Symbolic Computation of Conservation Laws of Nonlinear Partial Differential Equations

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Examples of Conservation Laws

Example 1: Traffic Flow

Modeling the density of cars (Bressan, 2009)

\( u(x, t) \) density of cars on a highway (e.g., number of cars per 100 meters).

\( s(u) \) mean (equilibrium) speed of the cars (depends on the density).
Change in number of cars in segment \([a, b]\) equals the difference between cars entering at \(a\) and leaving at \(b\) during time interval \([t_1, t_2]\):

\[
\int_a^b \left( u(x, t_2) - u(x, t_1) \right) \, dx = \int_{t_1}^{t_2} \left( J(a, t) - J(b, t) \right) \, dt
\]

\[
\int_a^b \left( \int_{t_1}^{t_2} u_t(x, t) \, dt \right) \, dx = -\int_{t_1}^{t_2} \left( \int_a^b J_x(x, t) \, dx \right) \, dt
\]

where \(J(x, t) = u(x, t)s(u(x, t))\) is the traffic flow (e.g., in cars per hour) at location \(x\) and time \(t\).
Then, $\int_{a}^{b} \int_{t_1}^{t_2} (u_t + J_x) \, dt \, dx = 0$ holds $\forall (a, b, t_1, t_2)$

Yields the conservation law:

\[
\begin{align*}
  u_t + [s(u)u]_x &= 0 \\
  D_t \rho + D_x J &= 0
\end{align*}
\]

$\rho = u$ is the conserved density;

$J(u) = s(u)u$ is the associated flux.

A simple Lighthill-Whitham-Richards model:

\[
s(u) = s_{\text{max}} \left(1 - \frac{u}{u_{\text{max}}} \right), \quad 0 \leq u \leq u_{\text{max}}
\]

$s_{\text{max}}$ is posted road speed, $u_{\text{max}}$ is the jam density.
Example 2: Fluid and Gas Dynamics

Euler equations for a compressible, non-viscous fluid:

\[ \rho_t + \nabla \cdot (\rho \mathbf{u}) = 0 \]
\[ (\rho \mathbf{u})_t + \nabla \cdot (\mathbf{u} \otimes (\rho \mathbf{u})) + \nabla p = 0 \]
\[ E_t + \nabla \cdot ((E + p) \mathbf{u}) = 0 \]

or, in components

\[ \rho_t + \nabla \cdot (\rho \mathbf{u}) = 0 \]
\[ (\rho u_i)_t + \nabla \cdot (\rho u_i \mathbf{u} + p \mathbf{e}_i) = 0 \quad (i = 1, 2, 3) \]
\[ E_t + \nabla \cdot ((E + p) \mathbf{u}) = 0 \]

Express conservation of mass, momentum, energy.
⊗ is the dyadic product.

ρ is the mass density.

\( u = u_1 \mathbf{e}_1 + u_2 \mathbf{e}_2 + u_3 \mathbf{e}_3 \) is the velocity.

\( p \) is the pressure \( p(\rho, e) \).

\( E \) is the energy density per unit volume:

\[
E = \frac{1}{2} \rho |\mathbf{u}|^2 + \rho e.
\]

e is internal energy density per unit of mass (related to temperature).
Notation – Computations on the Jet Space

• Independent variables \( \mathbf{x} = (x, y, z) \)

• Dependent variables \( \mathbf{u} = (u^{(1)}, u^{(2)}, \ldots, u^{(j)}, \ldots, u^{(N)}) \)

In examples: \( \mathbf{u} = (u, v, \theta, h, \ldots) \)

• Partial derivatives \( u_{kx} = \frac{\partial^k u}{\partial x^k}, \ u_{kx \ l} y = \frac{\partial^{k+l} u}{\partial x^k y^l}, \) etc.

Examples: \( u_{xxxxxx} = u_{5x} = \frac{\partial^5 u}{\partial x^5} \)
\( u_{xx \ yyyyy} = u_{2x \ 4y} = \frac{\partial^6 u}{\partial x^2 y^4} \)

• Differential functions

Example: \( f = uvv_x + x^2 u^3 v_x + u_x v_{xx} \)
• **Total derivatives:** \( D_t, D_x, D_y, \ldots \)

**Example:** Let \( f = uvv_x + x^2u_x^3v_x + u_xv_{xx} \)

Then

\[
D_x f = \frac{\partial f}{\partial x} + u_x \frac{\partial f}{\partial u} + u_{xx} \frac{\partial f}{\partial u_x} + v_x \frac{\partial f}{\partial v} + v_{xx} \frac{\partial f}{\partial v_x} + v_{xxx} \frac{\partial f}{\partial v_{xx}}
\]

\[
= 2xu_x^3v_x + u_x(vv_x) + u_{xx}(3x^2u_x^2v_x + v_{xx}) + v_x(uv_x) + v_{xx}(uv + x^2u_x^3) + v_{xxx}(u_x)
\]

\[
= 2xu_x^3v_x + vu_xv_x + 3x^2u_x^2v_xu_{xx} + u_{xx}v_{xx} + uv_x^2 + uvv_{xx} + x^2u_x^3v_{xx} + u_xv_{xxx}
\]
Conservation Laws for Nonlinear PDEs

- **System of evolution equations of order** $M$
  \[
  \mathbf{u}_t = \mathbf{F}(\mathbf{u}^{(M)}(\mathbf{x}))
  \]
  with $\mathbf{u} = (u, v, w, \ldots)$ and $\mathbf{x} = (x, y, z)$.

- **Conservation law in** $(1+1)$-dimensions
  \[
  D_t \rho + D_x J = 0
  \]
  where the dot means evaluated on the PDE. Conserved density $\rho$ and flux $J$.

  \[
  P = \int_{-\infty}^{\infty} \rho \, dx = \text{constant in time}
  \]
  if $J$ vanishes at $\pm \infty$. 
• Conservation law in (2+1)-dimensions

\[
D_t \rho + \nabla \cdot \mathbf{J} = D_t \rho + D_x J_1 + D_y J_2 \overset{\cdot}{=} 0
\]

Conserved density \( \rho \) and flux \( \mathbf{J} = (J_1, J_2) \).

• Conservation law in (3+1)-dimensions

\[
D_t \rho + \nabla \cdot \mathbf{J} = D_t \rho + D_x J_1 + D_y J_2 + D_z J_3 \overset{\cdot}{=} 0
\]

Conserved density \( \rho \) and flux \( \mathbf{J} = (J_1, J_2, J_3) \).
Reasons for Computing Conservation Laws

• Conservation of physical quantities (linear momentum, mass, energy, electric charge, ...).

• Testing of complete integrability and application of Inverse Scattering Transform.

• Testing of numerical integrators.

• Study of quantitative and qualitative properties of PDEs (Hamiltonian structure, recursion operators, ...).

