MODIFIED NODAL CUBIC SPLINE COLLOCATION FOR POISSON’S EQUATION

ABEER ALI ABUSHAMA † AND BERNARD BIALECKI‡

Abstract. We present a new modified nodal cubic spline collocation scheme for solving the Dirichlet problem for Poisson’s equation on the unit square. We prove existence and uniqueness of a solution of the scheme and show how the solution can be computed on an \((N + 1) \times (N + 1)\) uniform partition of the square with cost \(O(N^2 \log N)\) using a direct fast Fourier transform method. Using two comparison functions, we derive an optimal fourth order error bound in the continuous maximum norm. We compare our scheme with other modified nodal cubic spline collocation schemes, in particular, the one proposed by Houstis et al. in [8]. We believe that our paper gives the first correct convergence analysis of a modified nodal cubic spline collocation for solving partial differential equations.

Key words. nodal collocation, cubic splines, convergence analysis, interpolants

AMS subject classifications. 65N35, 65N12, 65N15, 65N22

1. Introduction. De Boor [7] proved that classical nodal cubic spline collocation for solving two-point boundary value problems is only second–order accurate and no better. For two-point boundary value problems, Archer [2] and independently Daniel and Swartz [6] developed a modified nodal cubic spline collocation (MNCSC) scheme which is fourth order accurate. The approximate solution in this scheme satisfies higher-order perturbations of the ordinary differential equation at the partition nodes. Based on the method of [2] and [6], Houstis et al. [8] derived a fourth order MNCSC scheme for solving elliptic boundary value problems on rectangles. For the Helmholtz equation, a direct fast Fourier transform (FFT) algorithm for solving this scheme was proposed recently in [3].

In this paper, we consider the Dirichlet boundary value problem for Poisson’s equation

\[
\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega,
\]

where \(\Delta\) denotes the Laplacian, \(\Omega = (0,1) \times (0,1)\), and \(\partial \Omega\) is the boundary of \(\Omega\). Let \(\rho_x = \{x_i\}_{i=0}^{N+1}\) be a uniform partition of \([0,1]\) in the \(x\)-direction such that \(x_i = ih,\) \(i = 0, \ldots, N + 1\), where \(h = 1/(N + 1)\). For the sake of simplicity, we assume that a uniform partition \(\rho_y = \{y_j\}_{j=0}^{N+1}\) of \([0,1]\) in the \(y\)-direction is such that \(y_j = x_j\). Let \(S_3\) be the space of cubic splines defined by

\[
S_3 = \{ v \in C^2[0,1] : v|_{[x_{i-1},x_i]} \in P_3, i = 1, \ldots, N + 1 \},
\]

where \(P_3\) denotes the set of all polynomials of degree \(\leq 3\), and let

\[
S^D = \{ v \in S_3 : v(0) = v(1) = 0 \}.
\]

Our MNCSC scheme for solving (1.1) is formulated as follows: Find \(u_h \in S^D \otimes S^D\) such that

\[
\Delta u_h(x_i, y_j) - \frac{h^2}{6} D^2_x D^2_y u_h(x_i, y_j) = f(x_i, y_j) - \frac{h^2}{12} \Delta f(x_i, y_j),
\]

\(i, j = 0, \ldots, N + 1\).

†Department of Mathematical and Computer Sciences, Colorado School of Mines, Golden, Colorado 80401-1887, U.S.A. (ashama@mines.edu)
‡Department of Mathematical and Computer Sciences, Colorado School of Mines, Golden, Colorado 80401-1887, U.S.A. (bbialeck@mines.edu)
The scheme (1.2) is motivated by the fourth order finite difference method for (1.1), see, for example, equation (7) in section 4.5 of [9]. Using \( u_h = u = 0 \) on \( \partial \Omega \) and (1.1), we see that (1.2) is equivalent to:

\[
2D_x^2 D_y^2 u_h(x_i, y_j) = \Delta f(x_i, y_j), \quad i, j = 0, N + 1,
\]

(1.3)

\[
D_x^2 u_h(x_i, y_j) - \frac{h^2}{6} D_x^2 D_y^2 u_h(x_i, y_j) = f(x_i, y_j) - \frac{h^2}{12} \Delta f(x_i, y_j),
\]

(1.4)

\[
D_y^2 u_h(x_i, y_j) - \frac{h^2}{6} D_x^2 D_y^2 u_h(x_i, y_j) = f(x_i, y_j) - \frac{h^2}{12} \Delta f(x_i, y_j),
\]

(1.5)

\[
\Delta u_h(x_i, y_j) - \frac{h^2}{6} D_x^2 D_y^2 u_h(x_i, y_j) = f(x_i, y_j) - \frac{h^2}{12} \Delta f(x_i, y_j), \quad i, j = 1, \ldots, N.
\]

(1.6)

The scheme (4.2)–(4.4) of [8] for (1.1) is: Find \( u_h \in S^D \otimes S^D \) satisfying (1.3) and

\[
D_x^2 u_h(x_i, y_j) = f(x_i, y_j), \quad i = 0, N + 1, \quad j = 1, \ldots, N,
\]

(1.7)

\[
D_y^2 u_h(x_i, y_j) = f(x_i, y_j), \quad i = 1, \ldots, N, \quad j = 0, N + 1,
\]

(1.8)

\[
(L_x + L_y) u_h(x_i, y_j) = f(x_i, y_j), \quad i, j = 1, \ldots, N,
\]

(1.9)

where, for \( i, j = 1, \ldots, N \),

\[
L_x v(x_i, y_j) = \frac{1}{12} \left[ D_x^2 v(x_{i-1}, y_j) + 10 D_x^2 v(x_i, y_j) + D_x^2 v(x_{i+1}, y_j) \right],
\]

\[
L_y v(x_i, y_j) = \frac{1}{12} \left[ D_y^2 v(x_i, y_{j-1}) + 10 D_y^2 v(x_i, y_j) + D_y^2 v(x_i, y_{j+1}) \right].
\]

(1.10)

Our scheme and that of [8] are identical at the corners of \( \overline{\Omega} \). However, they are different at the remaining partition nodes. While (1.4)–(1.6) involve perturbations of both the left- and right-hand sides, (1.9) involves a perturbation of the left-hand side only. Numerical results show that our scheme exhibits superconvergence phenomena while that of [8] does not.

An outline of this paper is as follows. We give preliminaries in section 2. The matrix-vector form of our scheme, an existence and uniqueness proof of its solution, and a direct FFT algorithm for solving the scheme are presented in section 3. In section 4, using two comparison functions, we derive a fourth order error bound in the continuous maximum norm. In section 5, we give convergence analysis of the scheme in [4] that consists of (1.3)–(1.5) and (1.9). We also explain why convergence analysis of the scheme (1.3) and (1.7)–(1.9), given in [8], is incorrect. This is why, we believe, our paper gives the first correct convergence analysis of MNCSC for solving partial differential equations. Section 6 includes numerical results obtained using our scheme.
2. Preliminaries. We extend the uniform partition \( \rho_x = \{ x_i \}_{i=0}^{N+1} \) outside of \([0,1]\) using \( x_i = ih, i = -3, -2, -1, N + 2, N + 3, N + 4 \), and introduce \( I_i = [x_{i-1}, x_i], i = -2, \ldots, N + 4 \). Let \( \{ B_m \}_{m=-1}^{N+2} \) be the basis for \( S_3 \) defined by

\[
B_m(x) = \begin{cases} 
  g_1(\frac{x-x_{m-2}}{h}), & x \in I_{m-1}, \\
  g_2(\frac{x-x_{m-1}}{h}), & x \in I_m, \\
  g_2(\frac{x_{m+1}-x}{h}), & x \in I_{m+1}, \\
  g_1(\frac{x_{m+2}-x}{h}), & x \in I_{m+2}, \\
  0, & \text{otherwise},
\end{cases}
\]

where

\[
g_1(x) = x^3, \quad g_2(x) = 1 + 3x + 3x^2 - 3x^3.
\]

The basis functions are such that, for \( m = 0, \ldots, N + 1 \),

\[
B_m(x) = \begin{cases} 
  1, & m = 0, \\
  4, & m = 1, \\
  6h^2, & m = 2, \\
  -12/h^2, & m = 3, \\
  6/h^2, & m = 4, \\
  6, & m = 5, \\
  0, & \text{otherwise},
\end{cases}
\]

Let \( \{ B_m \}_{m=-1}^{N+2} \) be the basis for \( S^D \) defined by

\[
B_0^D = B_0 - 4B_{-1}, \quad B_1^D = B_1 - B_{-1}, \\
B_m^D = B_m, \quad m = 2, \ldots, N - 1, \\
B_0^D = B_N - B_{N+2}, \quad B_{N+1}^D = B_{N+1} - 4B_{N+2}.
\]

It follows from (2.3) that

\[
B_0^D(x_1) = 1, \quad B_1^D(x_1) = 4, \quad B_2^D(x_2) = 1, \\
B_0^D(x_{N-1}) = 1, \quad B_1^D(x_N) = 4, \quad B_{N+1}^D(x_N) = 1,
\]

\[
B_0^D(x_i) - \frac{h^2}{6} [B_m^D]''(x_i) = \begin{cases} 
  6, & m = i, \\
  0, & m \neq i,
\end{cases} \quad m = 0, \ldots, N + 1.
\]

Throughout the paper, \( C \) denotes a generic positive constant that is independent of \( u \) and \( h \).

