text{Div } \mathbf{F}$
- Theorem (divergence test): $f = \text{Div } \mathbf{F}$ iff $\mathcal{L}_{u^{(j)}(\mathbf{x})}^{(0)} f \equiv 0, \quad j = 1, 2, \dots, N$

Tools from the Calculus of Variations

- Definition: a differential function f is *exact* iff $f = D_x F$
- Theorem (exactness test): $f = D_x F$ iff $\mathcal{L}_{u^{(j)}(x)}^{(0)} f \equiv 0, \quad j = 1, 2, \dots, N$
- Definition: a differential function f is a *divergence* if $f = \text{Div } \mathbf{F}$
- Theorem (divergence test): $f = \text{Div } \mathbf{F}$ iff $\mathcal{L}_{u^{(j)}(\mathbf{x})}^{(0)} f \equiv 0, \quad j = 1, 2, \dots, N$

The Euler operator annihilates divergences!

Formulas for Euler operators (variational derivatives):

$$\begin{aligned}\mathcal{L}_{u^{(j)}(x)}^{(0)} &= \sum_{k=0}^{M_x^{(j)}} (-D_x)^k \frac{\partial}{\partial u_{kx}^{(j)}} \\ &= \frac{\partial}{\partial u^{(j)}} - D_x \frac{\partial}{\partial u_x^{(j)}} + D_x^2 \frac{\partial}{\partial u_{2x}^{(j)}} - D_x^3 \frac{\partial}{\partial u_{3x}^{(j)}} + \dots\end{aligned}$$

$$\begin{aligned}\mathcal{L}_{u^{(j)}(x,y)}^{(0,0)} &= \sum_{k_x=0}^{M_x^{(j)}} \sum_{k_y=0}^{M_y^{(j)}} (-D_x)^{k_x} (-D_y)^{k_y} \frac{\partial}{\partial u_{k_x x k_y y}^{(j)}} \\ &= \frac{\partial}{\partial u^{(j)}} - D_x \frac{\partial}{\partial u_x^{(j)}} - D_y \frac{\partial}{\partial u_y^{(j)}} \\ &\quad + D_x^2 \frac{\partial}{\partial u_{2x}^{(j)}} + D_x D_y \frac{\partial}{\partial u_{xy}^{(j)}} + D_y^2 \frac{\partial}{\partial u_{2y}^{(j)}} - D_x^3 \frac{\partial}{\partial u_{3x}^{(j)}} \dots\end{aligned}$$

$$\mathcal{L}_{u^{(j)}(x,y,z)}^{(0,0,0)}$$

$$= \sum_{k_x=0}^{M_x^{(j)}} \sum_{k_y=0}^{M_y^{(j)}} \sum_{k_z=0}^{M_z^{(j)}} (-D_x)^{k_x} (-D_y)^{k_y} (-D_z)^{k_z} \frac{\partial}{\partial u_{k_x x k_y y k_z z}^{(j)}}$$

$$= \frac{\partial}{\partial u^{(j)}} - D_x \frac{\partial}{\partial u_x^{(j)}} - D_y \frac{\partial}{\partial u_y^{(j)}} - D_z \frac{\partial}{\partial u_z^{(j)}}$$

$$+ D_x^2 \frac{\partial}{\partial u_{2x}^{(j)}} + D_y^2 \frac{\partial}{\partial u_{2y}^{(j)}} + D_z^2 \frac{\partial}{\partial u_{2z}^{(j)}}$$

$$+ D_x D_y \frac{\partial}{\partial u_{xy}^{(j)}} + D_x D_z \frac{\partial}{\partial u_{xz}^{(j)}} + D_y D_z \frac{\partial}{\partial u_{yz}^{(j)}} - \dots$$

Application: Testing Exactness

Consider, for **example**,

$$f = 3u_x v^2 \sin u - u_x^3 \sin u - 6v v_x \cos u + 2u_x u_{2x} \cos u + 8v_x v_{2x}$$

for $u(x)$ and $v(x)$

- f is exact
- After integration by parts (by hand):

$$F = \int f dx = 4v_x^2 + u_x^2 \cos u - 3v^2 \cos u$$

- Exactness test with Euler operator:

$$f = 3u_x v^2 \sin u - u_x^3 \sin u - 6v v_x \cos u + 2u_x u_{2x} \cos u + 8v_x v_{2x}$$

$$\begin{aligned} \mathcal{L}_{u(x)}^{(0)} f &= \frac{\partial f}{\partial u} - D_x \frac{\partial f}{\partial u_x} + D_x^2 \frac{\partial f}{\partial u_{2x}} \\ &= 3u_x v^2 \cos u - u_x^3 \cos u + 6v v_x \sin u - 2u_x u_{2x} \sin u \\ &\quad - D_x [3v^2 \sin u - 3u_x^2 \sin u + 2u_{2x} \cos u] + D_x^2 [2u_x \cos u] \\ &= 3u_x v^2 \cos u - u_x^3 \cos u + 6v v_x \sin u - 2u_x u_{2x} \sin u \\ &\quad - [3u_x v^2 \cos u + 6v v_x \sin u - 3u_x^3 \cos u - 6u_x u_{2x} \sin u \\ &\quad - 2u_x u_{2x} \sin u + 2u_{3x} \cos u] \\ &\quad + [-2u_{3x} \cos u - 6u_x u_{2x} \sin u + 2u_{3x} \cos u] = 0 \end{aligned}$$

Similarly, $\mathcal{L}_{v(x)}^{(0)} f \equiv 0$

Inverting D_x and Div

Problem Statement

- In 1D:

Example: For $u(x)$ and $v(x)$

$$f = 3u_x v^2 \sin u - u_x^3 \sin u - 6v v_x \cos u + 2u_x u_{2x} \cos u + 8v_x v_{2x}$$

- Find $F = \int f dx$ so, $f = D_x F$

- Result (by hand):

$$F = 4v_x^2 + u_x^2 \cos u - 3v^2 \cos u$$

Inverting D_x and Div

Problem Statement

- In 1D:

Example: For $u(x)$ and $v(x)$

$$f = 3u_x v^2 \sin u - u_x^3 \sin u - 6v v_x \cos u + 2u_x u_{2x} \cos u + 8v_x v_{2x}$$

- Find $F = \int f dx$ so, $f = D_x F$
- Result (by hand):

$$F = 4v_x^2 + u_x^2 \cos u - 3v^2 \cos u$$

Mathematica cannot compute this integral!