• Verify the closure of a model.
Examples of PDEs with Conservation Laws

Example 1: KdV Equation

\[ \frac{\partial u}{\partial t} + 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \quad \text{or} \quad u_t + 6uu_x + u_{xxx} = 0 \]

shallow water waves, ion-acoustic waves in plasmas

Diederik Korteweg  
Gustav de Vries
Dilation Symmetry

\[ u_t + 6uu_x + u_{xxx} = 0 \]

has dilation (scaling) symmetry \((x, t, u) \rightarrow (x/\lambda, t/\lambda^3, \lambda^2 u)\)

\(\lambda\) is an arbitrary parameter.

Notion of weight: \(W(x) = -1\), thus, \(W(D_x) = 1\).

\[ W(t) = -3, \text{ hence, } W(D_t) = 3. \]

\[ W(u) = 2. \]

Notion of rank (total weight of a monomial).

Examples: \(\text{Rank}(u^3) = \text{Rank}(3u_x^2) = 6.\)

\[ \text{Rank}(u^3u_{xx}) = 10. \]
Key Observation: Scaling Invariance

Every term in a density has the same fixed rank.

Every term in a flux has some other fixed rank.

The conservation law

$$D_t \rho + D_x J \dot{=} 0$$

is uniform in rank.

Hence,

$$\text{Rank}(\rho) + \text{Rank}(D_t) = \text{Rank}(J) + \text{Rank}(D_x)$$
• **First six (of infinitely many) conservation laws:**

\[
\begin{align*}
D_t(u) + D_x(3u^2 + u_{xx}) &\doteq 0 \\
D_t(u^2) + D_x(4u^3 - u_x^2 + 2uu_{xx}) &\doteq 0 \\
D_t\left(u^3 - \frac{1}{2}u_x^2\right) &+ D_x\left(\frac{9}{2}u^4 - 6uu_x^2 + 3u^2u_{xx} + \frac{1}{2}u_{xx}^2 - uu_xu_{xx}\right) &\doteq 0 \\
D_t\left(u^4 - 2uu_x^2 + \frac{1}{5}u_{xx}^2\right) + D_x\left(\frac{24}{5}u^5 - 18uu_x^2 + 4u^3u_{xx} + 2u_x^2u_{xx} + \frac{16}{5}uu_{xx}^2 - 4uu_xu_{xxx} - \frac{1}{5}u_{xxx}^2 + \frac{2}{5}u_{xx}u_{4x}\right) &\doteq 0
\end{align*}
\]
\[ D_t \left( u^5 - 5 u^2 u_x + uu_{xx} - \frac{1}{14} u_{xxx} \right) \]
\[ + D_x \left( 5u^6 - 40u^3 u_x - \ldots - \frac{1}{7} u_{xxx} u_{5x} \right) \dot{=} 0 \]
\[ D_t \left( u^6 - 10 u^3 u_x - \frac{5}{6} u_x^4 + 3 u^2 u_{xx} \right) \]
\[ + \frac{10}{21} u_{xx}^3 - \frac{3}{7} uu_{xxx}^2 + \frac{1}{42} u_{4x}^2 \]
\[ + D_x \left( \frac{36}{7} u^7 - 75u^4 u_x - \ldots + \frac{1}{21} u_{4x} u_{6x} \right) \dot{=} 0 \]

- Third conservation law: Gerald Whitham, 1965
- Fourth and fifth: Norman Zabusky, 1965-66
- Seventh (sixth thru tenth): Robert Miura, 1966
Robert Miura
• Conservation law explicitly dependent on $t$ and $x$:

\[
D_t \left( tu^2 - \frac{1}{3} xu \right) + D_x \left( 4tu^3 - xu^2 + \frac{1}{3}u_x - tu_x^2 + 2tux_{xx} - \frac{1}{3}ux_{xx} \right) = 0
\]
• First five: IBM 7094 computer with FORMAC (1966) → storage space problem!

IBM 7094 Computer
• First eleven densities: Control Data Computer CDC-6600 computer (2.2 seconds) → large integers problem!

Control Data CDC-6600
Other Scaling Invariant Quantities

• **Generalized Symmetries:**

\[ G(x, t, u^{(N)}) \] is an \( N \)th order generalized symmetry iff it leaves \( u_t = F(x, t, u^{(M)}) \) invariant for the replacement \( u \rightarrow u + \epsilon G, \; u_{ix} \rightarrow u_{ix} + \epsilon D_x G \), within order \( \epsilon \):

\[
D_t(u + \epsilon G) \overset{\dot{\cdot}}{=} F(u + \epsilon G)
\]

must hold up to order \( \epsilon \).

• **Defining equation:**

\[
D_t G \overset{\dot{\cdot}}{=} F'(u)[G]
\]

\( F(u)'[G] \) is the Fréchet derivative of \( F(u) \) in the direction of \( G \):
\[ F'(u)[G] = \frac{\partial}{\partial \epsilon} F(u + \epsilon G) \bigg|_{\epsilon=0} = \sum_{i=0}^{M} \left( D^i_x G \right) \frac{\partial F}{\partial u_{ix}} \]

- First 4 generalized symmetries of \( u_t = 6uu_x + u_{3x} \)

\[
G^{(1)} = u_x \\
G^{(2)} = 6uu_x + u_{xxx} \\
G^{(3)} = 30u^2u_x + 20uu_xu_xx + 10uu_{xxx} + u_{5x} \\
G^{(4)} = 140u^3u_x + 70u^3_x + 280uu_xu_xx + 70u^2u_{xxx} + 70uu_{xxx} + 42u_xu_{4x} + 14uu_5x + u_{7x}
\]

Generalized symmetries are invariant under the scaling symmetry.
• Recursion Operator:

A recursion operator \( \mathcal{R} \) connects symmetries

\[
G^{(j+s)} = \mathcal{R}G^{(j)}, \quad j = 1, 2, \ldots
\]

\( s \) is seed (\( s = 1 \) in simplest case).

• Defining equation:

\[
D_t\mathcal{R} + [\mathcal{R}, F'(u)] \dot{=} 0
\]

Explicitly,

\[
\frac{\partial \mathcal{R}}{\partial t} + \mathcal{R}'[F] + \mathcal{R} \circ F'(u) - F'(u) \circ \mathcal{R} \dot{=} 0
\]

where \( \circ \) stands for composition, and \( \mathcal{R}'[F] \) is the
Fréchet derivative of $\mathcal{R}$ in the direction of $F$:

$$\mathcal{R}'[F] = \sum_{i=0}^{n} (D^i_x F) \frac{\partial \mathcal{R}}{\partial u_{ix}}$$

