Symbolic Computation of Exact Solutions of Nonlinear Wave Equations

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I. INTRODUCTION

Symbolic Software for Soliton Theory

- Painlevé test for single ODE or PDE
- Lie symmetries of systems of PDEs
- Solitary wave solutions via real exponential approach
- Test for and construction of soliton solutions via Hirota's method

All the above programs are written in MACSYMA

- Hirota's Direct Method
  allows to construct exact soliton solutions of
  - nonlinear evolution equations
  - wave equations
  - coupled systems
- Conditions for existence of soliton solutions (integrability)
- Algorithm
- MACSYMA implementation
- Syntax of the Code
- Examples:
  - Korteweg-de Vries equation (KdV)
  - Kadomtsev-Petviashvili equation (KP)
  - Sawada-Kotera equation (SK)
- Single equation
- Coupled system (two equations): work in progress

II. HIROTA'S METHOD

Hirota's method requires:

- a clever change of dependent variable
- the introduction of a novel differential operator
- a perturbation expansion to solve the resulting bilinear equation

Korteweg-de Vries equation

$$u_t + 6uu_x + u_{3x} = 0$$

Substitute

$$u(x,t) = 2\frac{\partial^2 \ln f(x,t)}{\partial x^2}$$

Integrate with respect to x

$$f f_{xt} - f_x f_t + f f_{4x} - 4 f_x f_{3x} + 3 f_{2x}^2 = 0$$

Bilinear form

$$B(f\cdot f) \stackrel{\text{def}}{=} \left(D_x D_t + D_x^4\right)(f\cdot f) = 0$$

New operator

$$D_x^m \, D_t^n (f \cdot g) = (\partial x - \partial x')^m (\partial t - \partial t')^n f(x,t) \, g(x',t')|_{x'=x, t'=t}$$

Expansion

$$f = 1 + \sum_{n=1}^{\infty} \epsilon^n \, f_n$$

Substitute f into the bilinear equation

Collect powers in ϵ (book keeping parameter)

$$\begin{aligned}
O(\epsilon^0) &: \; B(1 \cdot 1) = 0 \\
O(\epsilon^1) &: \; B(1 \cdot f_1 + f_1 \cdot 1) = 0 \\
O(\epsilon^2) &: \; B(1 \cdot f_2 + f_1 \cdot f_1 + f_2 \cdot 1) = 0 \\
O(\epsilon^3) &: \; B(1 \cdot f_3 + f_1 \cdot f_2 + f_2 \cdot f_1 + f_3 \cdot 1) = 0 \\
O(\epsilon^4) &: \; B(1 \cdot f_4 + f_1 \cdot f_3 + f_2 \cdot f_2 + f_3 \cdot f_1 + f_4 \cdot 1) = 0 \\
O(\epsilon^n) &: \; B(\sum_{j=0}^{n} f_j \cdot f_{n-j}) = 0 \qquad \text{with } \; f_0 = 1
\end{aligned}$$

If the original PDE admits a N-soliton solution
then the expansion will truncate at level $n = N$ provided

$$f_1 = \sum_{i=1}^{N} \exp(\theta_i) = \sum_{i=1}^{N} \exp\left(k_i\, x - \omega_i\, t + \delta_i\right)$$

k_i, ω_i and δ_i are constants

Dispersion law

$$\omega_i = k_i^3 \qquad i = 1, 2, ..., N$$

Consider the case N=3
Terms generated by $B(f_1, f_1)$ justify

$$\begin{aligned}
f_2 &= a_{12}\, \exp(\theta_1 + \theta_2) + a_{13}\, \exp(\theta_1 + \theta_3) + a_{23}\, \exp(\theta_2 + \theta_3) \\
&= a_{12}\, \exp\left[(k_1 + k_2)\, x - (\omega_1 + \omega_2)\, t + (\delta_1 + \delta_2)\right] \\
&+ a_{13}\, \exp\left[(k_1 + k_3)\, x - (\omega_1 + \omega_3)\, t + (\delta_1 + \delta_3)\right] \\
&+ a_{23}\, \exp\left[(k_2 + k_3)\, x - (\omega_2 + \omega_3)\, t + (\delta_2 + \delta_3)\right]
\end{aligned}$$

