Conservation of Energy for Unforced Spring-Mass Systems

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Conserved Quantities

A conserved quantity is some function (say, $E$) of the dependent variable(s) that is constant along a trajectory of the system (in this case, the solution to an ODE). Moreover, they are an important tool for the study of dynamical systems and are especially helpful with the qualitative analysis of non-linear systems. With that in mind, we explore the conservation of energy for a simple linear oscillator:

$$m \frac{d^2y}{dt^2} + b \frac{dy}{dt} + ky = 0,$$

where $m$ is the mass, $b$ is the damping coefficient, and $k$ is the spring constant.
Finding an Expression for the Total Energy

Kinetic energy, $K$, will be given by one-half of the mass times the square of the velocity:

$$K = \frac{1}{2} m \left( \frac{dy}{dt} \right)^2$$

Potential energy, $U$, is the integral over the displacement of the applied force due to the spring:

$$U = \int_{0}^{y} k \eta \, d\eta = \frac{1}{2} ky^2$$

Finally, the total energy, $E$, is the sum of the kinetic and potential energies:

$$E = \frac{1}{2} m \left( \frac{dy}{dt} \right)^2 + \frac{1}{2} ky^2$$
Undamped Spring-Mass System

We begin with the ODE for an unforced, undamped spring-mass system:

\[ my'' + ky = 0 \]

Next, let \( v = y' \). Thus, \( v' = y'' = \frac{-k}{m} y \). Then, we can write the second order equation as a system of first order equations:

\[ y' = v \]
\[ v' = \frac{-k}{m} y \]

Hence, \( E = \frac{1}{2} mv^2 + \frac{1}{2} ky^2 \).
Next we check $\frac{dE}{dt}$ to see how energy changes w.r.t. time:

$$\frac{dE}{dt} = \frac{d}{dt} \left( \frac{1}{2} mv^2 + \frac{1}{2} ky^2 \right)$$

$$= m v v' + k y y'$$

$$= m v \frac{-k}{m} y + k y v$$

$$= 0$$

Thus, the total energy is constant for all time.
Figure: Numerical solution to the ODE with $m = 1$, $k = 3$, $y(0) = y'(0) = 0.65$
**Undamped Spring-Mass System**

**Figure:** Total energy plotted as a surface with a particular numerical example plotted as a parametric curve.
Damped Spring-Mass System

We begin with the ODE for an unforced, damped spring-mass system:

\[ my'' + by' + ky = 0 \]

Next, let \( v = y' \). Thus, \( v' = y'' = \frac{-k}{m}y - \frac{b}{m}v \). Then, we can write the second order equation as a system of first order equations:

\[
\begin{align*}
y' &= v \\
v' &= \frac{-k}{m}y - \frac{b}{m}v
\end{align*}
\]

As before, \( E = \frac{1}{2}mv^2 + \frac{1}{2}ky^2 \)
Damped Spring-Mass System

Next we check \( \frac{dE}{dt} \) to see how energy changes w.r.t. time:

\[
\frac{dE}{dt} = \frac{d}{dt} \left( \frac{1}{2}mv^2 + \frac{1}{2}ky^2 \right)
\]

\[
= mvv' + kyy'
\]

\[
= mv \left( \frac{-k}{m}y - \frac{b}{m}v \right) + kyv
\]

\[
= -bv^2 \leq 0
\]

Thus, the total energy is decreasing for all time, i.e., the total energy is **not** conserved.
Damped Spring-Mass System

Figure: Numerical solution to the ODE with \( m = 1, k = 3, b = 0.25, \)
\( y(0) = y'(0) = 0.65 \)
Damped Spring-Mass System

**Figure:** Total energy plotted as a surface with a particular numerical example plotted as a parametric curve.
Damped Spring-Mass System

Figure: Total energy plotted as a surface with a particular numerical example plotted as a parametric curve.
Figure: The motion of a forced spring mass system. Note the difference in scale of the transient response and the steady state solution.