**Lemma 2.1.** \( \{ B_m^D \}_{m=0}^{N+1} \) of (2.4) satisfy \( \max_{x \in [0,1]} |B_m^D(x)| \leq C, m = 0, \ldots, N + 1 \).

**Proof.** For each fixed \( m = -1, \ldots, N + 2 \), using \( I_i = [x_{i-1}, x_i] \) and \( x_i = ih \), we have

\[
0 \leq \frac{x-x_{m-2}}{h} \leq 1, \quad x \in I_{m-1}, \quad 0 \leq \frac{x-x_{m-1}}{h} \leq 1, \quad x \in I_m, \\
0 \leq \frac{x_{m+1}-x}{h} \leq 1, \quad x \in I_{m+1}, \quad 0 \leq \frac{x_{m+2}-x}{h} \leq 1, \quad x \in I_{m+2}.
\]

Equations (2.2) and (2.8) give

\[
|g_1(\frac{x-x_{m-2}}{h})| \leq 1, \quad x \in I_{m-1}, \quad |g_2(\frac{x-x_{m-1}}{h})| \leq 7, \quad x \in I_m, \\
|g_2(\frac{x_{m+1}-x}{h})| \leq 7, \quad x \in I_{m+1}, \quad |g_1(\frac{x_{m+2}-x}{h})| \leq 1, \quad x \in I_{m+2}.
\]
Using (2.1) and (2.9), we see that $\max_{x \in [0,1]} |B_m(x)| \leq C, m = -1, \ldots, N + 2$. Hence the required inequality follows from (2.4) which implies that each $B_m^D$ is a linear combination of at most two of the functions $\{B_n\}_{n=-1}^{N+2}$.

For $\{B_m^D\}_{m=0}^{N+1}$ of (2.4), we introduce $N \times N$ matrices $A$ and $B$ defined by

$$A = (a_{i,m})_{i,m=1}^N, \quad a_{i,m} = [B_m^D]^m(x_i), \quad B = (b_{j,n})_{j,n=1}^N, \quad b_{j,n} = B_n^D(y_j).$$

It follows from (2.4), (2.3), (2.5), and (2.6) that

$$A = 6h^{-2}T, \quad B = T + 6I,$$

where $I$ is the identity matrix and the $N \times N$ matrix $T$ is given by

$$T = \begin{bmatrix}
-2 & 1 & \cdot & \cdot & \cdot \\
1 & -2 & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
1 & -2 & 1 & \cdot & \cdot \\
& & & & \\
\end{bmatrix}.$$

**Lemma 2.2.** If $[u_1, \ldots, u_N]^T = [v_1, \ldots, v_N]^T$, where $B$ is defined in (2.10), then $\max_{1 \leq i \leq N} |u_i| \leq C \max_{1 \leq i \leq N} |v_i|$.

**Proof.** It follows from (2.10), (2.11), and (2.12) that $|b_{i,j}| - \sum_{i \neq j} |b_{i,j}| \geq 2, i = 1, \ldots, N$. Hence the required result follows, for example, from the discussion on page 21 in [1].

In what follows, $[u_{1,1}, \ldots, u_{N,N}]^T$ is the short notation for

$$[u_{1,1}, u_{1,N}, u_{2,1}, \ldots, u_{2,N}, \ldots, u_{N,1}, \ldots, u_{N,N}]^T.$$

**Lemma 2.3.** If $u = [u_{1,1}, \ldots, u_{N,N}]^T$ and $v = [v_{1,1}, \ldots, v_{N,N}]^T$ are such that $(B \otimes B)u = v$, where $B$ is defined in (2.10), then $\max_{1 \leq i,j \leq N} |u_{i,j}| \leq C \max_{1 \leq i,j \leq N} |v_{i,j}|$.

**Proof.** Since $B \otimes B = (B \otimes I)(I \otimes B)$, we have

$$v = (B \otimes I)w, \quad w = (I \otimes B)u.$$

Using (2.13) and Lemma 2.2, we obtain

$$\max_{1 \leq i,j \leq N} |u_{i,j}| \leq C \max_{1 \leq i,j \leq N} |w_{i,j}|, \quad \max_{1 \leq i,j \leq N} |w_{i,j}| \leq C \max_{1 \leq i,j \leq N} |v_{i,j}|,$$

which imply the required inequality. \qed

It is well known (see Theorem 4.5.2 of [10]) that for $T$ of (2.12), we have

$$QTQ = \Lambda, \quad QQ = I,$$

where the $N \times N$ matrices $\Lambda$ and $Q$ are given by

$$\Lambda = \text{diag}(\lambda_i)_{i=1}^N, \quad \lambda_i = -4\sin^2 \frac{i\pi}{2(N+1)},$$

$$Q = (q_{i,j})_{i,j=1}^N, \quad q_{i,j} = \left(\frac{2}{N+1}\right)^{1/2} \sin \frac{i j \pi}{N+1}.$$

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Lemma 2.4. If \( \mathbf{v} = [v_{1,1}, \ldots, v_{N,N}]^T \) and \( \mathbf{w} = [w_{1,1}, \ldots, w_{N,N}]^T \) are such that

\[
\begin{bmatrix}
\frac{T}{h^2} \otimes I + I \otimes \frac{T}{h^2} + \frac{h^2}{6} \left( \frac{T}{h^2} \otimes \frac{T}{h^2} \right)
\end{bmatrix} \mathbf{v} = \mathbf{w},
\]

where \( T \) is the matrix defined in (2.12), then \( \max_{1 \leq i,j \leq N} v_{i,j}^2 \leq C h^2 \sum_{i=1}^{N} \sum_{j=1}^{N} w_{i,j}^2. \)

Proof. The matrix in (2.17) arises in the fourth order finite difference method for (1.1). Hence the desired result follows, for example, from the last unnumbered equation on page 296 in [9]. \( \square \)

Finally, we observe that the matrix-vector form of

\[
\phi_{i,j} = \sum_{m=1}^{N} \sum_{n=1}^{N} c_{1,m}^{(1)} c_{j,n}^{(2)} \psi_{m,n}, \quad i, j = 1, \ldots, N,
\]

is

\[
\phi = (C_1 \otimes C_2) \psi,
\]

where \( C_1 = \left( c_{1,m}^{(1)} \right)_{m=1}^{N}, \quad C_2 = \left( c_{j,n}^{(2)} \right)_{j=1}^{N}, \) and

\[
\phi = [\phi_{1,1}, \ldots, \phi_{N,N}]^T, \quad \psi = [\psi_{1,1}, \ldots, \psi_{N,N}]^T.
\]

3. Matrix-Vector Form of Scheme. Since \( \dim(S^D \otimes S^D) = (N + 2)^2 \), the scheme (1.3)–(1.6) involves \( (N + 2)^2 \) equations in \( (N + 2)^2 \) unknowns. Using the basis \( \{B_m^{D}\}_{m=0}^{N+1} \) of (2.4) for the space \( S^D \), we have

\[
u_{h}(x, y) = \sum_{m=0}^{N+1} \sum_{n=0}^{N+1} u_{m,n} B_m^{D}(x) B_n^{D}(y).
\]

Substituting (3.1) into (1.3), we obtain

\[
2 \sum_{m=0}^{N+1} \sum_{n=0}^{N+1} u_{m,n} \left[ B_m^{D}(x_i) \right] \left[ B_n^{D}(y_j) \right] = f(x_i, y_j), \quad i, j = 0, N + 1.
\]

Using (2.6), we conclude that (3.2) gives

\[
u_{i,j} = \frac{h^4}{2592} \Delta f(x_i, y_j), \quad i, j = 0, N + 1.
\]

Substituting (3.1) into (1.4), we obtain

\[
\sum_{m=0}^{N+1} \sum_{n=0}^{N+1} u_{m,n} \left[ B_m^{D}(x_i) \right] \left( B_n^{D}(y_j) - \frac{h^2}{6} [B_n^{D}]''(y_j) \right)
\]

\[
= f(x_i, y_j) - \frac{h^2}{12} \Delta f(x_i, y_j), \quad i = 0, N + 1, \quad j = 1, \ldots, N.
\]

Using (2.6) and (2.7), we see that (3.4) gives

\[
u_{i,j} = -\frac{h^2}{216} f(x_i, y_j) + \frac{h^4}{2592} \Delta f(x_i, y_j), \quad i = 0, N + 1, \quad j = 1, \ldots, N.
\]
Using (3.5) and symmetry with respect to $x$ and $y$, we conclude that (1.5) gives

\[(3.6) \quad u_{i,j} = -\frac{h^2}{216} f(x_i, y_j) + \frac{h^4}{2592} \Delta f(x_i, y_j), \quad i = 1, \ldots, N, \quad j = 0, N + 1.\]

Substituting (3.1) into (1.6), we obtain

\[(3.7) \quad \sum_{m=0}^{N+1} \sum_{n=0}^{N+1} u_{m,n} \left( [B^{D}]''(x_i) B_n^D(y_j) + \left[ B_m^D(x_i) - \frac{h^2}{6} [B_m^D]''(x_i) \right] [B_n^D]''(y_j) \right) = f(x_i, y_j) - \frac{h^2}{12} \Delta f(x_i, y_j), \quad i, j = 1, \ldots, N.\]