Allows to calculate the constants a_{12}, a_{13} and a_{23}
One obtains

$$a_{ij} = \frac{\left(k_i - k_j\right)^2}{\left(k_i + k_j\right)^2} \qquad i, j = 1, 2, 3$$

$B(f_1 \cdot f_2 + f_2 \cdot f_1)$ motivates

$$\begin{aligned} f_3 &= b_{123} \exp(\theta_1 + \theta_2 + \theta_3) \\ &= b_{123} \exp\left[(k_1 + k_2 + k_3)x - (\omega_1 + \omega_2 + \omega_3)t + (\delta_1 + \delta_2 + \delta_3)\right] \end{aligned}$$

with

$$b_{123} = a_{12}\, a_{13}\, a_{23} = \frac{(k_1 - k_2)^2 (k_1 - k_3)^2 (k_2 - k_3)^2}{(k_1 + k_2)^2 (k_1 + k_3)^2 (k_2 + k_3)^2}$$

Subsequently, $f_i = 0$ for $i > 3$

Set $\epsilon = 1$

$$\begin{aligned} f &= 1 + \exp\theta_1 + \exp\theta_2 + \exp\theta_3 \\ &+ a_{12} \exp(\theta_1 + \theta_2) + a_{13} \exp(\theta_1 + \theta_3) + a_{23} \exp(\theta_2 + \theta_3) \\ &+ b_{123} \exp(\theta_1 + \theta_2 + \theta_3) \end{aligned}$$

Return to the original $u(x,t)$

$$u(x,t) = 2\frac{\partial^2 \ln f(x,t)}{\partial x^2}$$

III. Hirota conditions

Bilinear equation

$$P(D_x, D_t) f \cdot f = 0$$

P is an arbitrary polynomial

- Single soliton solution

$$f = 1 + e^{\theta}\ , \qquad \theta = kx - \omega t + \delta$$

  k, ω and δ are constants

  k and ω satisfy the dispersion law

$$P(k, -\omega) = 0$$

- Two soliton solution

$$f = 1 + e^{\theta_1} + e^{\theta_2} + a_{12} e^{\theta_1 + \theta_2}$$

$$\theta_i = k_i x - \omega_i t + \delta_i\ , \quad P(k_i, -\omega_i) = 0 \qquad i = 1, 2$$

$$a_{12} = -\frac{P(k_1 - k_2, -\omega_1 + \omega_2)}{P(k_1 + k_2, -\omega_1 - \omega_2)}$$

If a bilinear form is available then the equation always has at least a two-soliton solution provided

* $P(0,0) = 0$ no constant term
* $P(k,\omega) = P(-k,-\omega)$ P is even

- For the general N-soliton solution

$$f = \sum_{\mu=0,1} exp[\sum_{i>j}^{(N)} A_{ij}\mu_i\mu_j + \sum_{i=1}^{N} \mu_i\theta_i]$$

$$a_{ij} = \exp A_{ij} = -\frac{P(k_i - k_j, -\omega_i + \omega_j)}{P(k_i + k_j, -\omega_i - \omega_j)}$$

$$\begin{aligned} S[P,n] \; = \; & \sum_{\sigma=\pm 1} P(\sum_{i=1}^{n} \sigma_i k_{s_i}, -\sum_{i=1}^{n} \sigma_i \omega_{s_i}) \\ & \times \prod_{i>j}^{(n)} (\sigma_i k_{s_i} - \sigma_j k_{s_j}, -\sigma_i\omega_{s_i} + \sigma_j\omega_{s_j})\sigma_i\sigma_j = 0 \end{aligned}$$

$$n = 1,\ldots,N\,. \quad s_i \in \{1,\ldots,N\}\,, \quad k_i > k_j\,, \quad i > j$$