- Recursion operator (KdV equation):

$$\mathcal{R} = D_x^2 + 2uI + 2D_x uD_x^{-1} = D_x^2 + 4uI + 2u_x D_x^{-1}$$

- For example,

$$\mathcal{R} u_x = (D_x^2 + 2uI + 2D_x uD_x^{-1}) u_x = 6uu_x + u_{3x}$$

$$\mathcal{R}(6uu_x + u_{3x}) = (D_x^2 + 2uI + 2D_x uD_x^{-1})(6uu_x + u_{3x})$$

$$= 30u^2 u_x + 20u_x u_{2x} + 10u u_{3x} + u_{5x}$$

Recursion operator is invariant under the scaling symmetry.
• **Lax Pair:** Key idea: Replace $u_t + 6uu_x + u_{xxx} = 0$ with a compatible linear system (Lax pair):

$$
\psi_{xx} + (u - \lambda) \psi = 0 \\
\psi_t + 4\psi_{xxx} + 6u\psi_x + 3u_x\psi = 0
$$

$\psi$ is eigenfunction; $\lambda$ is constant eigenvalue ($\lambda_t = 0$) (isospectral)

• **Lax Pair** $(\mathcal{L}, \mathcal{M})$ in Operator Form:

$$
\mathcal{L}\psi = \lambda\psi \quad \text{and} \quad D_t\psi = \mathcal{M}\psi
$$

• Require compatibility of both equations

$$
\mathcal{L}_t\psi + (\mathcal{L}\mathcal{M} - \mathcal{M}\mathcal{L})\psi = 0
$$
• **Defining Equation:** \[ \mathcal{L}_t + [\mathcal{L}, \mathcal{M}] \equiv 0 \]

with commutator \([\mathcal{L}, \mathcal{M}] = \mathcal{LM} - \mathcal{ML} \).

Furthermore, \( \mathcal{L}_t \psi = [\mathcal{D}_t, \mathcal{L}] \psi = \mathcal{D}_t(\mathcal{L}\psi) - \mathcal{L}\mathcal{D}_t \psi \).

• **Lax pair for the KdV equation:**

\[ \mathcal{L} = \mathcal{D}_x^2 + u\mathbf{I} \]

\[ \mathcal{M} = -\left(4\mathcal{D}_x^3 + 6uD_x + 3ux\mathbf{I}\right) \]

Lax pair is invariant under the scaling symmetry.
Example 2: The Zakharov-Kuznetsov Equation

\[ u_t + \alpha uu_x + \beta (u_{xx} + u_{yy})_x = 0 \]

models ion-sound solitons in a low pressure uniform magnetized plasma.

• Conservation laws:

\[ D_t(u) + D_x \left( \frac{\alpha}{2} u^2 + \beta u_{xx} \right) + D_y \left( \beta u_{xy} \right) = 0 \]

\[ D_t(u^2) + D_x \left( \frac{2\alpha}{3} u^3 - \beta (u_x^2 - u_y^2) + 2\beta u(u_{xx} + u_{yy}) \right) + D_y \left( -2\beta u_x u_y \right) = 0 \]
More conservation laws (ZK equation):

\[ D_t \left( u^3 - \frac{3\beta}{\alpha}(u_x^2 + u_y^2) \right) + D_x \left( 3u^2\left( \frac{\alpha}{4}u^2 + \beta u_{xx} \right) - 6\beta u(u_x^2 + u_y^2) \right) + \frac{3\beta^2}{\alpha}(u_{xx}^2 - u_{yy}^2) - \frac{6\beta^2}{\alpha}(u_x(u_{xxx} + u_{xyy}) + u_y(u_{xx} + u_{yy})) \]

\[ + D_y \left( 3\beta u^2 u_{xy} + \frac{6\beta^2}{\alpha} u_{xy}(u_{xx} + u_{yy}) \right) = 0 \]

\[ D_t \left( tu^2 - \frac{2}{\alpha} xu \right) + D_x \left( t\left( \frac{2\alpha}{3} u^3 - \beta(u_x^2 - u_y^2) + 2\beta u(u_{xx} + u_{yy}) \right) - xu^2 + \frac{2\beta}{\alpha} u_{xx} + \frac{2\beta}{\alpha} u_x \right) + D_y \left( - 2\beta(tuxu_y + \frac{1}{\alpha} xu_{xy}) \right) = 0 \]
Methods for Computing Conservation Laws

• Use the Lax pair $L$ and $A$, satisfying $[L, A] = 0$. If $L = D_x + U$, $A = D_t + V$ then $V_x - U_t + [U, V] = 0$.
  $\hat{L} = TLT^{-1}$ gives the densities, $\hat{A} = TAT^{-1}$ gives the fluxes.

• Use Noether’s theorem (Lagrangian formulation) to generate conservation laws from symmetries (Ovsiannikov, Olver, Mahomed, Kara, etc.).

• Integrating factor methods (Anderson, Bluman, Anco, Cheviakov, Mason, Naz, etc.) require solving ODEs (or PDEs).
Proposed Algorithmic Method

- Density is linear combination of scaling invariant terms (in the jet space) with undetermined coefficients.
- Compute $D_t \rho$ with total derivative operator.
- Use variational derivative (Euler operator) to express exactness.
- Solve a (parametrized) linear system to find the undetermined coefficients.
- Use the homotopy operator to compute the flux (invert $D_x$ or $\text{Div}$).
• Work with linearly independent pieces in finite dimensional spaces.
• Use linear algebra, calculus, and variational calculus (algorithmic).
• Implement the algorithm in Mathematica.
**Tools from the Calculus of Variations**

**Differential Topology and Differential Geometry**

- **Definition:**
  
  A differential function \( f \) is exact iff \( f = \text{Div} \mathbf{F} \).
  
  Special case (1D): \( f = D_x F \).

- **Question:** How can one test that \( f = \text{Div} \mathbf{F} \)?

- **Theorem (exactness test):**
  
  \[
  f = \text{Div} \mathbf{F} \iff \mathcal{L}_{u(j)}(x)f \equiv 0, \quad j = 1, 2, \ldots, N.
  \]

  \( N \) is the number of dependent variables.

  The Euler operator annihilates divergences.
• Euler operator in 1D (variable $u(x)$):

$$\mathcal{L}_{u(x)} = \sum_{k=0}^{M} (-D_x)^k \frac{\partial}{\partial u_{kx}}$$

$$= \frac{\partial}{\partial u} - D_x \frac{\partial}{\partial u_x} + D_x^2 \frac{\partial}{\partial u_{xx}} - D_x^3 \frac{\partial}{\partial u_{xxx}} + \cdots$$

• Euler operator in 2D (variable $u(x, y)$):

$$\mathcal{L}_{u(x,y)} = \sum_{k=0}^{M_x} \sum_{\ell=0}^{M_y} (-D_x)^k (-D_y)^\ell \frac{\partial}{\partial u_{kx \ell y}}$$

$$= \frac{\partial}{\partial u} - D_x \frac{\partial}{\partial u_x} - D_y \frac{\partial}{\partial u_y}$$

$$+ D_x^2 \frac{\partial}{\partial u_{xx}} + D_x D_y \frac{\partial}{\partial u_{xy}} + D_y^2 \frac{\partial}{\partial u_{yy}} - D_x^3 \frac{\partial}{\partial u_{xxx}} + \cdots$$
Application: Testing Exactness