Moving the terms involving $\{u_{m,n}\}_{n=0}^{N+1}$, $m = 0, N + 1$, $\{u_{m,n}\}_{m=1}^{N}$, $n = 0, N + 1$, to the right-hand side of (3.7), we get

\[(3.8) \quad \sum_{m=1}^{N} \sum_{n=1}^{N} u_{m,n} \left( [B^{D}]''(x_i) B_n^D(y_j) + \left[ B_m^D(x_i) - \frac{h^2}{6} [B_m^D]''(x_i) \right] [B_n^D]''(y_j) \right) = p_{i,j}, \quad i, j = 1, \ldots, N,

where

\[
p_{i,j} = f(x_i, y_j) - \frac{h^2}{12} \Delta f(x_i, y_j)
\]

\[
- \sum_{m=0}^{N+1} \sum_{n=0}^{N+1} u_{m,n} \left( [B^{D}]''(x_i) B_n^D(y_j) + \left[ B_m^D(x_i) - \frac{h^2}{6} [B_m^D]''(x_i) \right] [B_n^D]''(y_j) \right)
\]

\[
- \sum_{m=1}^{N} \sum_{n=0}^{N+1} u_{m,n} \left( [B^{D}]''(x_i) B_n^D(y_j) + \left[ B_m^D(x_i) - \frac{h^2}{6} [B_m^D]''(x_i) \right] [B_n^D]''(y_j) \right).
\]

Using (2.18)–(2.19), we write (3.8) as

\[(3.9) \quad \left[ A \otimes B + \left( B - \frac{h^2}{6} A \right) \otimes A \right] u = p,
\]

where $u = [u_{1,1}, \ldots, u_{N,N}]^T$, $p = [p_{1,1}, \ldots, p_{N,N}]^T$, and $A$, $B$ are defined in (2.10).

Using (2.11), we see that

\[
A \otimes B + \left( B - \frac{h^2}{6} A \right) \otimes A = \frac{6}{h^2} \left[ 6T \otimes I + (6I + T) \otimes T \right],
\]

and hence the system (3.9) simplifies to

\[(3.10) \quad 6h^{-2} \left[ 6T \otimes I + (6I + T) \otimes T \right] u = p.
\]

We are now ready to prove existence and uniqueness of $u_h$ in $S^D \otimes S^D$ that satisfies (1.3)–(1.6).

**Theorem 3.1.** There exists unique $u_h$ in $S^D \otimes S^D$ satisfying (1.3)–(1.6).

**Proof.** Since the number of equations in (1.3)–(1.6) is equal to the number of unknowns, we assume that the right-hand side in (1.3)–(1.6) is zero, and show that $u_h = 0$ is the only solution of the resulting scheme. Using (3.1), (3.3), (3.5), and (3.6), we have

\[(3.11) \quad u_{m,n} = 0, \quad m = 0, N + 1, n = 0, \ldots, N + 1, \quad m = 1, \ldots, N, n = 0, N + 1.
\]
(3.12) \[ 6h^{-2}[6T \otimes I + (6I + T) \otimes T] = 36 \left[ \frac{T}{h^2} \otimes I + I \otimes \frac{T}{h^2} + \frac{h^2}{6} \left( \frac{T}{h^2} \otimes \frac{T}{h^2} \right) \right]. \]

Hence it follows from (3.10) with \( p \) replaced by \( 0 \), (3.12), and Lemma 2.4 that

(3.13) \[ u_{m,n} = 0, \quad m, n = 1, \ldots, N. \]

Equations (3.1), (3.11), and (3.13) give \( u_h = 0 \). \( \square \)

Using \( Q \) of (2.16), we see that (3.10) is equivalent to

(3.14) \[ 6h^{-2}(Q \otimes I)[6T \otimes I + (6I + T) \otimes T] (Q \otimes I)(Q^{-1} \otimes I)u = (Q \otimes I)p. \]

Introducing \( u' = (Q^{-1} \otimes I)u \) and \( p' = (Q \otimes I)p \), and using (3.14) and (2.14), we obtain

(3.15) \[ 6h^{-2} [6\Lambda \otimes I + (6I + \Lambda) \otimes T]u' = p', \]

where \( \Lambda \) is defined in (2.15). The system (3.15) reduces to the \( N \) independent systems

(3.16) \[ 6h^{-2} [6\lambda_i I + (6 + \lambda_i)T]u'_i = p'_i, \quad i = 1, \ldots, N, \]

where \( u'_i = [u'_{i,1}, \ldots, u'_{i,N}]^T \), \( p'_i = [p'_{i,1}, \ldots, p'_{i,N}]^T \), \( i = 1, \ldots, N \).

We have the following algorithm for solving (3.10):

Step 1. Compute \( p' = (Q \otimes I)p \).

Step 2. Solve the \( N \) systems in (3.16).

Step 3. Compute \( u = (Q \otimes I)u' \).

Since the entries of \( Q \) in (2.16) are given in terms of sines, steps 1 and 3 are performed each using FFTs at a cost \( O(N^2 \log N) \). In step 2, the systems are tridiagonal, so this step is performed at a cost \( O(N^2) \). Thus the total cost of the algorithm is \( O(N^2 \log N) \).

4. Convergence Analysis. In what follows, \( C(u) \) denotes a generic positive constant that is independent of \( h \), but depends on \( u \).

Our goal is to show that if \( u \) in \( C^6(\Omega) \) and \( u_h \) in \( S^D \otimes S^D \) are the solutions of (1.1) and (1.3)–(1.6), respectively, then

(4.1) \[ \|u - u_h\|_{C(\Omega)} \leq C(u)h^4, \]

where \( \|g\|_{C(\Omega)} = \max_{x \in \Omega} |g(x)| \) for \( g \) in \( C(\Omega) \).

To prove (4.1), for \( u \) in \( C^4(\Omega) \), we introduce two comparison functions, the spline interpolants \( S \) and \( Z \) in \( S^D \otimes S^D \) of \( u \) defined respectively by

(4.2) \[ D_x^2 D_y^2 S(x_i, y_j) = D_x^2 D_y^2 u(x_i, y_j), \quad i, j = 0, N + 1, \]

(4.3) \[ D_x^2 S(x_i, y_j) - \frac{h^2}{6} D_x^2 D_y^2 S(x_i, y_j) = D_x^2 u(x_i, y_j) - \frac{h^2}{12} D_x^4 u(x_i, y_j) \]

\[ - \frac{h^2}{6} D_x^2 D_y^2 u(x_i, y_j), \quad i = 0, N + 1, \quad j = 1, \ldots, N, \]
Comparing (4.11)–(4.13) and (4.2)–(4.4), we see that
\[ u \] and the other hand, (4.6)–(4.8) are a simplified, tensor product version of (4.2)–(4.4).

It follows from (1.1) that
\[ f = D_x^2 u + D_y^2 u, \quad \Delta f = D_x^4 u + D_y^4 u + 2D_x^2 D_y^2 u. \]

Hence, using (4.10), we see that (1.3)–(1.5) reduce, respectively, to
\[ D_x^2 D_y^2 u_h(x, y) = D_x^2 D_y^2 u(x, y), \quad i = 0, N + 1, \]

\[ D_x^2 D_y^2 u_h(x, y) = D_x^2 D_y^2 u(x, y), \quad i = 1, \ldots, N. \]

(4.9) \[ Z(x, y) = u(x, y), \quad i, j = 1, \ldots, N. \]

Comparing (4.11)–(4.13) and (4.2)–(4.4), we see that \( u_h \) and \( S \) are defined in the same way for \( i = 0, N + 1, j = 0, \ldots, N + 1 \), and \( i = 1, \ldots, N, j = 0, N + 1 \). On the other hand, (4.6)–(4.8) are a simplified, tensor product version of (4.2)–(4.4).

The triangle inequality gives
\[ \|u - u_h\|_{C^0(\Omega)} \leq \|u - Z\|_{C^0(\Omega)} + \|Z - S\|_{C^0(\Omega)} + \|S - u_h\|_{C^0(\Omega)}. \]

In what follows, we bound the three terms on the right-hand side of (4.14).
4.1. Bounding $\|u - Z\|_{C^2(\Omega)}$. We need the following results.

**Lemma 4.1.** Let the interpolant $I_xv$ in $S_3$ of $v$ in $C^2[0,1]$ be defined by

(4.15) $(I_xv)''(x_i) = v''(x_i), \ i = 0, N + 1, \ I_xv(x_i) = v(x_i), \ i = 0, \ldots, N + 1.$

Then

(4.16) \[ \max_{x \in [0,1]} |v(x) - I_xv(x)| \leq C \max_{x \in [0,1]} |v''(x)|h^2. \]

If $v \in C^4[0,1]$, then

(4.17) \[ \max_{x \in [0,1]} |v(x) - I_xv(x)| \leq C \max_{x \in [0,1]} |v^{(4)}(x)|h^4. \]

**Proof.** First we prove (4.16). Using the discussion on page 404 in [5], we have

(4.18) \[ I_xv(x) = v(x_i) + B_i(x - x_i) + C_i(x - x_i)^2 + D_i(x - x_i)^3, \ x \in [x_i, x_{i+1}], \]

for $i = 0, \ldots, N$, where

(4.19) \[ B_i = -\frac{h}{6}r_{i+1} - \frac{h}{3}r_i + \frac{1}{h}[v(x_{i+1}) - v(x_i)], \ C_i = \frac{r_i}{2}, \ D_i = \frac{1}{6h}(r_{i+1} - r_i), \]

and $r_i = (I_xv)''(x_i)$. Equations (4.18) and (4.19) give

(4.20) \[ I_xv(x) - v(x) = A_i(x) - \frac{h}{6}r_{i+1}(x - x_i) - \frac{h}{3}r_i(x - x_i) + \frac{r_i}{2}(x - x_i)^2 \]

\[ + \frac{1}{6h}(r_{i+1} - r_i)(x - x_i)^3, \ x \in [x_i, x_{i+1}], \]

where

(4.21) \[ A_i(x) = v(x_i) - v(x) + \frac{v(x_{i+1}) - v(x_i)}{h}(x - x_i), \ x \in [x_i, x_{i+1}]. \]