IV. ALGORITHM FOR THE HIROTA METHOD

- Blocks (functions)
  - Block 1: Dispersion law

$$B(1{\cdot}f_1 + f_1{\cdot}1) = 0$$

  - Block 2: Test the conditions for 2 or 3 soliton solution
  - Block 3: Construct a N-soliton solution ($N = 1, 2, 3$)
  - Block 4: Check two soliton solution (polynomials)
  - Block 5: Check three soliton solution (polynomials)
  - Block 6: Hirota operators Dx, Dy, Dt, Dxt
  - Block 7: Hirota's method
- Main Program
  Hirota(B,name,n,test_for_3soliton,
  check_coefficients,test_for_4soliton)
  $B(f, g)$: Bilinear operator for the PDE
  name: Name of the PDE
  n: N-soliton solution

test_for_3soliton: True or false for the testing the Hirota conditions
check_coefficients: True or false for checking the calculated coefficients of the 2 and 3 soliton solutions
test_for_4soliton: True or false for the testing the Hirota conditions

V. MACSYMA PROGRAM

The symbolic program calculates

- the one soliton solution
- checks conditions for a 3 or even a 4 soliton solution
- constructs the two and three soliton solutions
- recalculates a_{ij} based on the polynomial form P
- verifies if $b_{123} = a_{12}a_{13}a_{23}$

The user must

- select the value of N
- provide the bilinear operator B
- give a name for the PDE
- set true or false for 'test_for_3soliton', 'check_coefficients' and 'test_for_4soliton'

VI. EXAMPLES AND TEST CASES

- Korteweg-de Vries equation

$$u_t + 6uu_x + u_{3x} = 0$$

One uses

$$u(x,t) = 2\frac{\partial^2 \ln f(x,t)}{\partial x^2}$$

$$B(f,g) := Dxt[1,1](f,g) + Dx[4](f,g)$$

One obtains

$$\omega_i = k_i^3, \quad i = 1,2,3$$

and

$$a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}, \quad i,j = 1,2,3 \quad i > j$$

$$b_{123} = a_{12}a_{13}a_{23}$$

There also exists a four soliton solution

- Kadomtsev-Petviashvili equation

$$(u_t + 6uu_x + u_{3x})_x + 3u_{2y} = 0$$

Here

$$u(x,t) = 2\frac{\partial^2 \ln f(x,y,t)}{\partial x^2}$$

$$B(f,g) := Dxt[1,1](f,g) + Dx[4](f,g) + 3Dy[2](f,g)$$

In this case $\theta_i = k_i x + l_i y - \omega_i t$

$$\omega_i = \frac{3\,l_i^2 + k_i^4}{k_i}\,, \quad i = 1,2,3$$

and

$$a_{ij} = \frac{(k_i\,k_j^2 - k_i^2\,k_j - l_i\,k_j + l_j\,k_i)(k_i\,k_j^2 - k_i^2\,k_j + l_i\,k_j - l_j\,k_i)}{(k_i\,k_j^2 + k_i^2\,k_j + l_i\,k_j - l_j\,k_i)(k_i\,k_j^2 + k_i^2\,k_j - l_i\,k_j + l_j\,k_i)}$$

$$b_{123} = a_{12}a_{13}a_{23}$$

There also exists a four soliton solution

- Sawada-Kotera equation

$$u_t + 45u^2 u_x + 15u_x u_{2x} + 15uu_{3x} + u_{5x} = 0$$

One uses

$$u(x,t) = 2\frac{\partial^2 \ln f(x,t)}{\partial x^2}$$

and

$$B(f,g) := Dxt[1,1](f,g) + Dx[6](f,g)$$

Furthermore

$$\omega_i = k_i^5 \,, \qquad i = 1,2,3$$

$$a_{ij} = \frac{(k_i - k_j)^3 \, (k_i^3 + k_j^3)}{(k_i + k_j)^3 \, (k_i^3 - k_j^3)} \,, \qquad i,j = 1,2,3$$

$$b_{123} = a_{12} a_{13} a_{23}$$

There also exists a four soliton solution

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