Example:

\[ f = 8v_x v_{xx} - u_x^3 \sin u + 2u_x u_{xx} \cos u - 6vv_x \cos u + 3u_x v^2 \sin u \]

where \( u(x) \) and \( v(x) \)

- \( f \) is exact
- After integration by parts (by hand):

\[
F = \int f \, dx = 4v_x^2 + u_x^2 \cos u - 3v^2 \cos u
\]
• **Exactness test with Euler operator:**

\[
f = 8v_x v_{xx} - u_x^3 \sin u + 2u_x u_{xx} \cos u - 6vv_x \cos u + 3u_x v^2 \sin u
\]

\[
\mathcal{L}_{u(x)} f = \frac{\partial f}{\partial u} - D_x \frac{\partial f}{\partial u_x} + D_x^2 \frac{\partial f}{\partial u_{xx}} \equiv 0
\]

\[
\mathcal{L}_{v(x)} f = \frac{\partial f}{\partial v} - D_x \frac{\partial f}{\partial v_x} + D_x^2 \frac{\partial f}{\partial v_{xx}} \equiv 0
\]
• Question: How can one compute \( F = \text{Div}^{-1} f \) ?

• Theorem (integration by parts):
  
  • In 1D: If \( f \) is exact then

  \[
  F = D_x^{-1} f = \int f \, dx = \mathcal{H}_u(x)f
  \]

  • In 2D: If \( f \) is a divergence then

  \[
  F = \text{Div}^{-1} f = (\mathcal{H}^{(x)}_{u(x,y)}f, \mathcal{H}^{(y)}_{u(x,y)}f)
  \]

  The homotopy operator inverts total derivatives and divergences!
Two continuous functions are called homotopic if one can be “continuously deformed” into the other. Such a deformation is called a homotopy between the two functions.

\[ T(u_0, \lambda) = u_0 + \lambda(u - u_0) = (1 - \lambda)u_0 + \lambda u \]
• Homotopy Operator in 1D (variable \( x \)):

\[
\mathcal{H}_{u(x)} f = \int_0^1 \sum_{j=1}^N (I_{u(j)} f)[\lambda u] \frac{d\lambda}{\lambda}
\]

with integrand

\[
I_{u(j)} f = \sum_{k=1}^{M_x(j)} \left( \sum_{i=0}^{k-1} u_{i,x}^{(j)} (-D_x)^{k-(i+1)} \right) \frac{\partial f}{\partial u_{k,x}^{(j)}}
\]

\((I_{u(j)} f)[\lambda u]\) means that in \( I_{u(j)} f \) one replaces

\( u \rightarrow \lambda u, \ u_x \rightarrow \lambda u_x, \ etc. \)

More general: \( u \rightarrow \lambda (u - u_0) + u_0 \)

\( u_x \rightarrow \lambda (u_x - u_{x0}) + u_{x0} \ \ etc. \)
• Homotopy Operator in 2D (variables $x$ and $y$):

$$
\mathcal{H}_{u(x,y)}^{(x)} f = \int_0^1 \sum_{j=1}^N (I_{u(j)}^{(x)} f)[\lambda u] \frac{d\lambda}{\lambda}
$$

$$
\mathcal{H}_{u(x,y)}^{(y)} f = \int_0^1 \sum_{j=1}^N (I_{u(j)}^{(y)} f)[\lambda u] \frac{d\lambda}{\lambda}
$$

where for dependent variable $u(x, y)$

$$
\mathcal{I}_{u}^{(x)} f = \sum_{k=1}^{M_x} \sum_{\ell=0}^{M_y} \left( \sum_{i=0}^{k-1} \sum_{j=0}^{\ell} u_{ix,jy} \left( \sum_{i=0}^{k-1} \sum_{j=0}^{\ell} \frac{(i+j)(k+\ell-i-j-1)}{(k-i-1)(k+\ell)} \left( -D_x \right)^{k-i-1} \left( -D_y \right)^{\ell-j} \right) \frac{\partial f}{\partial u_{kx\ell y}} \right)
$$
Application 1: The KdV Equation

\[ u_t + 6uu_x + u_{xxx} = 0 \]

- Step 1: Compute the dilation symmetry

Set \((x, t, u) \rightarrow \left( \frac{x}{\lambda}, \frac{t}{\lambda^a}, \lambda^b u \right) = (\tilde{x}, \tilde{t}, \tilde{u})\)

Apply change of variables (chain rule)

\[ \lambda^{-(a+b)} \tilde{u}_\tilde{t} + \lambda^{-(2b+1)} \tilde{u}_\tilde{x} + \lambda^{-(b+3)} \tilde{u}_{3\tilde{x}} = 0 \]

Solve \(a + b = 2b + 1 = b + 3\).

Solution: \(a = 3\) and \(b = 2\)

\((x, t, u) \rightarrow \left( \frac{x}{\lambda}, \frac{t}{\lambda^3}, \lambda^2 u \right)\)
Compute the density of selected \textbf{rank}, say, 6.

- **Step 2: Determine the form of the density**

List powers of $u$, up to rank 6: $[u, u^2, u^3]$

Differentiate with respect to $x$ to increase the rank

- $u$ has weight 2 $\rightarrow$ apply $D_x^4$
- $u^2$ has weight 4 $\rightarrow$ apply $D_x^2$
- $u^3$ has weight 6 $\rightarrow$ no derivatives needed
Apply the $D_x$ derivatives

Remove total and highest derivative terms:

$$D_x^4 u \rightarrow \{u_{4x}\} \rightarrow \text{empty list}$$

$$D_x^2 u^2 \rightarrow \{u_x^2, uu_{xx}\} \rightarrow \{u_x^2\}$$

since $uu_{xx} = (uu_x)_x - u_x^2$

$$D_x^0 u^3 \rightarrow \{u^3\} \rightarrow \{u^3\}$$

Linearly combine the “building blocks”

Candidate density: $\rho = c_1 u^3 + c_2 u_x^2$
• **Step 3: Compute the coefficients** $c_i$

Compute

$$D_t \rho = \frac{\partial \rho}{\partial t} + \rho'(u)[u_t]$$

$$= \frac{\partial \rho}{\partial t} + \sum_{k=0}^{M} \frac{\partial \rho}{\partial u_{kx}} D_x^k u_t$$

$$= (3c_1 u^2 I + 2c_2 u_x D_x) u_t$$

Substitute $u_t$ by $-(6uu_x + u_{xxx})$

$$E = -D_t \rho = (3c_1 u^2 I + 2c_2 u_x D_x)(6uu_x + u_{xxx})$$

$$= 18c_1 u^3 u_x + 12c_2 u^3 x + 12c_2 uu_x u_{xx}$$

$$+ 3c_1 u^2 u_{xxx} + 2c_2 u_x u_{4x}$$
Apply the Euler operator (variational derivative)

\[ \mathcal{L}_u(x) = \frac{\delta}{\delta u} = \sum_{k=0}^{m} (-D_x)^k \frac{\partial}{\partial u_{kx}} \]

Here, \( E \) has order \( m = 4 \), thus

\[ \mathcal{L}_u(x) E = \frac{\partial E}{\partial u} - D_x \frac{\partial E}{\partial u_x} + D_x^2 \frac{\partial E}{\partial u_{xx}} - D_x^3 \frac{\partial E}{\partial u_{3x}} + D_x^4 \frac{\partial E}{\partial u_{4x}} \]

\[ = -18(c_1 + 2c_2)u_x u_{xx} \]

This term must vanish!