Using (4.20) and the triangle inequality, we obtain, for $x \in [x_i, x_{i+1}],$

(4.22) \[ |I_xv(x) - v(x)| \leq |A_i(x)| + h^2 \left( |r_i| + \frac{1}{3} |r_{i+1}| \right) \leq |A_i(x)| + \frac{4}{3}h^2 \max_{0 \leq i \leq N+1} |r_i|. \]

We introduce

\[ E = (e_{i,j})_{i,j=0}^{N+1} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 4 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & 4 & 1 & \cdots & 1 \end{bmatrix}, \quad r = [r_0, \ldots, r_{N+1}]^T, \quad p = [p_0, \ldots, p_{N+1}]^T, \]

where

(4.23) \[ p_i = \begin{cases} v''(x_i), & i = 0, N + 1, \\ h^{-2}[v(x_{i-1}) - 2v(x_i) + v(x_{i+1})], & i = 1, \ldots, N. \end{cases} \]

It follows from the discussion on pages 400 and 401 in [5] that $Er = p$. Since \[ |e_{i,i} - \sum_{i \neq j} |e_{i,j}| \geq 1, \ i, j = 0, \ldots, N + 1, \] the discussion on page 21 in [1] implies that

(4.24) \[ \max_{0 \leq i \leq N+1} |r_i| \leq C \max_{0 \leq i \leq N+1} |p_i|. \]
Using Taylor’s theorem, we obtain

\[(4.25) \quad |v(x_{i-1}) - 2v(x_i) + v(x_{i+1})| \leq Ch^2 \max_{x \in [0,1]} |v''(x)|, \quad i = 1, \ldots, N.\]

It follows from \((4.24), (4.23)\) and \((4.25)\), that

\[(4.26) \quad \max_{0 \leq i \leq N+1} |r_i| \leq C \max_{x \in [0,1]} |v''(x)|.\]

Using Taylor’s theorem to expand \(v(x)\), \(x \in [x_i, x_{i+1}]\), around \(x_i\), we have

\[(4.27) \quad v(x) = v(x_i) + (x - x_i)v'(x_i) + \frac{(x - x_i)^2}{2}v''(\xi_{i,x}), \quad x_i \leq \xi_{i,x} \leq x.\]

Using \((4.21), (4.27)\), and the triangle inequality, we obtain, for \(x \in [x_i, x_{i+1}]\),

\[(4.28) \quad |A_i(x)| = \left|\frac{(x - x_i)^2}{2}v''(\xi_{i,x}) - \frac{h}{2}(x - x_i)v''(\xi_{i,x+1})\right| \leq h^2 \max_{x \in [0,1]} |v''(x)|.\]

Inequality \((4.16)\) follows from \((4.22), (4.26)\), and \((4.28)\).

Proof. Let \(\{C_i\}_{i=0}^{N+3}\) be the basis for \(S_3\) such that

\[(4.30)\]

\[C_i(x_j) = \delta_{ij}, \quad i, j = 0, \ldots, N + 1,
C_i'(x_j) = 0, \quad i = 0, \ldots, N + 1, \quad j = 0, N + 1,
C_{N+2}(x_j) = C_{N+3}(x_j) = 0, \quad j = 0, \ldots, N + 1,
C_{N+2}(x_0) = C_{N+3}(x_{N+1}) = 1, \quad C_{N+2}(x_{N+1}) = C_{N+3}(x_0) = 0,\]

where \(\delta_{ij}\) is the Kronecker delta. Using \((4.15)\) and \((4.30)\), we have for \((x, y) \in \Omega\),

\[I_x(I_y u)(x, y)\]

\[= I_x \left[ \sum_{j=0}^{N+1} u(x, y_j)C_j(y) + D_y^2 u(x, y_0)C_{N+2}(y) + D_y^2 u(x, y_{N+1})C_{N+3}(y) \right] C_i(x)\]

\[= \sum_{j=0}^{N+1} \left[ \sum_{j=0}^{N+1} u(x, y_j)C_j(y) + D_y^2 u(x_i, y_0)C_{N+2}(y) + D_y^2 u(x_i, y_{N+1})C_{N+3}(y) \right] C_i(x)\]

\[+ \sum_{j=0}^{N+1} D_y^2 u(x, y_j)C_j(y) + D_y^2 D_y^2 u(x_0, y_0)C_{N+2}(y)\]

\[+ D_y^2 D_y^2 u(x_0, y_{N+1})C_{N+3}(y) \right] C_{N+2}(x) + \left[ \sum_{j=0}^{N+1} D_y^2 u(x_{N+1}, y_j)C_j(y)\right.\]

\[+ D_y^2 D_y^2 u(x_{N+1}, y_0)C_{N+2}(y) + D_y^2 D_y^2 u(x_{N+1}, y_{N+1})C_{N+3}(y) \right] C_{N+3}(x).\]
Since \( u = 0 \) on \( \partial \Omega \), all terms involving \( C_0(x) \), \( C_N+1(x) \), \( C_0(y) \), \( C_N+1(y) \) drop out which implies that \( I_x(I_y u) \in S^D \otimes S^D \). Using (4.30), we verify that \( I_x(I_y u) \) satisfies (4.6)–(4.9), that is, (4.6)–(4.9) hold with \( I_x(I_y u) \) in place of \( Z \). Hence, the uniqueness of the interpolant \( Z \) implies the first equation in (4.29). To prove the second equation in (4.29), we use (4.15) and (4.30) to see that for \( (x, y) \in \Pi \),

\[
I_y(D^2u)(x, y) = \sum_{j=0}^{N+1} D_x^2u(x, y_j)C_j(y) + D_y^2D_x^2u(x, y_0)C_{N+2}(y) + D_y^2D_x^2u(x, y_{N+1})C_{N+3}(y).
\]

From (4.33) and (4.34), we have

\[
\|u - Z\|_{C(\Pi)} \leq \|u - I_x u\|_{C(\Pi)} + \|I_x(u - I_y u) - (u - I_y u)\|_{C(\Pi)}.
\]

For any fixed \( y \) in \([0, 1]\), \( I_x u(\cdot, y) \) is the cubic spline interpolant of \( u(\cdot, y) \). Using this, symmetry with respect to \( x \) and \( y \), and (4.17), we have

\[
\|u - I_x u\|_{C(\Pi)} \leq C(u)h^4, \quad \|u - I_y u\|_{C(\Pi)} \leq C(u)h^4.
\]

For any fixed \( y \) in \([0, 1]\), \( I_x u(\cdot, y) \) is the cubic spline interpolant of \((u - I_y u)(\cdot, y)\). Hence it follows from (4.16) that

\[
\|I_x(u - I_y u) - (u - I_y u)\|_{C(\Pi)} \leq C\|D^2_x(u - I_y u)\|_{C(\Pi)}h^2.
\]

Using (4.29) and (4.16), we obtain

\[
\|D^2_x(u - I_y u)\|_{C(\Pi)} = \|D^2_x u - I_y(D^2_x u)\|_{C(\Pi)} \leq C\|D^2_x D^2_y u\|_{C(\Pi)}h^2.
\]

Combining (4.33) and (4.34), we have

\[
\|I_x(u - I_y u) - (u - I_y u)\|_{C(\Pi)} \leq C(u)h^4.
\]

The desired inequality now follows from (4.31), (4.32), and (4.35). \( \square \)

4.2. Bounding \( \|Z - S\|_{C(\Pi)} \). We start by proving the following lemma.

**Lemma 4.3.** If \( u \in C^4(\Pi) \) and

\[
S(x, y) = \sum_{m=0}^{N+1} \sum_{n=0}^{N+1} s_{m,n} B^D_m(x) B^D_n(y), \quad Z(x, y) = \sum_{m=0}^{N+1} \sum_{n=0}^{N+1} z_{m,n} B^D_m(x) B^D_n(y),
\]

