Introduction to Wavelets in Image Processing
Pyramid Representation

- Recall that we can create a multi-resolution pyramid of images.
- At each level, we just store the differences (residuals) between the image at that level and the predicted image from the next level.
- We can reconstruct the image by just adding up all the residuals.
- Advantage: residuals are easier to store.
FIGURE 7.3 Two image pyramids and their statistics: (a) a Gaussian (approximation) pyramid and (b) a Laplacian (prediction residual) pyramid.
Wavelets

• Wavelets are a more general way to represent and analyze multiresolution images
• Can also be applied to 1D signals
• Very useful for
  – image compression (e.g., in the JPG-2000 standard)
  – removing noise
Wavelet Analysis

• Motivation
  – Sometimes we care about both frequency as well as time
  – Example: Music

  (a)  (b)  (c)  (d)

  – Time domain operations tell us “when”
  – Fourier domain operations tell us “frequency”
from Matlab help page on wavelets
Continuous Wavelet Transform

• Define a function $\psi(x)$
  – assume $\psi(x)$ band-limited and its dc component = 0

• Create scaled and shifted versions of $\psi(x)$

$$\psi_{s,\tau}(x) = \frac{1}{\sqrt{s}} \psi\left(\frac{x - \tau}{s}\right)$$

• Example:
Example of scaling

\begin{align*}
f(t) &= \psi(t) \quad ; \quad a = 1 \\
f(t) &= \psi(2t) \quad ; \quad a = \frac{1}{2} \\
f(t) &= \psi(4t) \quad ; \quad a = \frac{1}{4}
\end{align*}
Continuous Wavelet Transform

• Define the continuous wavelet transform of $f(x)$:

$$W_{\phi}(s, \tau) = \int_{-\infty}^{\infty} f(x)\psi_{s,\tau}(x) \, dx$$

• This transforms a continuous function of one variable into a continuous function of two variables: translation and scale

• The wavelet coefficients measure how closely correlated the wavelet is with each section of the signal

• For compact representation, choose a wavelet that matches the shape of the image components
  – Example: Haar wavelet for black and white drawings
Example

Low value for $W_\psi(s,\tau)$

Higher value of $W_\psi(s,\tau_2)$

Different scale
$f(x)$

$\psi(x)$

"mexican hat" wavelet
Matlab Demo

• Run “wavemenu”
  – Choose “Continuous wavelet 1D”
  – Choose “Example analysis” -> “frequency breakdown with mexh”
  – Look at magnitude of coefficients (right click on coefficients to select scale, then hit the button “new coefficients line”)
Inverse Transform

- Inverse continuous wavelet transform

\[
f(x) = \frac{1}{C_{\psi}} \int_0^\infty \int_{-\infty}^\infty W_\psi(s, \tau) \frac{\psi_{s, \tau}(x)}{s^2} d\tau \, ds
\]

- where

\[
C_{\psi} = \int_{-\infty}^\infty \frac{|\Psi(\mu)|}{|\mu|} d\mu
\]

- and \( \Psi(\mu) \) is the Fourier transform of \( \psi(x) \)
Discrete Wavelet Transform

- Don’t need to calculate wavelet coefficients at every possible scale
- Can choose scales based on powers of two, and get equivalent accuracy
  \[ \psi_{j,k}(x) = 2^{j/2} \psi(2^j x - k) \]
- We can represent a discrete function \( f(n) \) as a weighted summation of wavelets \( \psi(n) \), plus a coarse approximation \( \phi(n) \)
  \[
  f(n) = \frac{1}{\sqrt{M}} \sum_k W_\phi (j_0, k) \phi_{j_0,k}(n) + \frac{1}{\sqrt{M}} \sum_{j=j_0}^{\infty} \sum_k W_\psi (j, k) \psi_{j,k}(n)
  \]
  where \( j_0 \) is an arbitrary starting scale, and \( n = 0,1,2, \ldots M \)

<table>
<thead>
<tr>
<th>“Approximation” coefficients</th>
<th>“Detail” coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>( W_\phi (j_0, k) = \frac{1}{\sqrt{M}} \sum_x f(x) \phi_{j_0,k}(x) )</td>
<td>( W_\psi (j, k) = \frac{1}{\sqrt{M}} \sum_x f(x) \psi_{j,k}(x) )</td>
</tr>
</tbody>
</table>
Comparison with CWT

• Usually you don’t need to compute the continuous transform.

• A signal (with finite energy) can be reconstructed from the discrete transform.

From Matlab help page on wavelets.
Harr scaling functions

Harr wavelet functions
Example

- A function can be represented by a sum of approximation plus detail

\[
f(x) = f_a(x) + f_d(x)
\]

\[
f_a(x) = \frac{3\sqrt{2}}{4} \phi_{0,0}(x) - \frac{\sqrt{2}}{8} \phi_{0,2}(x)
\]

\[
f_d(x) = \frac{-\sqrt{2}}{4} \psi_{0,0}(x) - \frac{\sqrt{2}}{8} \psi_{0,2}(x)
\]
Matlab Demos

- “wavemenu”
- Do 1D discrete wavelet transform on noisy doppler signal, show denoising
Decomposition at level 5: \( s = a_5 + d_5 + d_4 + d_3 + d_2 + d_1 \).
Expanding to Two Dimensions

\[ W_\psi(j + 1, m, n) \]

\[ W_\psi(j, m, n) \]

\[ W_\psi^V(j, m, n) \]

\[ W_\psi^H(j, m, n) \]

\[ W_\psi^D(j, m, n) \]
a(m,n): approximation

d^V(m,n): detail in vertical

d^H(m,n): detail in horizontal

d^D(m,n): detail in diagonal
FIGURE 7.8 (a) A discrete wavelet transform using Haar basis functions. Its local histogram variations are also shown; (b)–(d) Several different approximations (64 × 64, 128 × 128, and 256 × 256) that can be obtained from (a).
Use of Wavelets in Processing

• Approach:
  – Compute the 2D wavelet transform
  – Alter the transform
  – Compute the inverse transform

• Examples:
  – De-noising
  – Compression
  – Image fusion
Figure 14–36  Wavelet transform image fusion: (a), (b) images taken at different focus settings; (c) fused image; (d) MRI image; (e) PET image; (f) fused image (Courtesy Henry Hui Li, reprinted by permission from [28])
Matlab Examples ("wavemenu")

• De-noising
  – Choose “SWT de-noising 2D”
  – Set threshold value to zero out coefficients below the threshold

• Compression
  – Choose “Wavelet coefficients selection 2D”

• Fusion
  – Choose “Image fusion”
Original Image - size = (256, 256)

Synthesized Image

Original Decomposition at level 5

Modified Decomposition at level 5
Summary / Questions

• Wavelets represent the scale of features in an image, as well as their position.
  – Can also be applied to 1D signals.

• They are useful for a number of applications including image compression.

• We can use them to process images:
  – Compute the 2D wavelet transform
  – Alter the transform
  – Compute the inverse transform

• What are some other applications of wavelet processing?