So, \( c_2 = -\frac{1}{2}c_1 \). Set \( c_1 = 1 \) then \( c_2 = -\frac{1}{2} \).

Hence, the final form density is

\[ \rho = u^3 - \frac{1}{2}u_x^2 \]
• **Step 4: Compute the flux** \( J \)

**Method 1: Integrate by parts (simple cases)**

Now,

\[
E = 18u^3 u_x + 3u^2 u_{xxx} - 6u^3_x - 6uu_x u_{xx} - u_x u_{xxxx}
\]

Integration of \( D_x J = E \) yields the flux

\[
J = \frac{9}{2} u^4 - 6uu_x^2 + 3u^2 u_{xx} + \frac{1}{2} u_{xx}^2 - u_x u_{xxxx}
\]
Method 2: Use the homotopy operator

\[ J = D_x^{-1} E = \int E \, dx = \mathcal{H}_{u(x)} E = \int_0^1 (I_u E)[\lambda u] \frac{d\lambda}{\lambda} \]

with integrand

\[ I_u E = \sum_{k=1}^{M} \left( \sum_{i=0}^{k-1} u_{ix} (-D_x)^{k-(i+1)} \right) \frac{\partial E}{\partial u_{kx}} \]
Here $M = 4$, thus

\[ I_u E = (uI)\left(\frac{\partial E}{\partial u_x}\right) + (u_x I - uD_x)\left(\frac{\partial E}{\partial u_{xx}}\right) \]

\[ + (u_{xx} I - u_x D_x + uD_x^2)\left(\frac{\partial E}{\partial u_{xxx}}\right) \]

\[ + (u_{xxx} I - u_{xx} D_x + u_x D_x^2 - uD_x^3)\left(\frac{\partial E}{\partial u_{4x}}\right) \]

\[ = (uI)(18u^3 + 18u_x^2 - 6uu_{xx} - u_{xxxx}) \]

\[ + (u_x I - uD_x)(-6uu_x) \]

\[ + (u_{xx} I - u_x D_x + uD_x^2)(3u^2) \]

\[ + (u_{xxx} I - u_{xx} D_x + u_x D_x^2 - uD_x^3)(-u_x) \]

\[ = 18u^4 - 18uu_x^2 + 9u^2u_{xx} + u_{xx}^2 - 2u_x u_{xxx} \]

Note: correct terms but incorrect coefficients!
Finally,

\[
\begin{align*}
J &= \mathcal{H}_{u(x)}E = \int_0^1 (I_u E)[\lambda u] \frac{d\lambda}{\lambda} \\
&= \int_0^1 \left( 18\lambda^3 u^4 - 18\lambda^2 uu_x^2 + 9\lambda^2 u^2 u_{xx} + \lambda u_{xx}^2 - 2\lambda uu_x u_{xxx} \right) d\lambda \\
&= \frac{9}{2} u^4 - 6uu_x^2 + 3u^2 u_{xx} + \frac{1}{2} u_{xx}^2 - uu_x u_{xxx}
\end{align*}
\]

Final form of the flux:

\[
J = \frac{9}{2} u^4 - 6uu_x^2 + 3u^2 u_{xx} + \frac{1}{2} u_{xx}^2 - uu_x u_{xxx}
\]
Application 2: Zakharov-Kuznetsov Equation

\[ u_t + \alpha uu_x + \beta (u_{xx} + u_{yy})_x = 0 \]

• Step 1: Compute the dilation invariance

ZK equation is invariant under scaling symmetry

\[(t, x, y, u) \rightarrow \left( \frac{t}{\lambda^3}, \frac{x}{\lambda}, \frac{y}{\lambda}, \lambda^2 u \right) = (\tilde{t}, \tilde{x}, \tilde{y}, \tilde{u})\]

\(\lambda\) is an arbitrary parameter.

• Hence, the weights of the variables are

\[ W(u) = 2, \quad W(D_t) = 3, \quad W(D_x) = 1, \quad W(D_y) = 1. \]
• A conservation law is invariant under the scaling symmetry of the PDE.

\[ W(u) = 2, \ W(D_t) = 3, \ W(D_x) = 1, \ W(D_y) = 1. \]

For example,

\[
\begin{align*}
D_t \left( u^3 - \frac{3\beta}{\alpha} (u_x^2 + u_y^2) \right) + D_x & \left( 3u^2 \left( \frac{\alpha}{4} u^2 + \beta u_{xx} \right) - 6\beta u (u_x^2 + u_y^2) \right) \\
& + \frac{3\beta^2}{\alpha} (u_{xx}^2 - u_{xy}^2) - \frac{6\beta^2}{\alpha} (u_x (u_{xxx} + u_{xyy}) + u_y (u_{xxy} + u_{yyy})) \\
+ D_y & \left( 3\beta u^2 u_{xy} + \frac{6\beta^2}{\alpha} u_{xy} (u_{xx} + u_{yy}) \right) = 0
\end{align*}
\]

\[ \text{Rank } (\rho) = 6, \quad \text{Rank } (J) = 8. \]

\[ \text{Rank } (\text{conservation law}) = 9. \]
Compute the density of selected rank, say, 6.

• Step 2: Construct the candidate density

For example, construct a density of rank 6.

Make a list of all terms with rank 6:

\[ \{u^3, u_x^2, uu_{xx}, u_y^2, uu_{yy}, u_xu_y, uu_{xy}, u_4x, u_3xy, u_2x2y, u_x3y, u_4y\} \]

Remove divergences and divergence-equivalent terms.