are defined by (4.2)–(4.5) and (4.6)–(4.9), respectively, then

\[
|s_{m,n} - z_{m,n}| \leq C(u)h^4, \quad m, n = 0, \ldots, N + 1.
\]
Next we prove the required inequality for \( m = 0, n = 1, \ldots, N \). Using (4.7), we have
\[
D_x^2(S - Z)(x_0, y_j) = D_x^2 S(x_0, y_j) - D_x^2 u(x_0, y_j), \quad j = 1, \ldots, N.
\]
It follows from (4.36), (4.37), and (2.6) that
\[
D_x^2(S - Z)(x_0, y_j) = -36h^{-2} \sum_{n=1}^{N} (s_{0,n} - z_{0,n}) B_n^D(y_j), \quad j = 1, \ldots, N.
\]
Using (4.44), (4.43), and (4.45), we obtain, for \( j = 1, \ldots, N \),
\[
D_x^2 S(x_0, y_j) = -36h^{-2} \sum_{n=0}^{N+1} s_{0,n} B_n^D(y_j) = -36h^{-2}(s_{0,j-1} + 4s_{0,j} + s_{0,j+1}).
\]
Substituting (4.39) and (4.40) into (4.38), and multiplying through by \(-h^2/36\), we have
\[
\sum_{n=1}^{N} (s_{0,n} - z_{0,n}) B_n^D(y_j) = s_{0,j-1} + 4s_{0,j} + s_{0,j+1} + \frac{h^2}{36} D_x^2 u(x_0, y_j)
\]
for \( j = 1, \ldots, N \). Using (4.2), (4.3), and following the derivations of (3.3) from (1.3) and (3.5) from (1.4), we obtain
\[
s_{0,j} = \frac{h^4}{1296} D_x^2 D_y^2 u(x_0, y_j), \quad j = 0, N + 1,
\]
and
\[
s_{0,j} = -\frac{h^2}{216} \left[ D_x^2 u(x_0, y_j) - \frac{h^2}{12} D_x^4 u(x_0, y_j) + \frac{h^2}{6} D_x^2 D_y^2 u(x_0, y_j) \right]
\]
for \( j = 1, \ldots, N \). Since \( u = 0 \) on \( \partial \Omega \), (4.42) is the same as (4.43) with \( j = 0, N + 1 \). This observation and (4.43) imply that for \( j = 1, \ldots, N \), we have
\[
s_{0,j\pm 1} = -\frac{h^2}{216} \left[ D_x^2 u(x_0, y_{j\pm 1}) - \frac{h^2}{12} D_x^4 u(x_0, y_{j\pm 1}) + \frac{h^2}{6} D_x^2 D_y^2 u(x_0, y_{j\pm 1}) \right].
\]
Using Taylor’s theorem, we obtain
\[
D_x^2 u(x_0, y_{j\pm 1}) = D_x^2 u(x_0, y_j) \pm h D_x^2 D_y u(x_0, y_j) + \frac{h^2}{2} D_x^2 D_y^2 u(x_0, \xi^\pm_j),
\]
where \( y_{j-1} \leq \xi^-_j \leq y_j, y_j \leq \xi^+_j \leq y_{j+1} \). Using (4.44), (4.43), and (4.45), we obtain
\[
\left| s_{0,j-1} + 4s_{0,j} + s_{0,j+1} + \frac{h^2}{36} D_x^2 u(x_0, y_j) \right| \leq C(u) h^4, \quad j = 1, \ldots, N.
\]
It follows from (4.46) that (4.41) is a system in \( \{s_{0,n} - z_{0,n}\}_{n=1}^{N} \) with the matrix \( B \) defined in (2.10) and with each entry on the right-hand side bounded in absolute value by \( C(u) h^4 \). Hence, Lemma 2.2 implies
\[
\max_{1 \leq n \leq N} |s_{0,n} - z_{0,n}| \leq C(u) h^4.
\]
Using (4.47) and symmetry with respect to $x$ and $y$, we also have

$$\max_{1 \leq n \leq N} |s_{N+1,n} - z_{N+1,n}| \leq C(u)h^4, \quad \max_{1 \leq m \leq N} |s_{m,n} - z_{m,n}| \leq C(u)h^4, \quad n = 0, N + 1. \tag{4.48}$$

Finally we prove the required inequality for $m, n = 1, \ldots, N$. Using (4.5) and (4.9), we have $(S - Z)(x_i, y_j) = 0$, $i, j = 1, \ldots, N$, which, by (4.36) and (4.37), can be written as

$$\sum_{m=1}^{N} \sum_{n=1}^{N} (s_{m,n} - z_{m,n})B_m^D(x_i)B_n^D(y_j) = d_{i,j}, \quad i, j = 1, \ldots, N, \tag{4.49}$$

where

$$d_{i,j} = \left( \sum_{m=0,N+1}^{N} \sum_{n=1}^{N} + \sum_{m=1}^{N} \sum_{n=0,N+1}^{N} \right) (z_{m,n} - s_{m,n})B_m^D(x_i)B_n^D(y_j).$$

Since for any fixed $i, j$, each of the above double sums reduces to at most three terms, using the triangle inequality, (4.47), (4.48), and Lemma 2.1, we obtain

$$|d_{i,j}| \leq C(u)h^4, \quad i, j = 1, \ldots, N. \tag{4.50}$$

It follows from (2.18)–(2.19) that (4.49) is a system in \{\(z_{m,n} - s_{m,n}\)\}_{m,n=1}^{N} with the matrix $B \otimes B$, where $B$ is defined in (2.10). Hence, for $m, n = 1, \ldots, N$, the required inequality follows from (4.50) and Lemma 2.3.

**Theorem 4.2.** If $u \in C^4(\overline{\Omega})$ and $S, Z$ in $S^D \otimes S^D$ are defined by (4.2)–(4.5) and (4.6)–(4.9), respectively, then $\|Z - S\|_{C(\overline{\Omega})} \leq C(u)h^4$.

**Proof.** Since $Z - S$ is continuous on $\overline{\Omega}$, there is $(x_*, y_*)$ in $\overline{\Omega}$ such that

$$\|Z - S\|_{C(\overline{\Omega})} = |(Z - S)(x_*, y_*)|. \tag{4.51}$$

Hence, (4.36) and the triangle inequality imply

$$\|Z - S\|_{C(\overline{\Omega})} \leq \sum_{m=0}^{N+1} \sum_{n=0}^{N+1} |s_{m,n} - z_{m,n}| |B_m^D(x_*)||B_n^D(y_*)|. \tag{4.52}$$

Since the above double sum reduces to at most nine terms, the required inequality follows from Lemmas 4.3 and 2.1.

**4.3. Bounding $\|S - u_h\|_{C(\overline{\Omega})}$ and $\|u - u_h\|_{C(\overline{\Omega})}$**. We need the following results.

**Lemma 4.4.** If $u \in C^6(\overline{\Omega})$ and $S$ in $S^D \otimes S^D$ is defined by (4.2)–(4.5), then for $i = 0, N + 1$, $j = 1, \ldots, N$,

$$\left| D_x^2 D_y^2 S(x_i, y_j) - D_x^2 D_y^2 u(x_i, y_j) \right| \leq C(u)h^2, \tag{4.51}$$

$$\left| D_x^2 S(x_i, y_j) - D_x^2 u(x_i, y_j) + \frac{h^2}{12} D_x^4 u(x_i, y_j) \right| \leq C(u)h^4. \tag{4.52}$$
Proof. We prove (4.51) for $i = 0$; for $i = N + 1$, (4.51) follows by symmetry with respect to $x$. Using (4.36), we obtain

$$D^2_x D^2_y S(x_0, y_j) = \sum_{m=0}^{N+1} \sum_{n=0}^{N+1} s_{m,n} [B_m^D]''(x_0) [B_n^D]''(y_j), \quad j = 1, \ldots, N,$$

and hence (2.4), (2.3), and (2.6) imply

$$(4.53) \quad D^2_x D^2_y S(x_0, y_j) = -216 h^{-4}(s_{0,j-1} - 2s_{0,j} + s_{0,j+1}), \quad j = 1, \ldots, N.$$  

Equations (4.53), (4.43), and (4.44) give, for $j = 1, \ldots, N$,

$$D^2_x D^2_y S(x_0, y_j) = -D^2_x D^2_y u(x_0, y_j) + h^{-2} \left[ D^2_x u(x_0, y_{j-1}) - 2D^2_x u(x_0, y_j) + D^2_x u(x_0, y_{j+1}) \right] \tag{4.54}$$

$$- \frac{1}{12} [D^4_y u(x_0, y_{j-1}) - 2D^4_y u(x_0, y_j) + D^4_y u(x_0, y_{j+1})] \tag{4.54}$$

$$- \frac{1}{6} \left[ D^2_x D^2_y u(x_0, y_{j-1}) - 2D^2_x D^2_y u(x_0, y_j) + D^2_x D^2_y u(x_0, y_{j+1}) \right].$$

Using Taylor’s theorem, we obtain

$$D^4_x u(x_0, y_{j\pm 1}) = D^4_x u(x_0, y_j) \pm h D^4_x D_y u(x_0, y_j) + \frac{h^2}{2} D^4_x D^4_y u(x_0, y_j) \tag{4.55}$$

$$+ \frac{h^3}{3!} D^4_x D^3_y u(x_0, y_j) + \frac{h^4}{4!} D^4_x D^4_y u(x_0, y_j),$$

$$D^4_y u(x_0, y_{j\pm 1}) = D^4_y u(x_0, y_j) \pm h D^4_y D_x u(x_0, y_j) + \frac{h^2}{2} D^4_x D^4_y u(x_0, y_j) \tag{4.56}$$

where $y_{j-1} \leq \xi_j^-, \eta_j^-, \kappa_j^+ \leq y_j \leq \xi_j^+, \eta_j^+, \kappa_j^+ \leq y_{j+1}$. Equations (4.55)–(4.57) give

$$|h^{-2} \left[ D^2_x u(x_0, y_{j-1}) - 2D^2_x u(x_0, y_j) + D^2_x u(x_0, y_{j+1}) \right] - D^2_x D^2_y u(x_0, y_j)| \leq C(u) h^2,$$

$$|D^4_x u(x_0, y_{j-1}) - 2D^4_x u(x_0, y_j) + D^4_x u(x_0, y_{j+1})| \leq C(u) h^2,$$

$$|D^4_y u(x_0, y_{j-1}) - 2D^4_y u(x_0, y_j) + D^4_y u(x_0, y_{j+1})| \leq C(u) h^2,$$

and hence (4.51) for $i = 0$ follows from (4.54) and the triangle inequality. Using (4.3) and (4.51), we obtain (4.52).