- In 2D or 3D:

Example: For $u(x, y)$ and $v(x, y)$

$$f = uv_y - u_x v_y - u_y v_x + u_{xy} v_x$$

- Find $\mathbf{F} = \text{Div}^{-1} f$ so, $f = \text{Div } \mathbf{F}$
- Result (by hand):

$$\tilde{\mathbf{F}} = (uv_y - u_x v_y, -u v_x + u_x v_x)$$

- In 2D or 3D:

Example: For $u(x, y)$ and $v(x, y)$

$$f = u_x v_y - u_{2x} v_y - u_y v_x + u_{xy} v_x$$

- Find $\mathbf{F} = \text{Div}^{-1} f$ so, $f = \text{Div } \mathbf{F}$
- Result (by hand):

$$\tilde{\mathbf{F}} = (u v_y - u_x v_y, -u v_x + u_x v_x)$$

Mathematica cannot do this!

- In 2D or 3D:

Example: For $u(x, y)$ and $v(x, y)$

$$f = u_x v_y - u_{2x} v_y - u_y v_x + u_{xy} v_x$$

- Find $\mathbf{F} = \text{Div}^{-1} f$ so, $f = \text{Div } \mathbf{F}$
- Result (by hand):

$$\tilde{\mathbf{F}} = (u v_y - u_x v_y, -u v_x + u_x v_x)$$

Mathematica cannot do this!

Can this be done without integration by parts?

- In 2D or 3D:

Example: For $u(x, y)$ and $v(x, y)$

$$f = u_x v_y - u_{2x} v_y - u_y v_x + u_{xy} v_x$$

- Find $\mathbf{F} = \text{Div}^{-1} f$ so, $f = \text{Div } \mathbf{F}$
- Result (by hand):

$$\tilde{\mathbf{F}} = (u v_y - u_x v_y, -u v_x + u_x v_x)$$

Mathematica cannot do this!

Can this be done without integration by parts?

Can this be reduced to single integral in one variable?

- In 2D or 3D:

Example: For $u(x, y)$ and $v(x, y)$

$$f = u_x v_y - u_{2x} v_y - u_y v_x + u_{xy} v_x$$

- Find $\mathbf{F} = \text{Div}^{-1} f$ so, $f = \text{Div } \mathbf{F}$
- Result (by hand):

$$\tilde{\mathbf{F}} = (u v_y - u_x v_y, -u v_x + u_x v_x)$$

Mathematica cannot do this!

Can this be done without integration by parts?

Can this be reduced to single integral in one variable?

Yes! With the Homotopy operator

Tools from Differential Geometry

Higher Euler Operators

- In 1D (with variable x):

$$\mathcal{L}_{u^{(j)}(x)}^{(i)} = \sum_{k=i}^{M_x^{(j)}} \binom{k}{i} (-D_x)^{k-i} \frac{\partial}{\partial u_{kx}^{(j)}}$$

Examples for component $u(x)$:

$$\mathcal{L}_{u(x)}^{(1)} = \frac{\partial}{\partial u_x} - 2D_x \frac{\partial}{\partial u_{2x}} + 3D_x^2 \frac{\partial}{\partial u_{3x}} - 4D_x^3 \frac{\partial}{\partial u_{4x}} + \dots$$

$$\mathcal{L}_{u(x)}^{(2)} = \frac{\partial}{\partial u_{2x}} - 3D_x \frac{\partial}{\partial u_{3x}} + 6D_x^2 \frac{\partial}{\partial u_{4x}} - 10D_x^3 \frac{\partial}{\partial u_{5x}} + \dots$$

$$\mathcal{L}_{u(x)}^{(3)} = \frac{\partial}{\partial u_{3x}} - 4D_x \frac{\partial}{\partial u_{4x}} + 10D_x^2 \frac{\partial}{\partial u_{5x}} - 20D_x^3 \frac{\partial}{\partial u_{6x}} + \dots$$

- In 2D (with variables x and y):

$$\mathcal{L}_{u^{(j)}(x,y)}^{(i_x,i_y)} = \sum_{k_x=i_x}^{M_x^{(j)}} \sum_{k_y=i_y}^{M_y^{(j)}} \binom{k_x}{i_x} \binom{k_y}{i_y} (-D_x)^{k_x-i_x} (-D_y)^{k_y-i_y} \frac{\partial}{\partial u_{k_x x k_y y}^{(j)}}$$

Examples for component $u(x, y)$:

$$\mathcal{L}_{u(x,y)}^{(1,0)} = \frac{\partial}{\partial u_x} - 2D_x \frac{\partial}{\partial u_{2x}} - D_y \frac{\partial}{\partial u_{xy}} + 3D_x^2 \frac{\partial}{\partial u_{3x}} + \dots$$

$$\mathcal{L}_{u(x,y)}^{(0,1)} = \frac{\partial}{\partial u_y} - 2D_y \frac{\partial}{\partial u_{2y}} - D_x \frac{\partial}{\partial u_{yx}} + 3D_y^2 \frac{\partial}{\partial u_{3y}} + \dots$$

$$\mathcal{L}_{u(x,y)}^{(1,1)} = \frac{\partial}{\partial u_{xy}} - 2D_x \frac{\partial}{\partial u_{2xy}} - 2D_y \frac{\partial}{\partial u_{x2y}} + 3D_x^2 \frac{\partial}{\partial u_{3xy}} + \dots$$

$$\mathcal{L}_{u(x,y)}^{(2,1)} = \frac{\partial}{\partial u_{2xy}} - 3D_x \frac{\partial}{\partial u_{3xy}} - 2D_y \frac{\partial}{\partial u_{2x2y}} + 6D_x^2 \frac{\partial}{\partial u_{4xy}} + \dots$$