Candidate density of rank 6:

\[ \rho = c_1u^3 + c_2u_x^2 + c_3u_y^2 + c_4u_xu_y \]
• Step 3: Compute the undetermined coefficients

Compute

\[ D_t \rho = \frac{\partial \rho}{\partial t} + \rho'(u)[u_t] \]

\[ = \frac{\partial \rho}{\partial t} + \sum_{k=0}^{M_x} \sum_{\ell=0}^{M_y} \frac{\partial \rho}{\partial u_{kx \ell y}} D_x^k D_y^\ell u_t \]

\[ = \left( 3c_1 u^2 I + 2c_2 u_x D_x + 2c_3 u_y D_y + c_4 (u_y D_x + u_x D_y) \right) u_t \]

Substitute \( u_t = -\left( \alpha uu_x + \beta (u_{xx} + u_{yy})_x \right) \).
\[ E = -D_t \rho = 3c_1 u^2 (\alpha u u_x + \beta (u_{xx} + u_{xy})_x) + 2c_2 u_x (\alpha u u_x + \beta (u_{xx} + u_{yy})_x)_x + 2c_3 u_y (\alpha u u_x + \beta (u_{xx} + u_{yy})x)_y + \beta (u_{xx} + u_{yy})_x y + c_4 (u_y (\alpha u u_x + \beta (u_{xx} + u_{yy})_x)_x + u_x (\alpha u u_x + \beta (u_{xx} + u_{yy})_x)_y) \]

Apply the Euler operator (variational derivative)

\[
\mathcal{L}_{u(x,y)} E = \sum_{k=0}^{M_x} \sum_{\ell=0}^{M_y} (-D_x)^k (-D_y)^\ell \frac{\partial E}{\partial u_{kx \ell y}}
\]

\[
= -2 \left( (3c_1 \beta + c_3 \alpha) u_x u_{yy} + 2(3c_1 \beta + c_3 \alpha) u_y u_{xy} + 2c_4 \alpha u_x u_{xy} + c_4 \alpha u_y u_{xx} + 3(3c_1 \beta + c_2 \alpha) u_x u_{xx} \right)
\equiv 0
\]
Solve a parameterized linear system for the $c_i$:

$$3c_1\beta + c_3\alpha = 0, \quad c_4\alpha = 0, \quad 3c_1\beta + c_2\alpha = 0$$

Solution:

$$c_1 = 1, \quad c_2 = -\frac{3\beta}{\alpha}, \quad c_3 = -\frac{3\beta}{\alpha}, \quad c_4 = 0$$

Substitute the solution into the candidate density

$$\rho = c_1 u^3 + c_2 u_x^2 + c_3 u_y^2 + c_4 u_x u_y$$

Final density of rank 6:

$$\rho = u^3 - \frac{3\beta}{\alpha} (u_x^2 + u_y^2)$$
Step 4: Compute the flux

Use the homotopy operator to invert \( \text{Div} \):

\[
\mathbf{J} = \text{Div}^{-1} \mathbf{E} = \left( \mathcal{H}_{u(x,y)}^{(x)} \mathbf{E}, \mathcal{H}_{u(x,y)}^{(y)} \mathbf{E} \right)
\]

where

\[
\mathcal{H}_{u(x,y)}^{(x)} \mathbf{E} = \int_0^1 (I_u^{(x)} \mathbf{E})[\lambda u] \frac{d\lambda}{\lambda}
\]

with

\[
\mathcal{I}_u^{(x)} \mathbf{E} = \sum_{k=1}^{M_x} \sum_{\ell=0}^{M_y} \left( \sum_{i=0}^{k-1} \sum_{j=0}^{\ell} u_{ixjy} \frac{(i+j)(k+\ell-i-j-1)}{(k+i-1)\binom{k+\ell}{k}} \right) \left( -D_x \right)^{k-i-1} \left( -D_y \right)^{\ell-j} \frac{\partial \mathbf{E}}{\partial u_{kx\ell y}}
\]

Similar formulas for \( \mathcal{H}_{u(x,y)}^{(y)} \mathbf{E} \) and \( \mathcal{I}_u^{(y)} \mathbf{E} \).
Let $A = \alpha uu_x + \beta (u_{xxx} + u_{xyy})$ so that

$$E = 3u^2 A - \frac{6\beta}{\alpha} u_x A_x - \frac{6\beta}{\alpha} u_y A_y$$

Then,

$$J = \left( \mathcal{H}_{u(x,y)}^{(x)} E, \mathcal{H}_{u(x,y)}^{(y)} E \right) = \left( \frac{3\alpha}{4} u^4 + \beta u^2 (3u_{xx} + 2u_{yy}) - 2\beta u(3u_x^2 + u_y^2) + \frac{3\beta^2}{4\alpha} u(u_{2x2y} + u_{4y}) - \frac{\beta^2}{\alpha} u_x(\frac{7}{2} u_{xyy} + 6u_{xxx}) - \frac{\beta^2}{\alpha} u_y(4u_{xxy} + \frac{3}{2} u_{yyy}) + \frac{\beta^2}{\alpha} (3u_{xx}^2 + \frac{5}{2} u_{xy}^2 + \frac{3}{4} u_{yy}^2) + \frac{5\beta^2}{4\alpha} u_{xx} u_{yy}, \beta u^2 u_{xy} - 4\beta uu_x u_y - \frac{3\beta^2}{4\alpha} u(u_{3y} + u_{3xy}) - \frac{\beta^2}{4\alpha} u_x(13u_{xxy} + 3u_{yyy}) - \frac{5\beta^2}{4\alpha} u_y(u_{xxx} + 3u_{xyy}) + \frac{9\beta^2}{4\alpha} u_{xy}(u_{xx} + u_{yy}) \right)$$
However, $\text{Div}^{-1}E$ is not unique.

Indeed, $\mathbf{J} = \tilde{\mathbf{J}} + \mathbf{K}$, where $\mathbf{K} = (D_y \theta, -D_x \theta)$ is a curl term.

For example,

$$\theta = 2\beta u^2 u_y + \frac{\beta^2}{4\alpha} \left( 3u(u_{xxy} + u_{yyy}) + 10u_x u_{xy} + 5u_y(3u_{yy} + u_{xx}) \right)$$

Shorter flux:

$$\tilde{\mathbf{J}} = \mathbf{J} - \mathbf{K}$$

$$= \left( 3u^2 (\frac{\alpha}{4} u^2 + \beta u_{xx}) - 6\beta u(u_x^2 + u_y^2) + \frac{3\beta^2}{\alpha} \left( u_{xx}^2 - u_{yy}^2 \right) \right.$$

$$\left. - \frac{6\beta^2}{\alpha} (u_x (u_{xxx} + u_{xyy}) + u_y (u_{xxy} + u_{yyy})) \right)$$

$$3\beta u^2 u_{xy} + \frac{6\beta^2}{\alpha} u_{xy}(u_{xx} + u_{yy})$$
Software Demonstration

Software packages in Mathematica

Codes are available via the Internet:
URL: http://inside.mines.edu/~whereman/
**Additional Examples**

- **Manakov-Santini system**
  \[
  u_{tx} + u_{yy} + (u u_x)_x + v_x u_{xy} - u_{xx} v_y = 0
  
  v_{tx} + v_{yy} + u v_{xx} + v_x v_{xy} - v_y v_{xx} = 0
  \]

- **Conservation laws for Manakov-Santini system:**
  \[
  D_t \left( f u_x v_x \right) + D_x \left( f (u u_x v_x - u_x v_x v_y - u_y v_y) \right) - f' y (u_t + u u_x - u_x v_y) + D_y \left( f (u_x v_y + u_y v_x + u_x v_x^2) \right) + f' (u - y u_y - y u_x v_x) = 0
  \]

where \( f = f(t) \) is arbitrary.
Conservation laws – continued:

\[
D_t \left( f(2u + v_x^2 - y u_x v_x) \right) + D_x \left( f(u^2 + uv_x^2 + uy) - v_y^2 - v_x^2 v_y - y(uu_x v_x - u_x v_x v_y - u_y v_y) \right) \\
- f'y(v_t + uv_x - v_x v_y) + (f' - 2fx)y^2(u_t + uu_x - u_x v_y) \\
+ D_y \left( f(v_x^3 + 2v_x v_y - u_x v - y(u_x v_x^2 + u_x v_y + u_y v_x)) \\
+ f'(v - y(2u + v_y + v_x^2)) + (f'y^2 - 2fx)(u_x v_x + uy) \right) = 0
\]

where \( f = f(t) \) is arbitrary.