**Lemma 4.5.** If $u \in C^5(\overline{\Omega})$ and $S$ in $S^D \otimes S^D$ is defined by (4.2)–(4.5), then, for $i, j = 1, \ldots, N$, we have

$$|D^2_x S(x_i, y_j) - D^2_x u(x_i, y_j) + \frac{h^2}{12} D^4_x u(x_i, y_j)| \leq C(u) h^4,$$  

$$|D^2_y S(x_i, y_j) - D^2_y u(x_i, y_j) + \frac{h^2}{12} D^4_y u(x_i, y_j)| \leq C(u) h^4,$$  

$$|D^2_x D^2_y S(x_i, y_j) - D^2_x D^2_y u(x_i, y_j)| \leq C(u) h^2.$$
Proof. First we prove (4.58). For $i = 0, \ldots, N + 1, j = 1, \ldots, N,$ we introduce

$$d_{i,j} = D_x^2S(x_i, y_j) - \left[ D_x^2u(x_i, y_j) - \frac{h^2}{12} D_x^4u(x_i, y_j) \right]. \quad (4.61)$$

Then

$$d_{i-1,j} + 4d_{i,j} + d_{i+1,j} = \phi_{i,j} - \psi_{i,j}, \quad i, j = 1, \ldots, N,$$

where

$$\phi_{i,j} = D_x^2S(x_{i-1}, y_j) + 4D_x^2S(x_i, y_j) + D_x^2S(x_{i+1}, y_j) - 6 \left[ D_x^2u(x_i, y_j) + \frac{h^2}{12} D_x^4u(x_i, y_j) \right], \quad (4.63)$$

$$\psi_{i,j} = D_x^2u(x_{i-1}, y_j) - \frac{h^2}{12} D_x^4u(x_{i-1}, y_j) + 4 \left[ D_x^2u(x_i, y_j) - \frac{h^2}{12} D_x^4u(x_i, y_j) \right] + D_x^2u(x_{i+1}, y_j) - \frac{h^2}{12} \left[ D_x^4u(x_{i-1}, y_j) + 10D_x^4u(x_i, y_j) + D_x^4u(x_{i+1}, y_j) \right]. \quad (4.64)$$

Since $S(\cdot, y_j) \in S_3, (2.1.7)$ in [1], (4.5), and $S = u = 0$ on $\partial \Omega$, imply that

$$D_x^2S(x_{i-1}, y_j) + 4D_x^2S(x_i, y_j) + D_x^2S(x_{i+1}, y_j) = 6h^{-2} \left[ u(x_{i-1}, y_j) - 2u(x_i, y_j) + u(x_{i+1}, y_j) \right], \quad i, j = 1, \ldots, N. \quad (4.65)$$

Using Taylor’s theorem, we obtain

$$u(x_{i+1}, y_j) = u(x_i, y_j) + hD_xu(x_i, y_j) + \frac{h^2}{2} D_x^2u(x_i, y_j) + \frac{h^3}{3!} D_x^3u(x_i, y_j) + \frac{h^4}{4!} D_x^4u(x_i, y_j) + \frac{h^5}{5!} D_x^5u(x_i, y_j) + \frac{h^6}{6!} D_x^6u(x_i, y_j),$$

where $x_{i-1} \leq x_i \leq x_{i+1}$, and hence

$$|h^{-2} [u(x_{i-1}, y_j) - 2u(x_i, y_j) + u(x_{i+1}, y_j)] - D_x^2u(x_i, y_j) + \frac{h^2}{12} D_x^4u(x_i, y_j)| \leq C(u)h^4, \quad i, j = 1, \ldots, N. \quad (4.66)$$

Using (4.63), (4.65), and (4.66), we obtain

$$|\phi_{i,j}| \leq C(u)h^4, \quad i, j = 1, \ldots, N. \quad (4.67)$$

Using Taylor’s theorem, we obtain

$$D_x^2u(x_{i+1}, y_j) = D_x^2u(x_i, y_j) + hD_x^2u(x_i, y_j) + \frac{h^2}{2} D_x^4u(x_i, y_j) + \frac{h^3}{3!} D_x^5u(x_i, y_j) + \frac{h^4}{4!} D_x^6u(x_i, y_j),$$

$$D_x^4u(x_{i+1}, y_j) = D_x^4u(x_i, y_j) + hD_x^4u(x_i, y_j) + \frac{h^2}{2} D_x^6u(x_i, y_j).$$
where \(x_{i-1} \leq \xi_i^-, \eta_i^- \leq x_i, \ x_i \leq \xi_i^+, \eta_i^+ \leq x_{i+1}\), and hence (4.64) gives

\[
(4.68) \quad |\psi_{i,j}| \leq C(u)h^4, \quad i, j = 1, \ldots, N.
\]

Using (4.61) and (4.52), we have

\[
(4.69) \quad |d_{i,j}| \leq C(u)h^4, \quad i = 0, N + 1, \quad j = 1, \ldots, N.
\]

It follows from (4.67)–(4.69) that moving \(d_{0,j}\) and \(d_{N+1,j}\) to the right-hand side of (4.62), we obtain, for each \(j = 1, \ldots, N\), a system in \(\{d_{i,j}\}_{i=1}^N\) with the matrix \(B\) of (2.10)–(2.12), and with each entry on the right-hand side bounded in absolute value by \(C(u)h^4\). Hence (4.58) follows from (4.61) and Lemma 2.2, and (4.59) follows from (4.58) by symmetry with respect to \(x\) and \(y\).

Next we prove (4.60). Since \(S(x, \cdot) \in S_3\) for \(x \in [0, 1]\), (2.1.7) in [1] gives

\[
(4.70) \quad D_x^2D_y^2S(x, y_{j-1}) + 4D_x^2D_y^2S(x, y_j) + D_y^2S(x, y_{j+1}) = 6h^{-2}[S(x, y_{j-1}) - 2S(x, y_j) + S(x, y_{j+1})], \quad j = 1, \ldots, N, \quad x \in [0, 1].
\]

Differentiating (4.70) twice with respect to \(x\), we obtain, for \(j = 1, \ldots, N, \ x \in [0, 1]\),

\[
(4.71) \quad D_x^2D_y^2S(x, y_{j-1}) + 4D_x^2D_y^2S(x, y_j) + D_y^2D_x^2S(x, y_{j+1}) = 6h^{-2}[D_x^2S(x, y_{j-1}) - 2D_x^2S(x, y_j) + D_x^2S(x, y_{j+1})].
\]

Using (4.71) with \(x = x_{i-1}, x_i, x_{i+1}\), we obtain, for \(i, j = 1, \ldots, N, \)

\[
(4.72) \quad D_x^2D_y^2S(x_{i-1}, y_{j-1}) + 4D_x^2D_y^2S(x_{i-1}, y_j) + D_y^2D_x^2S(x_{i-1}, y_{j+1}) = 6h^{-2}[D_x^2S(x_{i-1}, y_{j-1}) - 2D_x^2S(x_{i-1}, y_j) + D_x^2S(x_{i-1}, y_{j+1})],
\]

\[
(4.73) \quad D_x^2D_y^2S(x_i, y_{j-1}) + 4D_x^2D_y^2S(x_i, y_j) + D_y^2D_x^2S(x_i, y_{j+1}) = 6h^{-2}[D_x^2S(x_i, y_{j-1}) - 2D_x^2S(x_i, y_j) + D_x^2S(x_i, y_{j+1})],
\]

\[
(4.74) \quad D_x^2D_y^2S(x_{i+1}, y_{j-1}) + 4D_x^2D_y^2S(x_{i+1}, y_j) + D_y^2D_x^2S(x_{i+1}, y_{j+1}) = 6h^{-2}[D_x^2S(x_{i+1}, y_{j-1}) - 2D_x^2S(x_{i+1}, y_j) + D_x^2S(x_{i+1}, y_{j+1})].
\]

Adding (4.72), (4.74) and (4.73) multiplied through by \(4\), and using (4.65) and \(S = u = 0\) on \(\partial \Omega\), we obtain

\[
(4.75) \quad D_x^2D_y^2S(x_{i-1}, y_{j-1}) + 4D_x^2D_y^2S(x_{i-1}, y_j) + D_y^2D_x^2S(x_{i-1}, y_{j+1}) + 4D_x^2D_y^2S(x_i, y_{j-1}) + 16D_y^2D_x^2S(x_i, y_j) + 4D_x^2D_y^2S(x_i, y_{j+1}) + D_x^2D_y^2S(x_{i+1}, y_{j-1}) + 4D_x^2D_y^2S(x_{i+1}, y_j) + D_x^2D_y^2S(x_{i+1}, y_{j+1}) = 36h^{-4}\alpha_{i,j}, \quad i, j = 1, \ldots, N,
\]

where

\[
(4.76) \quad \alpha_{i,j} = u(x_{i-1}, y_{j-1}) - 2u(x_{i-1}, y_j) + u(x_{i+1}, y_{j-1}) - 2u(x_{i+1}, y_j) + u(x_{i-1}, y_{j+1}) - 2u(x_i, y_{j+1}) + u(x_{i+1}, y_{j+1}).
\]