- In 3D (with variables $x, y,$ and z):

$$\mathcal{L}_{u^{(j)}(x,y,z)}^{(i_x, i_y, i_z)} = \sum_{k_x=i_x}^{M_x^{(j)}} \sum_{k_y=i_y}^{M_y^{(j)}} \sum_{k_z=i_z}^{M_z^{(j)}} \binom{k_x}{i_x} \binom{k_y}{i_y} \binom{k_z}{i_z} (-D_x)^{k_x-i_x} (-D_y)^{k_y-i_y} (-D_z)^{k_z-i_z} \frac{\partial}{\partial u_{k_x x k_y y k_z z}^{(j)}}$$

- Theorem:

[Kruskal et al, J. Math. Phys. **11** (1970) 952-960]

$$f = D_x^r F \quad \text{iff} \quad \mathcal{L}_{u^{(j)}(x)}^{(i)} f \equiv 0 \quad \text{for } i=0, 1, \dots, r-1, \\ j=1, \dots, N$$

Integration by Parts with Homotopy Operator

- Theorem (integration with homotopy operator):
 - In 1D: If f is exact then

$$F = D_x^{-1} f = \int f dx = \mathcal{H}_{\mathbf{u}(x)} f$$

- In 2D: If f is a divergence then

$$\mathbf{F} = \text{Div}^{-1} f = (\mathcal{H}_{\mathbf{u}(x,y)}^{(x)} f, \mathcal{H}_{\mathbf{u}(x,y)}^{(y)} f)$$

- In 3D: If f is a divergence then

$$\mathbf{F} = \text{Div}^{-1} f = (\mathcal{H}_{\mathbf{u}(x,y,z)}^{(x)} f, \mathcal{H}_{\mathbf{u}(x,y,z)}^{(y)} f, \mathcal{H}_{\mathbf{u}(x,y,z)}^{(z)} f)$$

Homotopy Operators

- In 1D (with variable x):

$$\mathcal{H}_{\mathbf{u}(x)} f = \int_0^1 \sum_{j=1}^N (I_{u^{(j)}} f)[\lambda \mathbf{u}] \frac{d\lambda}{\lambda}$$

with integrand

$$I_{u^{(j)}} f = \sum_{i=0}^{M_x^{(j)}-1} D_x^i \left(u^{(j)} \mathcal{L}_{u^{(j)}(x)}^{(i+1)} f \right)$$

N is the number of dependent variables and $(I_{u^{(j)}} f)[\lambda \mathbf{u}]$ means that in $I_{u^{(j)}} f$ one replaces $\mathbf{u}(x) \rightarrow \lambda \mathbf{u}(x)$, $\mathbf{u}_x(x) \rightarrow \lambda \mathbf{u}_x(x)$, etc.

Example: $\mathbf{u}(x) = (u^{(1)}(x), u^{(2)}(x)) = (u(x), v(x))$:

$$\mathcal{H}_{\mathbf{u}(x)} f = \int_0^1 (I_u f + I_v f) [\lambda \mathbf{u}] \frac{d\lambda}{\lambda}$$

where

$$I_u f = \sum_{i=0}^{M_x^{(1)}-1} D_x^i \left(u \mathcal{L}_{u(x)}^{(i+1)} f \right)$$

and

$$I_v f = \sum_{i=0}^{M_x^{(2)}-1} D_x^i \left(v \mathcal{L}_{v(x)}^{(i+1)} f \right)$$

- In 2D (with variables x and y):

$$\mathcal{H}_{\mathbf{u}(x,y)}^{(x)} f = \int_0^1 \sum_{j=1}^N (I_{u^{(j)}}^{(x)} f) [\lambda \mathbf{u}] \frac{d\lambda}{\lambda}$$

with

$$I_{u^{(j)}}^{(x)} f = \sum_{i_x=0}^{M_x^{(j)}-1} \sum_{i_y=0}^{M_y^{(j)}} \left(\frac{1 + i_x}{1 + i_x + i_y} \right) D_x^{i_x} D_y^{i_y} \left(u^{(j)} \mathcal{L}_{u^{(j)}(x,y)}^{(1+i_x, i_y)} f \right)$$

Analogous formulas for $\mathcal{H}_{\mathbf{u}(x,y)}^{(y)} f$ and $I_{u^{(j)}}^{(y)} f$

- In 3D (with variables $x, y,$ and z):

$$\mathcal{H}_{\mathbf{u}(x,y,z)}^{(x)} f = \int_0^1 \sum_{j=1}^N (I_{u^{(j)}}^{(x)} f) [\lambda \mathbf{u}] \frac{d\lambda}{\lambda}$$

with

$$I_{u^{(j)}}^{(x)} f =$$

$$\sum_{i_x=0}^{M_x^{(j)}-1} \sum_{i_y=0}^{M_y^{(j)}} \sum_{i_z=0}^{M_z^{(j)}} \left(\frac{1 + i_x}{1 + i_x + i_y + i_z} \right) D_x^{i_x} D_y^{i_y} D_z^{i_z} \left(u^{(j)} \mathcal{L}_{u^{(j)}(x,y,z)}^{(1+i_x, i_y, i_x)} f \right)$$