There are three additional conservation laws.
• (2+1)-dimensional Camassa-Holm equation

\[(\alpha u_t + \kappa u_x - u_{txx} + 3\beta uu_x - 2u_x u_{xx} - uu_{xxx})_x + u_{yy} = 0\]

Interchange \(t\) with \(y\)

\[(\alpha u_y + \kappa u_x - u_{xxy} + 3\beta uu_x - 2u_x u_{xx} - uu_{xxx})_x + u_{tt} = 0\]

Set \(v = u_t\) to get

\[u_t = v\]

\[v_t = -\alpha u_{xy} - \kappa u_{xx} + u_{3xy} - 3\beta u_x^2 - 3\beta uu_{xx} + 2u_{xx}^2\]

\[+ 3u_x u_{xxx} + uu_{4x}\]
• Conservation laws for the Camassa-Holm equation

\[\begin{align*}
D_t (fu) + D_x \left( \frac{1}{\alpha} f \left( \frac{3\beta}{2} u^2 + \kappa u - \frac{1}{2} u_x^2 - uu_{xx} - u_{tx} \right) \\
+ \left( \frac{1}{2} f'y^2 - \frac{1}{\alpha} fx \right) (\alpha u_t + \kappa u_x + 3\beta uu_x - 2u_xu_{xx} - uu_{xxx} - u_{txx}) \right) + D_y \left( \left( \frac{1}{2} f'y^2 - \frac{1}{\alpha} fx \right) u_y - f'y u \right) &= 0
\end{align*}\]

\[\begin{align*}
D_t (fyu) + D_x \left( \frac{1}{\alpha} fy \left( \frac{3\beta}{2} u^2 + \kappa u - \frac{1}{2} u_x^2 - uu_{xx} - u_{tx} \right) \\
+ y \left( \frac{1}{6} f'y^2 - \frac{1}{\alpha} fx \right) (\alpha u_t + \kappa u_x + 3\beta uu_x - 2u_xu_{xx} - uu_{xxx} - u_{txx}) \right) + D_y \left( y \left( \frac{1}{6} f'y^2 - \frac{1}{\alpha} fx \right) u_y + \left( \frac{1}{\alpha} fx - \frac{1}{2} f'y^2 \right) u \right) &= 0
\end{align*}\]

where \( f = f(t) \) is an arbitrary function.
Khoklov-Zabolotskaya equation describes e.g., sound waves in nonlinear media

\[
(u_t - uu_x)_x - u_{yy} - u_{zz} = 0
\]

Conservation law:

\[
\begin{align*}
D_t(fu) & - D_x\left(\frac{1}{2}fu^2 + (fx + g)(u_t - uu_x)\right) \\
+ D_y\left((fx + g)u_y - (f_yx + g_y)u\right) \\
+ D_z\left((fx + g)u_z - (f_zx + g_z)u\right) &= 0
\end{align*}
\]

under the constraints \(\Delta f = 0\) and \(\Delta g = f_t\)

where \(f = f(t, y, z)\) and \(g = g(t, y, z)\).
• Shallow water wave model (atmosphere)

\[ u_t + (u \cdot \nabla)u + 2 \Omega \times u + \nabla(\theta h) - \frac{1}{2} h \nabla \theta = 0 \]
\[ \theta_t + u \cdot (\nabla \theta) = 0 \]
\[ h_t + \nabla \cdot (uh) = 0 \]

where \( u(x, y, t), \theta(x, y, t) \) and \( h(x, y, t) \).

• In components:

\[ u_t + uu_x + vu_y - 2 \Omega v + \frac{1}{2} h \theta_x + \theta h_x = 0 \]
\[ v_t + uv_x + vv_y + 2 \Omega u + \frac{1}{2} h \theta_y + \theta h_y = 0 \]
\[ \theta_t + u \theta_x + v \theta_y = 0 \]
\[ h_t + hu_x + uh_x + hv_y + vh_y = 0 \]
First few conservation laws of SWW model:

\[ \rho_{(1)} = h \]
\[ \rho_{(2)} = h \theta \]
\[ \rho_{(3)} = h \theta^2 \]
\[ \rho_{(4)} = h (u^2 + v^2 + h\theta) \]
\[ \rho_{(5)} = \theta (2\Omega + v_x - u_y) \]

\[ \mathbf{J}^{(1)} = h \begin{pmatrix} u \\ v \end{pmatrix} \]
\[ \mathbf{J}^{(2)} = h \theta \begin{pmatrix} u \\ v \end{pmatrix} \]
\[ \mathbf{J}^{(3)} = h \theta^2 \begin{pmatrix} u \\ v \end{pmatrix} \]
\[ \mathbf{J}^{(4)} = h \begin{pmatrix} u (u^2 + v^2 + 2h\theta) \\ v (v^2 + u^2 + 2h\theta) \end{pmatrix} \]
\[ \mathbf{J}^{(5)} = \frac{1}{2} \theta \begin{pmatrix} 4\Omega u - 2uu_y + 2uv_x - h\theta_y \\ 4\Omega v + 2vv_x - 2vu_y + h\theta_x \end{pmatrix} \]
• More general conservation laws for SWW model:

\[ D_t \left( f(\theta)h \right) + D_x \left( f(\theta)hu \right) + D_y \left( f(\theta)hv \right) = 0 \]

\[ D_t \left( g(\theta)(2\Omega + v_x - u_x) \right) \]
\[ + D_x \left( \frac{1}{2} g(\theta)(4\Omega u - 2uu_y + 2uv_x - h\theta_y) \right) \]
\[ + D_y \left( \frac{1}{2} g(\theta)(4\Omega v - 2u_yv + 2vv_x + h\theta_x) \right) = 0 \]

for any functions \( f(\theta) \) and \( g(\theta) \).
• Kadomtsev-Petviashvili (KP) equation

\[ (u_t + \alpha uu_x + u_{xxx})_x + \sigma^2 u_{yy} = 0 \]

parameter \( \alpha \in \mathbb{R} \) and \( \sigma^2 = \pm 1 \).