Using (4.76) and the discussion on pages 290–292 in [9], we have

\[
(4.77) \quad |h^{-4}\alpha_{i,j} - D_x^2D_y^2u(x_i, y_j)| \leq C(u)h^2, \quad i, j = 1, \ldots, N.
\]
Equation (4.75) is equivalent to
\begin{align*}
D_2^2D_y^2(S-u)(x_{i-1},y_{j-1}) &+ 4D_2^2D_y^2(S-u)(y_{j-1}) \\
+2D_2^2D_y^2(S-u)(x_{i-1},y_{j+1}) &+ 4D_2^2D_y^2(S-u)(x_{i-1},y_{j+1})
\end{align*}
(A.78)
where
\begin{align*}
\beta_{i,j} &= D_2^2D_y^2u(x_{i-1},y_{j-1}) + 4D_2^2D_y^2u(x_{i-1},y_{j+1}) \\
&+ 4D_2^2D_y^2u(x_{i+1},y_{j+1}) + 16D_2^2D_y^2u(x_{i+1},y_{j+1}) \\
&+ 2D_2^2D_y^2u(x_{i+1},y_{j-1}) + 4D_2^2D_y^2u(x_{i+1},y_{j+1}) + D_2^2D_y^2u(x_{i+1},y_{j+1}).
\end{align*}
Using Taylor's theorem, we obtain
\begin{align*}
D_2^2D_y^2u(x_{i-1},y_{j+1}) &= D_2^2D_y^2u(x_{i-1},y_{j+1}) - hD_2^2D_y^2u(x_{i-1},y_{j+1}) \\
&+ h^2D_2^2D_y^2u(x_{i-1},y_{j+1}) + \epsilon_{i,j}^\pm,
\end{align*}
\begin{align*}
D_2^2D_y^2u(x_{i+1},y_{j+1}) &= D_2^2D_y^2u(x_{i+1},y_{j+1}) + hD_2^2D_y^2u(x_{i+1},y_{j+1}) \\
&+ h^2D_2^2D_y^2u(x_{i+1},y_{j+1}) + \mu_{i,j+1},
\end{align*}
where \(|\epsilon_{i,j}^\pm|, |\sigma_{i,j}^\pm|, |\mu_{i,j}|, |\nu_{i,j}^\pm| \leq C(u)h^2, i, j = 1, \ldots, N, and hence (4.79) gives
\begin{align*}
|\beta_{i,j} - 36D_2^2D_y^2u(x_{i-1},y_{j+1})| \leq C(u)h^2, i, j = 1, \ldots, N.
\end{align*}

It follows from (4.77) and (4.80) that the right-hand side of (4.78) is bounded by absolute value by \(C(u)h^2\). Using (4.2) and moving terms involving \(D_2^2D_y^2(S-u)(x_i,y_j)\), \(i = 0, N+1, j = 1, \ldots, N, i = 1, \ldots, N, j = 0, N+1,\) to the right-hand side of (4.78), we obtain a system in \(D_2^2D_y^2(S-u)(x_i,y_j)\) with the matrix \(B \otimes B\), where \(B\) is given in (2.10)-(2.12). By (4.51) and symmetry with respect to \(x\) and \(y\), each entry on the right-hand side in this system is bounded by absolute value by \(C(u)h^2\). Therefore, (4.60) follows from Lemma 2.3.

**Lemma 4.6.** If \(u \in C^3(\overline{\Omega})\) and
\begin{align*}
\sum_{m=0}^N \sum_{n=0}^N s_{m,n}B_m^D(x)B_n^D(y),
\end{align*}
are defined by (1.3)-(1.6) and (1.2)-(1.4), respectively, then
\begin{align*}
\max_{1 \leq m,n \leq N} \sum_{m,n} |s_{m,n} - u_{m,n}| \leq C(u)h^4, m, n = 1, \ldots, N.
\end{align*}

**Proof.** Using (4.2)-(4.4), (4.11)-(4.13), and following the derivations of (3.3) from (1.3), (3.5) from (1.4), and (3.6) from (1.5), we conclude that
\begin{align*}
s_{m,n} = u_{m,n}, m = 0, N+1, n = 0, \ldots, N+1, m = 1, \ldots, N, n = 0, N+1.
\end{align*}

We define \(\{w_{i,j}\}_{i,j=1}^N\) by
\begin{align*}
\Delta(S-u_h)(x_i,y_j) - \frac{h^2}{6}D_2^2D_y^2(S-u_h)(x_i,y_j) = w_{i,j}, i, j = 1, \ldots, N.
\end{align*}
Using (4.84), (1.6), and (4.10), we obtain
\[
w_{i,j} = D_y^2 S(x_i, y_j) + D_y^2 S(x_i, y_j) - \frac{h^2}{6} D_y^2 D_y^2 S(x_i, y_j) - D_y^2 u(x_i, y_j) - \frac{h^2}{12} [D_y^2 u(x_i, y_j) + D_y^2 u(x_i, y_j) + 2D_x^2 D_y^2 u(x_i, y_j)],
\]
and hence (4.58)–(4.60) and the triangle inequality imply that
\[
|w_{i,j}| \leq C(u)h^4, \quad i,j = 1, \ldots, N.
\]
Introducing \(v = [s_{1,1} - u_{1,1}, \ldots, s_{N,N} - u_{N,N}]^T\), \(w = [w_{1,1}, \ldots, w_{N,N}]^T\), using (4.84), (4.81), (4.83), and following the derivation of (3.10) from (1.6), we obtain
\[
6h^{-2} [6T \otimes I + (6I + T) \otimes T] v = w.
\]

Since \(h = 1/(N + 1)\), (4.85) gives \(h^2 \sum_{i,j=1}^{N} w_{i,j}^2 \leq C^2(u)h^{8}\) and hence (4.82) follows from (4.86), (3.12), and Lemma 2.4. □

**Theorem 4.3.** If \(u \in C^6(\Omega)\) and \(u_h\) and \(S\) in \(S^D \otimes S^D\) are defined by (1.3)–(1.6) and (4.2)–(4.5), respectively, then \(\|S - u_h\|^2_{C(\Omega)} \leq C(u)h^4\).

**Proof.** Since \(u_h - S\) is continuous on \(\Omega\), there is \((x_*, y_*)\) in \(\Omega\) such that
\[
\|u_h - S\|^2_{C(\Omega)} = |(S - u_h)(x_*, y_*)|.
\]
Hence, (3.1), (4.36), (4.83), and the triangle inequality give
\[
\|u_h - S\|^2_{C(\Omega)} \leq \sum_{m=1}^{N} \sum_{n=1}^{N} |s_{m,n} - u_{m,n}| \|B_m(x_*)\| \|B_n(y_*)\|.
\]
Since the above double sum reduces to at most nine terms, the desired result follows from Lemmas 4.6 and 2.1. □

**Theorem 4.4.** If \(u \in C^6(\Omega)\) and \(u_h\) in \(S^D \otimes S^D\) are the solutions of (1.1) and (1.3)–(1.6), respectively, then \(\|u - u_h\|^2_{C(\Omega)} \leq C(u)h^4\).

**Proof.** The required inequality follows from (4.14) and Theorems 4.1, 4.2, 4.3. □

**5. Other Schemes.** Consider the scheme for solving (1.1) formulated as follows:

Find \(u_h \in S^D \otimes S^D\) satisfying (1.3)–(1.5) and (1.9). This scheme is essentially the same as the scheme (4.1)–(4.3) in [4], except that (1.5) is replaced in [4] with
\[
\frac{1}{12} [13D_y^2 u_h(x_i, y_j) - 2D_y^2 u_h(x_i, y_{j+1}) + D_y^2 u_h(x_i, y_{j+2})] = f(x_i, y_j), \quad j = 0,
\]
\[
\frac{1}{12} [D_y^2 u_h(x_i, y_{j-2}) - 2D_y^2 u_h(x_i, y_{j-1}) + 13D_y^2 u_h(x_i, y_j)] = f(x_i, y_j), \quad j = N + 1,
\]
where \(i = 1, \ldots, N\). It follows from (3.1), the discussion in section 3, and (2.9) of [4] that the the matrix-vector form of (1.3)–(1.5) and (1.9) is
\[
(A \otimes B + B \otimes A)u = p,
\]
where \( u = [u_{1,1}, \ldots, u_{N,N}]^T \), \( p = [p_{1,1}, \ldots, p_{N,N}]^T \),

\[
p_{i,j} = f(x_i, y_j) - \sum_{m=0}^{N+1} \sum_{n=0}^{N+1} u_{m,n} \left[ L_x B_m^D(x_i) B_n^D(y_j) + B_m^D(x_i) L_y B_n^D(y_j) \right] \\
- \sum_{m=1}^{N} \sum_{n=0}^{N+1} u_{m,n} \left[ L_x B_m^D(x_i) B_n^D(y_j) + B_m^D(x_i) L_y B_n^D(y_j) \right],
\]

\( \{u_{i,j}\}_{j=0}^{N+1}, i = 0, N + 1, \{u_{i,j}\}_{i=1}^{N}, j = 0, N + 1 \), are given in (3.3), (3.5), (3.6), (5.2)

\[
A = \frac{1}{2h^2} (T^2 + 12T), \quad B = T + 6I,
\]

and \( T \) is defined in (2.12).