Analogous formulas for $\mathcal{H}_{\mathbf{u}(x,y,z)}^{(y)} f$, $\mathcal{H}_{\mathbf{u}(x,y,z)}^{(z)} f$,

$$I_{u^{(j)}}^{(y)} f, \text{ and } I_{u^{(j)}}^{(z)} f$$

Simplified Formula for Homotopy Operator

Homotopy Operator in 1D (with variable x)

$$\mathcal{H}_{\mathbf{u}(x)} f = \int_0^1 \sum_{j=1}^N (I_{u^{(j)}} f) [\lambda \mathbf{u}] \frac{d\lambda}{\lambda}$$

where

$$\begin{aligned} I_{u^{(j)}} f &= \sum_{i=0}^{M_x^{(j)}-1} D_x^i \left(u^{(j)} \mathcal{L}_{u^{(j)}(x)}^{(i+1)} f \right) \\ &= \sum_{i=0}^{M_x^{(j)}-1} u_{ix}^{(j)} \sum_{k=i+1}^{M_x^{(j)}} (-D_x)^{k-(i+1)} \frac{\partial f}{\partial u_{kx}^{(j)}} \end{aligned}$$

Application of Homotopy Operator in 1D

Example:

$$f = 3u_x v^2 \sin u - u_x^3 \sin u - 6v v_x \cos u + 2u_x u_{2x} \cos u + 8v_x v_{2x}$$

- Compute

$$\begin{aligned} I_u f &= u \frac{\partial f}{\partial u_x} + u_x \frac{\partial f}{\partial u_{2x}} - u D_x \frac{\partial f}{\partial u_{2x}} \\ &= 3uv^2 \sin u - uu_x^2 \sin u + 2u_x^2 \cos u \end{aligned}$$

- Similarly,

$$\begin{aligned}
 I_v f &= v \frac{\partial f}{\partial v_x} + v_x \frac{\partial f}{\partial v_{2x}} - v D_x \frac{\partial f}{\partial v_{2x}} \\
 &= -6v^2 \cos u + 8v_x^2
 \end{aligned}$$

- Finally,

$$\begin{aligned}
 F &= \mathcal{H}_{\mathbf{u}(x)} f = \int_0^1 (I_u f + I_v f) [\lambda \mathbf{u}] \frac{d\lambda}{\lambda} \\
 &= \int_0^1 \left(3\lambda^2 u v^2 \sin(\lambda u) - \lambda^2 u u_x^2 \sin(\lambda u) + 2\lambda u_x^2 \cos(\lambda u) \right. \\
 &\quad \left. - 6\lambda v^2 \cos(\lambda u) + 8\lambda v_x^2 \right) d\lambda \\
 &= 4v_x^2 + u_x^2 \cos u - 3v^2 \cos u
 \end{aligned}$$

Why does this work?

Sketch of Derivation and Proof

(in 1D with variable x , and for one component u)

Definition: Degree operator \mathcal{M}

$$\mathcal{M}f = \sum_{i=0}^M u_{ix} \frac{\partial f}{\partial u_{ix}} = u \frac{\partial f}{\partial u} + u_x \frac{\partial f}{\partial u_x} + u_{2x} \frac{\partial f}{\partial u_{2x}} + \cdots + u_{Mx} \frac{\partial f}{\partial u_{Mx}}$$

f is of order M in x

Example: $f = u^p u_x^q u_{3x}^r$ (p, q, r non-negative integers)

$$g = \mathcal{M}f = \sum_{i=0}^3 u_{ix} \frac{\partial f}{\partial u_{ix}} = (p + q + r) u^p u_x^q u_{3x}^r$$

Application of \mathcal{M} computes the total *degree*

Theorem (Inverse operator) $\mathcal{M}^{-1}g(u) = \int_0^1 g[\lambda u] \frac{d\lambda}{\lambda}$

Proof:

$$\frac{d}{d\lambda} g[\lambda u] = \sum_{i=0}^M \frac{\partial g[\lambda u]}{\partial \lambda u_{ix}} \frac{d\lambda u_{ix}}{d\lambda} = \frac{1}{\lambda} \sum_{i=0}^M u_{ix} \frac{\partial g[\lambda u]}{\partial u_{ix}} = \frac{1}{\lambda} \mathcal{M}g[\lambda u]$$

Integrate both sides with respect to λ

$$\begin{aligned} \int_0^1 \frac{d}{d\lambda} g[\lambda u] d\lambda &= g[\lambda u] \Big|_{\lambda=0}^{\lambda=1} = g(u) - g(0) \\ &= \int_0^1 \mathcal{M}g[\lambda u] \frac{d\lambda}{\lambda} = \mathcal{M} \int_0^1 g[\lambda u] \frac{d\lambda}{\lambda} \end{aligned}$$

Assuming $g(0) = 0$,

$$\mathcal{M}^{-1}g(u) = \int_0^1 g[\lambda u] \frac{d\lambda}{\lambda}$$

Example:

If $g(u) = (p + q + r) u^p u_x^q u_{3x}^r$, then

$$g[\lambda u] = (p + q + r) \lambda^{p+q+r} u^p u_x^q u_{3x}^r$$

Hence,

$$\begin{aligned} \mathcal{M}^{-1}g &= \int_0^1 (p + q + r) \lambda^{p+q+r-1} u^p u_x^q u_{3x}^r d\lambda \\ &= u^p u_x^q u_{3x}^r \lambda^{p+q+r} \Big|_{\lambda=0}^{\lambda=1} = u^p u_x^q u_{3x}^r \end{aligned}$$

Theorem: If f is an exact differential function, then

$$F = \mathcal{D}_x^{-1} f = \int f dx = \mathcal{H}_{u(x)} f$$

Proof: Multiply

$$\mathcal{L}_{u(x)}^{(0)} f = \sum_{k=0}^M (-\mathcal{D}_x)^k \frac{\partial f}{\partial u_{kx}}$$

by u to restore the degree.

Split off $u \frac{\partial f}{\partial u}$. Integrate by parts.