Equation be written as a conservation law

\[ D_t(u_x) + D_x(\alpha uu_x + u_{xxx}) + D_y(\sigma^2 u_y) = 0. \]

Exchange \( y \) and \( t \) and set \( u_t = v \)

\[ u_t = v \]
\[ v_t = -\frac{1}{\sigma^2}(u_{xy} + \alpha u_x^2 + \alpha uu_{xx} + u_{xxxx}) \]
• Examples of conservation laws for KP equation (explicitly dependent on $t$, $x$, and $y$)

\[ D_t(xu_x) + D_x \left( 3u^2 - uu_{xx} - 6xuu_x + xu_{xxx} \right) + D_y \left( \alpha xu_y \right) = 0 \]

\[ D_t(yu_x) + D_x \left( y(\alpha uu_x + uu_{xx}) \right) + D_y \left( \sigma^2 (yu_y - u) \right) = 0 \]

\[ D_t(\sqrt{tu}) + D_x \left( \frac{\alpha}{2} \sqrt{tu}^2 + \sqrt{tu} uu_x + \frac{\sigma^2 y^2}{4\sqrt{t}} u_t + \frac{\sigma^2 y^2}{4\sqrt{t}} uu_{xx} + \frac{\alpha \sigma^2 y^2}{4\sqrt{t}} uu_x - x\sqrt{tu} u_t - \alpha x\sqrt{tu} uu_x - x\sqrt{tu} uu_{xx} \right) \]

\[ + D_y \left( x\sqrt{tu} u_y + \frac{y^2 u_y}{4\sqrt{t}} - \frac{yu}{2\sqrt{t}} \right) = 0 \]
• More general conservation laws for KP equation:

\[
D_t(fu) + D_x\left(f\left(\frac{\alpha}{2}u^2 + u_{xx}\right)\right)
+ \left(\frac{\sigma^2}{2}f'y^2 - fx\right)(u_t + \alpha uu_x + u_{3x})
\]

\[
+ D_y\left(\left(\frac{1}{2}f'y^2 - \sigma^2 fx\right)u_y - f'yu\right) = 0
\]

\[
D_t(fyu) + D_x\left(fy\left(\frac{\alpha}{2}u^2 + u_{xx}\right)\right)
+ y\left(\frac{\sigma^2}{6}f'y^2 - fx\right)(u_t + \alpha uu_x + u_{3x})
\]

\[
+ D_y\left(y\left(\frac{1}{6}f'y^2 - \sigma^2 fx\right)u_y + (\sigma^2 fx - \frac{1}{2}f'y^2)u\right) = 0
\]

where \(f(t)\) is arbitrary function.
• Potential KP equation

Replace $u$ by $u_x$ and integrate with respect to $x$.

$$u_{xt} + \alpha u_x u_{xx} + u_{xxxx} + \sigma^2 u_{yy} = 0$$

• Examples of conservation laws
  (not explicitly dependent on $x, y, t$):

$$D_t (u_x) + D_x \left( \frac{\alpha}{2} u_x^2 + u_{xxx} \right) + D_y \left( \sigma^2 u_y \right) = 0$$
$$D_t \left( u_x^2 \right) + D_x \left( \frac{2\alpha}{3} u_x^3 - u_{xx}^2 + 2u_x u_{xxx} - \sigma^2 u_{yy} \right) + D_y \left( 2\sigma^2 u_x u_y \right) = 0$$
Conservation laws for pKP equation – continued:

\[ D_t(u_x u_y) + D_x \left( \alpha u_x^2 u_y + u_t u_y + 2u_{xxx} u_y - 2u_{xx} u_{xy} \right) \]
\[ + D_y \left( \sigma^2 u_y^2 - \frac{1}{3} u_x^3 - u_t u_x + u_{xx}^2 \right) = 0 \]

\[ D_t \left( 2\alpha uu_x u_{xx} + 3uu_{4x} - 3\sigma^2 u_y^2 \right) + D_x \left( 2\alpha u_t u_x^2 + 3u_t^2 \right) \]
\[ - 2\alpha uu_x u_{tx} - 3u_{tx} u_{xx} + 3u_t u_{xxx} + 3u_x u_{tx} u_{xx} - 3uu_t u_{xxx} \]
\[ + D_y \left( 6\sigma^2 u_t u_y \right) = 0 \]

Various generalizations exist.
Generalized Zakharov-Kuznetsov equation

\[ u_t + \alpha u^n u_x + \beta (u_{xx} + u_{yy}) x = 0 \]

where \( n \) is rational, \( n \neq 0 \).

Conservation laws:

\[
D_t (u) + D_x \left( \frac{\alpha}{n+1} u^{n+1} + \beta u_{xx} \right) + D_y (\beta u_{xy}) = 0
\]

\[
D_t (u^2) + D_x \left( \frac{2\alpha}{n+2} u^{n+2} - \beta (u_x^2 - u_y^2) + 2\beta u (u_{xx} + u_{yy}) \right) \\
+ D_y \left( -2\beta u_x u_y \right) = 0
\]
• Third conservation law for gZK equation:

\[
\begin{align*}
D_t & \left( u^{n+2} - \frac{(n+1)(n+2)\beta}{2\alpha} (u_x^2 + u_y^2) \right) \\
& + D_x \left( \frac{(n+2)\alpha}{2(n+1)} u^{2(n+1)} + (n + 2)\beta u^{n+1} u_{xx} \right) \\
& - (n + 1)(n + 2)\beta u^n (u_x^2 + u_y^2) + \frac{(n+1)(n+2)\beta^2}{2\alpha} (u_{xx} - u_{yy}) \\
& - \frac{(n+1)(n+2)\beta^2}{\alpha} (u_x (u_{xxx} + u_{xyy}) + u_y (u_{xxy} + u_{yyy})) \\
& + D_y \left( (n + 2)\beta u^{n+1} u_{xy} + \frac{(n+1)(n+2)\beta^2}{\alpha} u_{xy}(u_{xx} + u_{yy}) \right) = 0.
\end{align*}
\]
Conclusions and Future Work

• The power of Euler and homotopy operators:
  ▶ Testing exactness
  ▶ Integration by parts: $D_x^{-1}$ and $\text{Div}^{-1}$

• Integration of non-exact expressions

Example: $f = u_x v + uv_x + u^2 u_{xx}$

$$\int f \, dx = uv + \int u^2 u_{xx} \, dx$$

• Use other homotopy formulas (moving terms amongst the components of the flux; prevent curl terms)
• Broader class of PDEs (beyond evolution type)

**Example:** short pulse equation (nonlinear optics)

\[ u_{xt} = u + (u^3)_{xx} = u + 6uu_x^2 + 3u^2u_{xx} \]

with non-polynomial conservation law

\[ D_t \left( \sqrt{1 + 6u_x^2} \right) - D_x \left( 3u^2 \sqrt{1 + 6u_x^2} \right) = 0 \]

• Continue the implementation in *Mathematica*

• Software: [http://inside.mines.edu/~whereman](http://inside.mines.edu/~whereman)
Thank You
Publications