**Lemma 5.1.** Assume \( A, B \) are as in (5.2) and \( v = [v_{1,1}, \ldots, v_{N,N}]^T \), \( w = [w_{1,1}, \ldots, w_{N,N}]^T \) are such that \( (A \otimes B + B \otimes A) v = w \). Then

\[
\max_{1 \leq i, j \leq N} v_{i,j}^2 \leq C h^2 \sum_{i=1}^{N} \sum_{j=1}^{N} w_{i,j}^2.
\]

**Proof.** It follows from (5.2) that

\[
A \otimes B + B \otimes A = \frac{1}{2h^2} (T^2 \otimes T) + \frac{3}{h^2} (T^2 \otimes I) + \frac{6}{h^2} (T \otimes T) + \frac{36}{h^2} (T \otimes I) \\
+ \frac{1}{2h^2} (T \otimes T^2) + \frac{3}{h^2} (I \otimes T^2) + \frac{6}{h^2} (T \otimes T) + \frac{36}{h^2} (I \otimes T) = 36 [r(T) + s(T)],
\]

where for an \( N \times N \) matrix \( P \),

\[
r(P) = h^{-2} (P \otimes I + I \otimes P),
\]

\[
s(P) = \frac{1}{3h^2} (P \otimes P) + \frac{1}{12h^2} (P^2 \otimes P + P \otimes P^2) + \frac{1}{12h^2} (P^2 \otimes I + I \otimes P^2).
\]

First, we will show that

\[
([r(T) + s(T)]z, z) \leq \frac{2}{9} (r(T)z, z), \quad z \in R^{N^2},
\]

where \((\cdot, \cdot)\) is the standard inner product in \( R^{N^2} \). It follows from (2.14) and \( Q^T = Q \) for \( Q \) of (2.16) that

\[
(r(T)z, z) = ([Q \otimes Q]r(\Lambda)[Q \otimes Q]z, z) = (r(\Lambda)[Q \otimes Q]z, [Q \otimes Q]z),
\]

\[
(s(T)z, z) = ([Q \otimes Q]s(\Lambda)[Q \otimes Q]z, z) = (s(\Lambda)[Q \otimes Q]z, [Q \otimes Q]z),
\]

where \( \Lambda \) is given in (2.15). Hence (5.6) is equivalent to

\[
([r(\Lambda) + s(\Lambda)]z, z) \leq \frac{2}{9} (r(\Lambda)z, z), \quad z \in R^{N^2},
\]

which, by (5.4), (5.5), and (2.15), is in turn equivalent to

\[
g(\lambda_i, \lambda_j) \leq 0, \quad i, j = 1, \ldots, N,
\]
where
\[ g(x, y) = \frac{7}{9} (x + y) + \frac{1}{3} xy + \frac{1}{72} (x^2 y + xy^2) + \frac{1}{12} (x^2 + y^2). \]

It follows from (2.15) that \(-4 \leq \lambda_i \leq 0, \ i = 1, \ldots, N\). Hence, (5.7) follows from
\[ g(x, y) \leq 0, \quad x, y \in [-4, 0], \]
which is established using elementary calculus.

Theorem 4.3. Hence Lemma 5.1 implies (4.82) and the desired result follows from the proof of (4.83) that the matrix-vector form of (5.10) is
\[ (1.3) - (1.5) \text{ and } (1.9), \text{ respectively, then } L_{x} \text{ and } \mathcal{S}(x) \text{ are \ positive \ definite. \ Hence, } (5.6) \text{ and } (5.7) \text{ on page 135 in [9] imply that} \]
\[ \| r(T) \|_2 \leq \frac{9}{2} \| r(T) + s(T) \|_2, \quad z \in \mathbb{R}^{2}. \]

where \( \| \cdot \|_2 \) is the two vector norm. It is known (see, for example, the embedding theorem on page 281 in [9]) that
\[ \max_{1 \leq i, j \leq N} z_{i,j}^2 \leq \frac{1}{4} h^2 \| r(T) \|_2^2, \quad z = [z_{1,1}, \ldots, z_{N,N}]^T \in \mathbb{R}^{2}. \]

Hence the desired result follows from (5.9), (5.8), and (5.3).

Theorem 5.1. If \( u \in C^6(\mathbb{R}) \) and \( u_h \) and \( S \) are defined by (1.3)–(1.5) and (1.9), and (4.2)–(4.5), respectively, then
\[ \| S - u_h \|_{C(\mathbb{R})} \leq C(u) h^4. \]

Proof. Following the proof of Lemma 4.6, we define \( \{ w_{i,j} \}_{i,j=1}^N \) by
\[ (L_x + L_y)(S - u_h)(x_i, y_j) = w_{i,j}, \quad i, j = 1, \ldots, N. \]

Using (5.10), (1.9), (1.1), we obtain
\[ w_{i,j} = L_x S(x_i, y_j) - D_x^2 u(x_i, y_j) + L_y S(x_i, y_j) - D_y^2 u(x_i, y_j). \]

Equations (1.10), (4.58), and (4.52) give, for \( i, j = 1, \ldots, N, \)
\[ L_x S(x_i, y_j) - D_x^2 u(x_i, y_j) = D_x^2 S(x_i, y_j) - D_x^2 u(x_i, y_j) \]
\[ + \frac{1}{12} [D_x^2 S(x_{i-1}, y_j) - 2 D_x^2 S(x_i, y_j) + D_x^2 S(x_{i+1}, y_j)] \]
\[ = - \frac{h^2}{12} D_x^2 u(x_i, y_j) + \frac{1}{12} [D_x^2 u(x_{i-1}, y_j) - 2 D_x^2 u(x_i, y_j) + D_x^2 u(x_{i+1}, y_j)] \]
\[ - \frac{h^4}{144} [D_x^2 u(x_{i-1}, y_j) - 2 D_x^2 u(x_i, y_j) + D_x^4 u(x_{i+1}, y_j)] + \epsilon_{i,j}, \]
where \( |\epsilon_{i,j}| \leq C(u) h^4, \ i, j = 1, \ldots, N. \) Hence Taylor’s theorem and similar considerations for \( L_y S(x_i, y_j) - D_y^2 u(x_i, y_j) \) show that (4.85) holds. It follows from (4.81) and (4.83) that the matrix-vector form of (5.10) is
\[ (A \otimes B + B \otimes A) v = w, \]
where \( A, B \) are as in (5.2), \( v = [s_{1,1} - u_{1,1}, \ldots, s_{N,N} - u_{N,N}]^T, \ w = [w_{1,1}, \ldots, w_{N,N}]^T. \)

Hence Lemma 5.1 implies (4.82) and the desired result follows from the proof of Theorem 4.3.

Theorem 5.2. If \( u \in C^6(\mathbb{R}) \) and \( u_h \) in \( S^D \otimes S^D \) are the solutions of (1.1), and (1.3)–(1.5) and (1.9), respectively, then
\[ \| u - u_h \|_{C(\mathbb{R})} \leq C(u) h^4. \]
Proof. The required inequality follows from (4.14) and Theorems 4.1, 4.2, 5.1. □

It is claimed in Theorem 4.1 of [8] that for the scheme (1.3) and (1.7)–(1.9), one has (5.11) provided that \( u \in C^6(\Omega) \). The proof of this claim in [8] is based on using \( Z \) defined in (4.6)–(4.9) as a comparison function. It is claimed, for example, in Lemma 2.1 of [8] that \( Z \) has properties (4.58) and (4.59), that is, (4.58) and (4.59) hold with \( Z \) in place of \( S \). Unfortunately, numerical examples indicate that such property does not hold even in one dimensional case. Specifically, for \( u(x) = x(x-1)e^x \) and \( Z \in \mathcal{S}^D \) such that

\[
Z(x_i) = u(x_i), \quad i = 1, \ldots, N, \quad Z''(x_i) = u''(x_i), \quad i = 0, N + 1,
\]

we only have

\[
\max_{1 \leq i \leq N} \left| Z''(x_i) - u''(x_i) + \frac{h^2}{12} u^{(4)}(x_i) \right| = Ch^2
\]

and not better. It should be noted that the convergence analysis of [6] for two-point boundary value problems involves the comparison function \( S \in \mathcal{S}^D \) defined by

\[
S(x_i) = u(x_i), \quad i = 1, \ldots, N, \quad S''(x_i) = u''(x_i) - \frac{h^2}{12} u^{(4)}(x_i), \quad i = 0, N + 1,
\]

which, in part, was motivation for the definition (4.2)–(4.5). The convergence analysis of the scheme (4.2)–(4.4) in [8] remains an open problem. We believe that such analysis may require proving stability not only with respect to the right-hand side but also with respect to the boundary conditions.

6. Numerical Results. We used scheme (1.3)–(1.6) and algorithm of section 3 to solve a test problem (1.1). The computations were carried out in double precision. We determined the nodal and global errors using the formulas

\[
\|w\|_h = \max_{0 \leq i,j \leq N+1} |w(x_i, y_j)|, \quad \|w\|_{C(\Omega)} \approx \max_{0 \leq i,j \leq 501} |w(t_i, t_j)|,
\]

where \( t_i = i/501, \ i = 1, \ldots, 501 \). Convergence rates were determined using the formula

\[
\text{rate} = \frac{\log(e_{N/2}/e_N)}{\log((N+1)/(N/2+1))},
\]

where \( e_N \) is the error corresponding to the partition \( \rho_x \times \rho_y \).

We took \( f \) in (1.1) corresponding to the exact solution

\[
u(x, y) = 3e^{xy}(x^2 - x)(y^2 - y).
\]

We see from the results in Tables 1 and 2 that the scheme (1.3)–(1.6) produces fourth order accuracy for \( u \) in both the discrete and the continuous maximum norms. We also observe superconvergence phenomena since the derivative approximations at the partition nodes are of order four.

REFERENCES

Table 1

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Table 2

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