Split off $u_x \frac{\partial f}{\partial u_x}$. Repeat the process.

Lastly, split off $u_{Mx} \frac{\partial f}{\partial u_{Mx}}$.

Explicitly,

$$\begin{aligned} u \mathcal{L}_{u(x)}^{(0)} f &= u \sum_{k=0}^M (-\mathcal{D}_x)^k \frac{\partial f}{\partial u_{kx}} \\ &= u \frac{\partial f}{\partial u} - \mathcal{D}_x \left(u \sum_{k=1}^M (-\mathcal{D}_x)^{k-1} \frac{\partial f}{\partial u_{kx}} \right) + u_x \sum_{k=1}^M (-\mathcal{D}_x)^{k-1} \frac{\partial f}{\partial u_{kx}} \\ &= u \frac{\partial f}{\partial u} + u_x \frac{\partial f}{\partial u_x} - \mathcal{D}_x \left(u \sum_{k=1}^M (-\mathcal{D}_x)^{k-1} \frac{\partial f}{\partial u_{kx}} \right. \\ &\quad \left. + u_x \sum_{k=2}^M (-\mathcal{D}_x)^{k-2} \frac{\partial f}{\partial u_{kx}} \right) + u_{2x} \sum_{k=2}^M (-\mathcal{D}_x)^{k-2} \frac{\partial f}{\partial u_{kx}} \\ &= \dots \end{aligned}$$

$$\begin{aligned}
&= u \frac{\partial f}{\partial u} + u_x \frac{\partial f}{\partial u_x} + \dots + u_{Mx} \frac{\partial f}{\partial u_{Mx}} \\
&\quad - \mathcal{D}_x \left(u \sum_{k=1}^M (-\mathcal{D}_x)^{k-1} \frac{\partial f}{\partial u_{kx}} + u_x \sum_{k=2}^M (-\mathcal{D}_x)^{k-2} \frac{\partial f}{\partial u_{kx}} \right. \\
&\quad \left. + \dots + u_{(M-1)x} \sum_{k=M}^M (-\mathcal{D}_x)^{k-M} \frac{\partial f}{\partial u_{kx}} \right) \\
&= \sum_{i=0}^M u_{ix} \frac{\partial f}{\partial u_{ix}} - \mathcal{D}_x \left(\sum_{i=0}^{M-1} u_{ix} \sum_{k=i+1}^M (-\mathcal{D}_x)^{k-(i+1)} \frac{\partial f}{\partial u_{kx}} \right) \\
&= \mathcal{M}f - \mathcal{D}_x \left(\sum_{i=0}^{M-1} u_{ix} \sum_{k=i+1}^M (-\mathcal{D}_x)^{k-(i+1)} \frac{\partial f}{\partial u_{kx}} \right) \\
&= 0
\end{aligned}$$

So,

$$\mathcal{M}f = \mathcal{D}_x \left(\sum_{i=0}^{M-1} u_{ix} \sum_{k=i+1}^M (-\mathcal{D}_x)^{k-(i+1)} \frac{\partial f}{\partial u_{kx}} \right)$$

Apply \mathcal{M}^{-1} and use $\mathcal{M}^{-1}\mathcal{D}_x = \mathcal{D}_x\mathcal{M}^{-1}$.

$$f = \mathcal{D}_x \left(\mathcal{M}^{-1} \sum_{i=0}^{M-1} u_{ix} \sum_{k=i+1}^M (-\mathcal{D}_x)^{k-(i+1)} \frac{\partial f}{\partial u_{kx}} \right)$$

Apply \mathcal{D}_x^{-1} and use the formula for \mathcal{M}^{-1}

$$\begin{aligned} F &= \mathcal{D}_x^{-1} f = \int_0^1 \left(\sum_{i=0}^{M-1} u_{ix} \sum_{k=i+1}^M (-\mathcal{D}_x)^{k-(i+1)} \frac{\partial f}{\partial u_{kx}} \right) [\lambda u] \frac{d\lambda}{\lambda} \\ &= \mathcal{H}_{u(x)} f \end{aligned}$$

using

$$\begin{aligned} I_u f &= \sum_{i=0}^{M-1} \mathcal{D}_x^i \left(u \mathcal{L}_{u(x)}^{(i+1)} f \right) \\ &= \sum_{i=0}^{M-1} \mathcal{D}_x^i \left(u \sum_{k=i+1}^M \binom{k}{i+1} (-\mathcal{D}_x)^{k-(i+1)} \frac{\partial f}{\partial u_{kx}} \right) \\ &= \sum_{i=0}^{M-1} u_{ix} \sum_{k=i+1}^M (-\mathcal{D}_x)^{k-(i+1)} \frac{\partial f}{\partial u_{kx}} \end{aligned}$$

The latter formula provides a **fast algorithm** to compute the integrand $I_u f$

Simplified Formula for Homotopy Operator

Homotopy Operator in 2D (with variables x and y)

$$\mathcal{H}_{\mathbf{u}(x,y)}^{(x)} f = \int_0^1 \sum_{j=1}^N (I_{u^{(j)}}^{(x)} f)[\lambda \mathbf{u}] \frac{d\lambda}{\lambda}$$

$$\mathcal{H}_{\mathbf{u}(x,y)}^{(y)} f = \int_0^1 \sum_{j=1}^N (I_{u^{(j)}}^{(y)} f)[\lambda \mathbf{u}] \frac{d\lambda}{\lambda}$$

where

$$\begin{aligned} \mathcal{I}_{u^{(j)}}^{(x)} f &= \sum_{i_x=0}^{M_x^{(j)}-1} \sum_{i_y=0}^{M_y^{(j)}} \binom{i_x+i_y}{i_x} u_{i_x x i_y y} \sum_{k_x=i_x+1}^{M_x^{(j)}} \sum_{k_y=i_y}^{M_y^{(j)}} \frac{\binom{k_x+k_y-i_x-i_y-1}{k_x-i_x-1}}{\binom{k_x+k_y}{k_x}} \\ &\quad (-\mathcal{D}_x)^{k_x-i_x-1} (-\mathcal{D}_y)^{k_y-i_y} \frac{\partial f}{\partial u_{k_x x k_y y}^{(j)}} \end{aligned}$$

and

$$\begin{aligned} \mathcal{I}_{u^{(j)}}^{(y)} f &= \sum_{i_x=0}^{M_x^{(j)}} \sum_{i_y=0}^{M_y^{(j)}-1} \binom{i_x+i_y}{i_y} u_{i_x x i_y y} \sum_{k_x=i_x}^{M_x^{(j)}} \sum_{k_y=i_y+1}^{M_y^{(j)}} \frac{\binom{k_x+k_y-i_x-i_y-1}{k_y-i_y-1}}{\binom{k_x+k_y}{k_y}} \\ &\quad (-\mathcal{D}_x)^{k_x-i_x} (-\mathcal{D}_y)^{k_y-i_y-1} \frac{\partial f}{\partial u_{k_x x k_y y}^{(j)}} \end{aligned}$$

Simplified Formula for Homotopy Operator

Homotopy Operator in 3D (with variables x, y , and z)

$$\mathcal{H}_{\mathbf{u}(x,y,z)}^{(x)} f = \int_0^1 \sum_{j=1}^N (I_{u^{(j)}}^{(x)} f) [\lambda \mathbf{u}] \frac{d\lambda}{\lambda}$$

$$\mathcal{H}_{\mathbf{u}(x,y,z)}^{(y)} f = \int_0^1 \sum_{j=1}^N (I_{u^{(j)}}^{(y)} f) [\lambda \mathbf{u}] \frac{d\lambda}{\lambda}$$

$$\mathcal{H}_{\mathbf{u}(x,y,z)}^{(z)} f = \int_0^1 \sum_{j=1}^N (I_{u^{(j)}}^{(z)} f) [\lambda \mathbf{u}] \frac{d\lambda}{\lambda}$$

where

$$\begin{aligned}
 & I_{u^{(j)}(x,y,z)}^{(x)} f \\
 &= \sum_{i_x=0}^{M_x^{(j)}-1} \sum_{i_y=0}^{M_y^{(j)}} \sum_{i_z=0}^{M_z^{(j)}} \binom{i_x+i_y+i_z}{i_x} \binom{i_y+i_z}{i_y} u_{i_x x i_y y i_z z}^{(j)} \\
 & \quad \sum_{k_x=i_x+1}^{M_x^{(j)}} \sum_{k_y=i_y}^{M_y^{(j)}} \sum_{k_z=i_z}^{M_z^{(j)}} \frac{\binom{k_x+k_y+k_z-i_x-i_y-i_z-1}{k_x-i_x-1} \binom{k_y+k_z-i_y-i_z}{k_y-i_y}}{\binom{k_x+k_y+k_z}{k_x} \binom{k_y+k_z}{k_y}} \\
 & \quad (-\mathcal{D}_x)^{k_x-i_x-1} (-\mathcal{D}_y)^{k_y-i_y} (-\mathcal{D}_z)^{k_z-i_z} \frac{\partial f}{\partial u_{k_x x k_y y k_z z}^{(j)}}
 \end{aligned}$$

Integrands $I_{u^{(j)}(x,y,z)}^{(y)} f$ and $I_{u^{(j)}(x,y,z)}^{(z)} f$ are similar

Application of Homotopy Operator in 2D

Example: $f = uv_y - u_{2x}v_y - u_yv_x + u_{xy}v_x$

Recall (by hand): $\tilde{\mathbf{F}} = (uv_y - u_xv_y, -uv_x + u_xv_x)$

- Compute

$$\begin{aligned} I_u^{(x)} f &= u \frac{\partial f}{\partial u_x} + u_x \frac{\partial f}{\partial u_{2x}} - u D_x \frac{\partial f}{\partial u_{2x}} \\ &\quad + \frac{1}{2} u_y \frac{\partial f}{\partial u_{xy}} - \frac{1}{2} u D_y \frac{\partial f}{\partial u_{xy}} \\ &= uv_y + \frac{1}{2} u_y v_x - u_x v_y + \frac{1}{2} u v_{xy} \end{aligned}$$

- Similarly,

$$I_v^{(x)} f = v \frac{\partial f}{\partial v_x} = -u_y v + u_{xy} v$$

- Hence,

$$\begin{aligned} F_1 &= \mathcal{H}_{\mathbf{u}(x,y)}^{(x)} f = \int_0^1 \left(I_u^{(x)} f + I_v^{(x)} f \right) [\lambda \mathbf{u}] \frac{d\lambda}{\lambda} \\ &= \int_0^1 \lambda \left(uv_y + \frac{1}{2} u_y v_x - u_x v_y + \frac{1}{2} u v_{xy} - u_y v + u_{xy} v \right) d\lambda \\ &= \frac{1}{2} uv_y + \frac{1}{4} u_y v_x - \frac{1}{2} u_x v_y + \frac{1}{4} u v_{xy} - \frac{1}{2} u_y v + \frac{1}{2} u_{xy} v \end{aligned}$$

- Analogously,

$$\begin{aligned}
 F_2 &= \mathcal{H}_{\mathbf{u}(x,y)}^{(y)} f = \int_0^1 \left(I_u^{(y)} f + I_v^{(y)} f \right) [\lambda \mathbf{u}] \frac{d\lambda}{\lambda} \\
 &= \int_0^1 \left(\lambda \left(-uv_x - \frac{1}{2}uv_{2x} + \frac{1}{2}u_xv_x \right) + \lambda (u_xv - u_{2x}v) \right) d\lambda \\
 &= -\frac{1}{2}uv_x - \frac{1}{4}uv_{2x} + \frac{1}{4}u_xv_x + \frac{1}{2}u_xv - \frac{1}{2}u_{2x}v
 \end{aligned}$$

- So,

$$\mathbf{F} = \begin{pmatrix} \frac{1}{2}uv_y + \frac{1}{4}u_yv_x - \frac{1}{2}u_xv_y + \frac{1}{4}uv_{xy} - \frac{1}{2}u_yv + \frac{1}{2}u_{xy}v \\ -\frac{1}{2}uv_x - \frac{1}{4}uv_{2x} + \frac{1}{4}u_xv_x + \frac{1}{2}u_xv - \frac{1}{2}u_{2x}v \end{pmatrix}$$

Let $\mathbf{K} = \tilde{\mathbf{F}} - \mathbf{F}$ then

$$\mathbf{K} = \begin{pmatrix} \frac{1}{2}uv_y - \frac{1}{4}u_yv_x - \frac{1}{2}u_xv_y - \frac{1}{4}uv_{xy} + \frac{1}{2}u_yv - \frac{1}{2}u_{xy}v \\ -\frac{1}{2}uv_x + \frac{1}{4}uv_{2x} + \frac{3}{4}u_xv_x - \frac{1}{2}u_xv + \frac{1}{2}u_{2x}v \end{pmatrix}$$

then $\text{Div } \mathbf{K} = 0$

- Also, $\mathbf{K} = (D_y\theta, -D_x\theta)$ with $\theta = \frac{1}{2}uv - \frac{1}{4}uv_x - \frac{1}{2}u_xv$
(*curl* in 2D)

Needed: Algorithm to remove curl terms!

Computation of Conservation Laws for SWW

Quick Recapitulation

- Conservation law in $(2 + 1)$ dimensions

$$\boxed{D_t \rho + \nabla \cdot \mathbf{J} = D_t \rho + D_x J_1 + D_y J_2 = 0} \quad (\text{on PDE})$$

conserved density ρ and flux $\mathbf{J} = (J_1, J_2)$

- **Example:** Shallow water wave (SWW) equations

$$u_t + uu_x + vu_y - 2\Omega v + \frac{1}{2}h\theta_x + \theta h_x = 0$$

$$v_t + uv_x + vv_y + 2\Omega u + \frac{1}{2}h\theta_y + \theta h_y = 0$$

$$\theta_t + u\theta_x + v\theta_y = 0$$

$$h_t + hu_x + uh_x + hv_y + vh_y = 0$$

- Typical density-flux pair:

$$\rho^{(5)} = v_x \theta - u_y \theta + 2\Omega \theta$$

$$\mathbf{J}^{(5)} = \frac{1}{2} \begin{pmatrix} 4\Omega u \theta - 2u u_y \theta + 2u v_x \theta - h \theta \theta_y \\ 4\Omega v \theta + 2v v_x \theta - 2v u_y \theta + h \theta \theta_x \end{pmatrix}$$

Algorithm

- **Step 1: Construct the form of the density**

The SWW equations are invariant under the scaling symmetries

$$(x, y, t, u, v, \theta, h, \Omega) \rightarrow (\lambda^{-1}x, \lambda^{-1}y, \lambda^{-2}t, \lambda u, \lambda v, \lambda\theta, \lambda h, \lambda^2\Omega)$$

and

$$(x, y, t, u, v, \theta, h, \Omega) \rightarrow (\lambda^{-1}x, \lambda^{-1}y, \lambda^{-2}t, \lambda u, \lambda v, \lambda^2\theta, \lambda^0 h, \lambda^2\Omega)$$

Construct a **candidate density**, for example,

$$\rho = c_1\Omega\theta + c_2u_y\theta + c_3v_y\theta + c_4u_x\theta + c_5v_x\theta$$

which is scaling invariant under *both* symmetries

- **Step 2: Determine the constants c_i**

Compute $E = -D_t \rho$ and remove time derivatives

$$\begin{aligned}
 E &= -\left(\frac{\partial \rho}{\partial u_x} u_{tx} + \frac{\partial \rho}{\partial u_y} u_{ty} + \frac{\partial \rho}{\partial v_x} v_{tx} + \frac{\partial \rho}{\partial v_y} v_{ty} + \frac{\partial \rho}{\partial \theta} \theta_t\right) \\
 &= c_4 \theta (uu_x + vu_y - 2\Omega v + \frac{1}{2} h \theta_x + \theta h_x)_x \\
 &\quad + c_2 \theta (uu_x + vu_y - 2\Omega v + \frac{1}{2} h \theta_x + \theta h_x)_y \\
 &\quad + c_5 \theta (uv_x + vv_y + 2\Omega u + \frac{1}{2} h \theta_y + \theta h_y)_x \\
 &\quad + c_3 \theta (uv_x + vv_y + 2\Omega u + \frac{1}{2} h \theta_y + \theta h_y)_y \\
 &\quad + (c_1 \Omega + c_2 u_y + c_3 v_y + c_4 u_x + c_5 v_x) (u \theta_x + v \theta_y)
 \end{aligned}$$

Require that

$$\mathcal{L}_{u(x,y)}^{(0,0)} E = \mathcal{L}_{v(x,y)}^{(0,0)} E = \mathcal{L}_{\theta(x,y)}^{(0,0)} E = \mathcal{L}_{h(x,y)}^{(0,0)} E \equiv 0$$

- Solution: $c_1 = 2, c_2 = -1, c_3 = c_4 = 0, c_5 = 1$ gives

$$\rho = 2\Omega\theta - u_y\theta + v_x\theta$$

- **Step 3: Compute the flux \mathbf{J}**

$$\begin{aligned} E = & \theta(u_x v_x + u v_{2x} + v_x v_y + v v_{xy} + 2\Omega u_x \\ & + \frac{1}{2}\theta_x h_y - u_x u_y - u u_{xy} - u_y v_y - u_{2y} v \\ & + 2\Omega v_y - \frac{1}{2}\theta_y h_x) \\ & + 2\Omega u \theta_x + 2\Omega v \theta_y - u u_y \theta_x \\ & - u_y v \theta_y + u v_x \theta_x + v v_x \theta_y \end{aligned}$$

Apply the 2D homotopy operator:

$$\mathbf{J} = (J_1, J_2) = \text{Div}^{-1} E = (\mathcal{H}_{\mathbf{u}(x,y)}^{(x)} E, \mathcal{H}_{\mathbf{u}(x,y)}^{(y)} E)$$

Compute

$$\begin{aligned} I_u^{(x)} E &= u \frac{\partial E}{\partial u_x} + u_x \frac{\partial E}{\partial u_{2x}} - u D_x \frac{\partial E}{\partial u_{2x}} + \frac{1}{2} u_y \frac{\partial E}{\partial u_{xy}} - \frac{1}{2} u D_y \frac{\partial E}{\partial u_{xy}} \\ &= uv_x \theta + 2\Omega u \theta + \frac{1}{2} u^2 \theta_y - uu_y \theta \end{aligned}$$

Similarly, compute

$$I_v^{(x)} E = vv_y \theta + \frac{1}{2} v^2 \theta_y + uv_x \theta$$

$$I_\theta^{(x)} E = \frac{1}{2} \theta^2 h_y + 2\Omega u \theta - uu_y \theta + uv_x \theta$$

$$I_h^{(x)} E = -\frac{1}{2} \theta \theta_y h$$

Next,

$$\begin{aligned} J_1 &= \mathcal{H}_{\mathbf{u}(x,y)}^{(x)} E \\ &= \int_0^1 \left(I_u^{(x)} E + I_v^{(x)} E + I_\theta^{(x)} E + I_h^{(x)} E \right) [\lambda \mathbf{u}] \frac{d\lambda}{\lambda} \\ &= \int_0^1 \left(4\lambda\Omega u\theta + \lambda^2 \left(3uv_x\theta + \frac{1}{2}u^2\theta_y - 2uu_y\theta + vv_y\theta \right. \right. \\ &\quad \left. \left. + \frac{1}{2}v^2\theta_y + \frac{1}{2}\theta^2 h_y - \frac{1}{2}\theta\theta_y h \right) \right) d\lambda \\ &= 2\Omega u\theta - \frac{2}{3}uu_y\theta + uv_x\theta + \frac{1}{3}vv_y\theta + \frac{1}{6}u^2\theta_y \\ &\quad + \frac{1}{6}v^2\theta_y - \frac{1}{6}h\theta\theta_y + \frac{1}{6}h_y\theta^2 \end{aligned}$$

Analogously,

$$\begin{aligned} J_2 &= \mathcal{H}_{\mathbf{u}(x,y)}^{(y)} E \\ &= 2\Omega v\theta + \frac{2}{3}vv_x\theta - vu_y\theta - \frac{1}{3}uu_x\theta - \frac{1}{6}u^2\theta_x - \frac{1}{6}v^2\theta_x \\ &\quad + \frac{1}{6}h\theta\theta_x - \frac{1}{6}h_x\theta^2 \end{aligned}$$

Hence,

$$\mathbf{J} = \frac{1}{6} \begin{pmatrix} 12\Omega u\theta - 4uu_y\theta + 6uv_x\theta + 2vv_y\theta + u^2\theta_y + v^2\theta_y - h\theta\theta_y + h_y\theta^2 \\ 12\Omega v\theta + 4vv_x\theta - 6vu_y\theta - 2uu_x\theta - u^2\theta_x - v^2\theta_x + h\theta\theta_x - h_x\theta^2 \end{pmatrix}$$

After removing the curl term (by hand)

$$\tilde{\mathbf{j}}^{(5)} = \frac{1}{2} \begin{pmatrix} 4\Omega u\theta - 2uu_y\theta + 2uv_x\theta - h\theta\theta_y \\ 4\Omega v\theta + 2vv_x\theta - 2vu_y\theta + h\theta\theta_x \end{pmatrix}$$

Needed: Algorithm to remove curl terms!

Conclusions and Future Work

- Usefulness of the homotopy operator
- Integration by parts, D_x^{-1} , and Div^{-1}
- **To do:** Integration of non-exact differential functions

Example: $f = u_x v + u v_x + u^2 u_{2x}$

$$\int f dx = uv + \int u^2 u_{2x} dx$$

- **To do:** Integration of parametrized differential functions

Example: $f = a u_x v + b u v_x$

$$\int f dx = uv \text{ if } a = b$$

-
- Algorithm to compute conservation laws in multi-dimensions
- **To do:** various PDEs (other than those of evolution type)
- **To do:** Full implementation in *Mathematica*

Implementation in Mathematica & Software

Codes are available via the Internet

URL: http://www.mines.edu/fs_home/whereman/
and via anonymous FTP from mines.edu in directory
[pub/papers/math_cs_dept/software/](ftp://mines.edu/pub/papers/math_cs_dept/software/)

Publications

1. W. Hereman, M. Colagrosso, R. Sayers, A. Ringler, B. Deconinck, M. Nivala, and M. S. Hickman, Continuous and Discrete Homotopy Operators and the Computation of Conservation Laws. In: Differential Equations with Symbolic Computation, Eds.: D. Wang and Z. Zheng, Birkhäuser Verlag, Basel (2005), Chapter 15, pp. 249-285.

2. W. Hereman, Symbolic computation of conservation laws of nonlinear partial differential equations in multi-dimensions, Int. J. Quan. Chem. **106**(1), 278-299 (2006).
3. W. Hereman, B. Deconinck, and L. D. Poole, Continuous and discrete homotopy operators: A theoretical approach made concrete, Math. Comput. Simul. (2006) submitted.